Strong Formulations for Mixed Integer Programs: Valid Inequalities and Extended Formulations

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Abstract

We examine progress over the last fifteen years in finding strong valid inequalities and tight extended formulations for simple mixed integer sets lying both on the "easy" and "hard" sides of the complexity frontier. Most progress has been made in studying sets arising from knapsack and single node flow sets, and a variety of sets motivated by different lot-sizing models. We conclude by citing briefly some of the more intriguing new avenues of research.

Keywords: Mixed Integer Programming, Strong Valid Inequalities, Convex Hull, Extended Formulations, Single Node Flow Sets, Lot-sizing

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1 Introduction

Looking back at a survey on strong formulations for Mixed Integer Programming (MIP) [62] published in 1989, it is striking on the one hand to see what has been learnt and what progress has been made, and on the other to see that many of the questions that need answering remain the same as fourteen years ago.

As before we are interested in sets (structures) that appear in general mixed integer programs, or in fairly generic problems such as fixed charge network flow, production planning and scheduling problems. The idea is then to study these sets, so as to develop either good a priori formulations, or strong valid inequalities that can be added dynamically as cutting planes.

Compared to fourteen years ago, several important changes can be observed:

- *branch-and-cut* and *branch-cut-and-price* have definitely replaced branch-and-bound as the algorithmic paradigm
- *cutting planes* have been an integral part of the commercial mixed integer programming systems such as Xpress [66] and Cplex [15] for about five years, and have proved very effective on a wide variety of problems
- most of the cuts, and even the separation algorithms or heuristics used in these systems are not new – Gomory mixed integer cutting planes [19] date from 1960, knapsack cover inequalities from the 70s, flow cover, path and mixed integer rounding inequalities from the 80s. What has changed is that much greater efforts are made to *detect these sets* within a general MIP, and the remarkable versatility of *mixed integer rounding* as a cut generation technique has become apparent [39]
- the importance of *superadditive lifting functions* to convert valid inequalities into strong valid inequalities has been made clear [23]
- the study of single item lot-sizing sets has been fundamentally modified by the study of special *"Wagner-Whitin" objective functions* leading to the study of simpler sets that can be viewed as the intersections of discrete lot-sizing sets [47]. This in turn has led to the study of *multiple MIP* [40] and *mixed MIP sets* [24], as well as to the study of *dynamic knapsack sets* [33]

- progress in developing tight extended formulations has indicated the need for smaller *approximations* of such sets [57]

- the idea of introducing *extended formulations on the fly* (or integer generating sets) even for difficult sets, see the integer basis method in [26, 27], is intriguing and suggests many new possibilities

- the lift-and-project approach [9], and its generalizations including positive semi-definite reformulations [36], the reformulation linearization technique [54], and recent generalizations [12, 30] provide both extended formulations and new cutting planes.

We now outline the contents of this paper. In Section 2 we briefly discuss the basic decomposition approach used in tackling an MIP, and the questions to be asked relative to a given MIP set X appearing as a superset or relaxation of a specific problem instance. We also introduce the basic two variable MIP set, and the mixed integer rounding inequality.

In Sections 3-6, we examine different families of MIP sets, on both the "easy" and "hard" sides of the complexity frontier. For each family we present a few basic results, as well as an indication of recent progress. In Section 3 we start on the "hard" side looking at knapsack and single node flow sets, and show an example of superadditive lifting. In Section 4 we look at knapsack and single node flow sets that are "easy" because the data is restricted (e.g. the arc capacities can only take one or two different values). In Section 5 we stay on the "easy" side and look at multiple and mixed MIR sets that have been derived from constant capacity lot-sizing problems. In Section 6 we look further at lot-sizing sets, considering both the uncapacitated case, and dynamic knapsack sets derived from the varying capacity "hard" case. In Section 7 we indicate briefly two other research directions that are being pursued: tight all integer reformulations for "on the fly" reformulation, and approximations of tight extended formulations. We terminate with a mention of a few related research topics of importance for MIP.

2 The Approach

For most practical MIP problems there is little chance of finding a fast algorithm or a "nice" description of the convex hull of the solution set Z. Therefore a natural strategy is to describe the solution set in terms of simpler sets that we know something about, or that appear amenable to study. In particular the case where $Z = \cap_{i=1}^{I} Z^i$ is the intersection of simple sets Z^i with some structure is of practical interest. For simplicity we suppose below that $I = 2$.

2.1 The Decomposition Approach

We suppose that the MIP to be solved is given in the form

$$\min\{cx + fy : (x, y) \in Z\}$$

where $Z = P \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2})$ and $P = \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : Ax + Gy \leq b\}$ is a formulation (polyhedron) of the set Z. We suppose that $Z = Z^1 \cap Z^2$ where each set Z^i has formulation P^i for $i = 1, 2$.

There are now two cases:

i) optimization over Z^i is *easy*, and a tight formulation $\tilde{P}^i = \text{conv}(Z^i)$ is known for Z^i

ii) optimization over Z^i is easy or hard, and an improved formulation $\tilde{P}^i$ with $\text{conv}(Z^i) \subset \tilde{P}^i \subset P^i$ is known for Z^i.

Clearly we now have at least implicitly an improved formulation for Z, namely

$$\tilde{P} = \tilde{P}^1 \cap \tilde{P}^2 \supseteq \text{conv}(Z^1) \cap \text{conv}(Z^2),$$

with equality holding in the ideal case.

The form of the improved formulations has an influence on the algorithmic approach taken. Now there are again two cases.

i) the improved formulation involves only the *original variables* x, y. In this case if the formulation is compact, it can be added *a priori* to the original formulation. If the formulation is large, a separation algorithm or heuristic is needed to generate valid inequalities for $\tilde{P}^i$ as *cutting planes*

ii) the improved formulation is a compact extended formulation with additional variables

$$Q^i = \{(x, y, w) : C^i x + D^i y + E^i w \geq b\}$$

with the property that

$$proj_{x,y} Q^i = \tilde{P}^i.$$

In this case, it can in principle be added *a priori* to the original formulation. However if the reformulation is in practice too large, an *approximation* to Q^i may be required.

2.2 Studying Simple MIP Sets

Suppose that we have isolated a set $X = Z^i$ to be studied. A variety of questions may be of interest. The first six are standard. Of these the first three deal with the Optimization problem, while the next three deal with the problem of Separation and of finding a description of $\text{conv}(X)$. Here it is important to remember the "polynomial equivalence" of Optimization and Separation [21]. The last three questions are relatively new.

1. Characterize the optimal solutions of $OPT(X, c)$, namely the optimization problem $\min\{cx : x \in X\}$.
2. What is the complexity of the problem $OPT(X, c)$
3. Describe an algorithm for $OPT(X, c)$
4. Describe a family $\mathcal{F}$ of valid inequalities for X or $\text{conv}(X)$
5. Describe a separation algorithm for $\text{conv}(X)$, or for the polyhedron described by the family $\mathcal{F}$
6. Describe an extended formulation for X, if possible providing a tight formulation of $\text{conv}(X)$
7. (Tight Formulations with Additional Constraints). Suppose that $P = \text{conv}(X)$, and $Y \subset X$ with $Y = X \cap Q$ where Q is a polyhedron. For which polyhedra Q is it true that

$$\mathrm{conv}(Y) = \mathrm{conv}(X) \cap Q?$$

8. (Approximate Extended Formulations). Given an extended formulation Q for $\mathrm{conv}(X)$, describe and characterize more compact relaxations of Q

9. (Integer Generating Sets). Describe an integer generating set for X, namely a reformulation
$$X = \{x : x = C\lambda, D\lambda = e, \lambda \in \mathbb{Z}^r_+\},$$
where the r columns of C "generate" X, and r is ideally not too large.

In Sections 3-6 we will be particularly interested in questions 4,6 and 7, whereas questions 8 and 9 will be discussed briefly in Section 7.

2.3 The Two Variable MIP Set

Here we consider the set

$$X^{MI} = \{(s, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^1 : s + y \geq b\}.$$

Proposition 1.
i) Suppose that $f = b - \lfloor b \rfloor > 0$. *The* simple mixed integer rounding *inequality*

$$s \geq f(\lceil b \rceil - y) \tag{1}$$

is valid for X^{MI}.

ii) The polyhedron

$$\begin{aligned} s + y &\geq b \\ s + fy &\geq f\lceil b \rceil \\ s &\geq 0 \end{aligned}$$

describes the convex hull of X^{MI}.

By taking $z = -y$ and $d = -b$, we obtain the result in a different form that is needed below.

Corollary 2. *The inequality* $s \geq (1 - f_d)(z - \lfloor d \rfloor)$ *is valid for* $X^{MI}_{\leq} = \{(s, z) \in \mathbb{R}^1_+ \times \mathbb{Z}^1 : z \leq d + s\}$ *where* $f_d = d - \lfloor d \rfloor$.

In the next four sections we examine different generalizations of the set X^{MI}. For each set examined, we derive valid inequalities, and if appropriate indicate whether these inequalities suffice to describe the convex hull, and/or give an extended formulation.

3 Knapsack and Single Node Flow Sets

We study three sets taht are simple mixed integer generalizations of 0-1 or integer knapsack sets.

1. *The Integer Knapsack with a Single Continuous Variable Set*

$$X^{IK} = \{(s, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^n_+ : \sum_{j=1}^{n} a_j y_j \leq b + s\}.$$

2. *The Binary Knapsack with a Single Continuous Variable Set*

$$X^{BK} = \{(s, y) \in \mathbb{R}^1_+ \times \{0, 1\}^n : \sum_{j=1}^{n} a_j y_j \leq b + s\}.$$

3. *The Binary Single Node Flow Set*

$$X^{SNF} = \{(x, y) \in \mathbb{R}^n_+ \times \{0, 1\}^n : \sum_{j \in N_1} x_j - \sum_{j \in N_2} x_j \leq b, x_j \leq a_j y_j \text{ for } j = 1, \ldots, n\},$$

where N_1, N_2 is a partition of N with $n_i = |N_i|$ for $i = 1, 2$ and $n = n_1 + n_2$.

In all three cases above, we assume that $a_j > 0$ for $j = 1, \ldots, n$.

3.1 The Integer Knapsack with a Single Continuous Variable Set

Here we consider the set

$$X^{IK} = \{(s, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^n_+ : \sum_{j=1}^{n} a_j y_j \leq b + s\}.$$

Let $f_j = a_j - \lfloor a_j \rfloor$, and $f_b = b - \lfloor b \rfloor$. Now setting $z = \sum_{j:f_j \le f_b} \lfloor a_j \rfloor y_j + \sum_{j:f_j > f_b} \lceil a_j \rceil y_j$, we see that

$$z \le \sum_{j=1}^{n} a_j y_j + \sum_{j:f_j > f_b} (1 - f_j) y_j \le b + s + \sum_{j:f_j > f_b} (1 - f_j) y_j = b + s'$$

where $s' = s + \sum_{j:f_j > f_b} (1 - f_j) y_j \in \mathbb{R}^1_+$ and $z \in \mathbb{Z}^1$. Now by Proposition 1, the simple MIR inequality $z \le \lfloor b \rfloor + \frac{s'}{1-f_b}$ is valid. Substituting for s' and z gives the following result.

Proposition 3. *[42] The mixed integer rounding (MIR) inequality*

$$\sum_{j=1}^{n} (\lfloor a_j \rfloor + \frac{(f_j - f_b)^+}{1 - f_b}) y_j \le \lfloor b \rfloor + \frac{s}{1 - f_b} \tag{2}$$

is valid for X^{IK}.

Note that with $\alpha = f_b$, the inequality (2) can also be written as

$$\sum_{j=1}^{n} F_\alpha(a_j) x_j + \bar{F}_\alpha(-1) s \le F_\alpha(b)$$

where $F_\alpha : \mathbb{R}^1 \to \mathbb{R}^1$ is a nondecreasing superadditive (MIR) function given by

$$F_\alpha(d) = d + \frac{(d - \lfloor d \rfloor - \alpha)^+}{1 - \alpha}$$

and $\bar{F}_\alpha(d) = \min[0, \frac{d}{1-\alpha}]$.

3.2 The Binary Knapsack with a Single Continuous Variable Set

Here we consider the set

$$X^{BK} = \{(s, y) \in \mathbb{R}^1_+ \times \{0,1\}^n : \sum_{j=1}^{n} a_j y_j \le b + s\}.$$

We first derive a family of strong valid inequalities, based on covers (or infeasible points).

Definition 1. *A set* $C \subseteq N$ *is a* cover *for* X^{BK} *if*
i) $\sum_{j \in C} a_j = b + \lambda$ *with* $\lambda > 0$, *and*
ii) if $k = \arg\max\{a_j : j \in C\}$, *then* $a_k > \lambda$.

Proposition 4. *The* continuous cover inequality

$$s + \sum_{j \in C} \min[a_j, \lambda](1 - y_j) \geq \lambda \tag{3}$$

is valid for X^{BK}.

We present a proof showing a typical use of the MIR inequalities, and the ease with which such inequalities can be generated.

Proof. As $x_j \geq 0$ and $a_j \geq 0$ for all $j \in N \setminus C$, we consider the relaxation

$$\{(s, y) \in \mathbb{R}^1_+ \times \{0, 1\}^n : \sum_{j \in C} a_j y_j \leq b + s\}.$$

Introducing the complementary variables $\bar{y}_j = 1 - y_j$ for $j \in C$, the constraint becomes

$$s + \sum_{j \in C} a_j \bar{y}_j \geq \lambda,$$

or after division by a_k

$$\sum_{j \in C} \frac{-a_j}{a_k} \bar{y}_j \leq \frac{-\lambda}{a_k} + \frac{s}{a_k}.$$

Now taking the set consisting of this constraint, $y_j \in \mathbb{Z}^1_+$ for $j \in C$ and $s \geq 0$, the resulting MIR inequality is

$$-\sum_{j \in C} \min[1, \frac{a_j}{\lambda}] \bar{y}_j \leq -1 + \frac{s}{\lambda},$$

which after multiplication by λ and substitution for $\bar{y}_j$ gives the required inequality. □

We now demonstrate the role of superadditive lifting in generating strong valid inequalities.

Consider again the set X^{BK}. We now temporarily set $y_j = 0$ for $j \in N \setminus C$ giving the set $\tilde{X}^{BK} = X^{BK} \cap \{y : y_j = 0 \text{ for } j \in N \setminus C\}$. Now the inequality (3) derived above is valid and facet-defining for conv($\tilde{X}^{BK}$). We convert it into a facet-defining inequality for conv(X^{BK}) by restoring the variables y_j for $j \in N \setminus C$ with appropriate coefficients. In particular it is necessary to calculate the "lifting" function

$$\phi_C(u) = \min\{s + \textstyle\sum_{j \in C} \min[a_j, \lambda](1 - y_j) - \lambda :$$
$$\textstyle\sum_{j \in C} a_j y_j - s \leq b - u, y_j \in \{0, 1\} \text{ for } j \in C, s \in \mathbb{R}^1_+\}.$$

Let $\tilde{C} = \{j \in C : a_j > \lambda\}$ and $r = |\tilde{C}| \geq 1$. Reorder the elements so that $\tilde{C} = \{a_1, \ldots a_r\}$ with $a_1 \geq \ldots \geq a_r > \lambda$, and let $A_j = \sum_{i=1}^{j} a_i$ for $j = 1, \ldots, r$.

Now $\phi_C(u)$ takes the following values:

$$\phi_C(u) = \begin{cases} \lambda(j-1) & \text{if } \ A_{j-1} \leq u \leq A_j - \lambda \\ \lambda(j-1) + u - (A_j - \lambda) & \text{if } \ A_j - \lambda \leq u \leq A_j \\ \lambda(r-1) + u - (A_j - \lambda) & \text{if } \ u \geq A_r - \lambda. \end{cases}$$

Because ϕ_C is *superadditive on* $\mathbb{R}^1_+$ (i.e. $\phi_C(u) + \phi_C(v) \leq \phi_C(u+v)$ for $u, v \in \mathbb{R}^1_+$), results in [23, 60] give that ϕ_C can be used to calculate all the missing coefficients for $j \in N \setminus C$ (so-called *sequence independent lifting*).

Theorem 5. *[39] The lifted cover inequality*

$$s + \sum_{j \in C} \min[a_j, \lambda](1 - y_j) \geq \lambda + \sum_{j \in N \setminus C} \phi_C(a_j) y_j$$

is valid and facet-defining for conv(X^{BK}).

Note that in general, if the lifting function is not superadditive, it is necessary to calculate functions $\phi_{C \cup \{j_1\}}, \phi_{C \cup \{j_1, j_2\}}, \ldots, \phi_{C \cup \{j_1, \ldots, j_r\}}$ where $j_1, \ldots, j_r$ is some ordering of the elements of $N \setminus C$.

3.3 The Binary Single Node Flow Set

Here we consider the set

$$X^{SNF} = \{(x, y) \in \mathbb{R}^n_+ \times \{0,1\}^n : \sum_{j \in N_1} x_j - \sum_{j \in N_2} x_j \leq b, \\ x_j \leq a_j y_j \text{ for } j = 1, \ldots, n\}.$$

We now extend the definition of a cover.

Definition 2. *A set $(C_1, C_2) \subseteq (N_1, N_2)$ is a* flow cover *for X^{SNF} if*
i) $\sum_{j \in C_1} a_j - \sum_{j \in C_2} a_j - b = \lambda > 0$, *and*
ii) $\bar{a} = \max_{j \in C_1} a_j > \lambda$.

Proposition 6. *[35] If (C_1, C_2) is a flow cover for X^{SNF} and (C_i, L_i, R_i) is a partition of N_i for $i = 1, 2$, the* MIR flow cover inequality

$$\sum_{j \in C_1} \{x_j + [a_j + \lambda F(-\tfrac{a_j}{\bar{a}})](1 - y_j)\} + \sum_{j \in L_1} x_j - \sum_{j \in L_1} [a_j - \lambda F(\tfrac{a_j}{\bar{a}})] y_j$$
$$\leq b + \sum_{j \in C_2} a_j - \sum_{j \in C_2} \lambda F(\tfrac{a_j}{\bar{a}})(1 - y_j) - \sum_{j \in L_2} \lambda F(-\tfrac{a_j}{\bar{a}}) y_j + \sum_{j \in R_2} x_j + s$$

is valid for X^{SNF}, where $F = F_\alpha$ with $\alpha = \frac{\bar{a} - \lambda}{\bar{a}}$.

Selecting a specific value of $\bar{a}$ leads to

Corollary 7. *If $\bar{a} = \max_{j \in C_1} a_j$, the MIR flow cover is at least as strong as the GFC2 inequality [58]*

$$\sum_{j \in C_1} x_j + \sum_{j \in C_1} (a_j - \lambda)^+ (1 - y_j) + \sum_{j \in L_1} x_j - \sum_{j \in L_1} (\max[a_j, \bar{a}] - \lambda) y_j$$
$$\leq b + \sum_{j \in C_2} a_j - \sum_{j \in C_2} \min[\lambda, (a_j - (\bar{a} - \lambda))^+](1 - y_j)$$
$$+ \sum_{j \in L_2} \max[a_j - (\bar{a} - \lambda), \lambda] y_j + \sum_{j \in R_2} x_j + s.$$

A wide variety of related models and extensions of knapsack and single node flow sets have been studied recently, in particular knapsack sets with a single continuous variable and bounded integer variables [5], the lifting of flow cover inequalities [22], and knapsack sets with 0-1 variables and bounded continuous variables [51, 52]. A recent survey [35] examines many of these models, as well as discussing superadditive lifting in some detail.

4 Single Node Flow Models with Restricted Data

Here we consider briefly three special cases of the models of Section 3 that have arisen in practical network design models and that are easy because the data (arc capacities) are restricted to just one or two values. Apart from being interesting in their own right, they may provide useful insights and/or relaxations in tackling more complicated sets.

Here we look at three sets.

1. *Single Node Flow with Constant Capacities*

$$X^{SNF-CC} = \{(x,y,s) \in \mathbb{R}^n_+ \times \{0,1\}^n \times \mathbb{R}^1_+ : \sum_{j \in N_1} x_j - \sum_{j \in N_2} x_j \le b + s,$$
$$x_j \le y_j \text{ for } j \in N_1 \cup N_2\}$$

the special case of a single node flow set X^{SNF} of Section 3.3 in which $a_j = 1$ for all $j \in N_1 \cup N_2$.

2. *Single Node Inflow with Divisible Capacities*

$$X^{SNF-DIV} = \{(x,y,s) \in \mathbb{R}^n_+ \times \mathbb{Z}^n_+ \times \mathbb{R}^1_+ : \sum_{j \in N_1} x_j \le b + s, x_j \le a_j y_j \text{ for } j \in N_1\}$$

with $1 = a_1|a_2|\ldots|a_n$, where $x|y$ means that y is an integer multiple of x.

3. *Single Node Inflow with Constant Variable Lower and Upper Bounds*

$$X^{SNF-CC-LB} = \{(x,y,s) \in \mathbb{R}^{n_1}_+ \times \mathbb{Z}^{n_1}_+ \times \mathbb{R}^1_+ : \sum_{j \in N_1} x_j \le b + s,$$
$$l_j y_j \le x_j \le a_j y_j \text{ for } j \in N_1\}$$

where the a_j, l_j take at most two distinct non-zero values, for example $l_j = 1, a_j = a$ for all j, or $l_j = 0, a_j \in \{1, a\}$ for all $j \in N_1 \cup N_2$.

4.1 Single Node Flow with Constant Capacities

This model has been studied in [45] for the case with only inflows, and more recently in [7] under the name cut-set polyhedra.

Proposition 8. *[7] Every nontrivial valid inequality for the set*

$$X^{SNF-CC}_{\ge} = \{(x,y,s) \in \mathbb{R}^n_+ \times Z^n_+ \times \mathbb{R}^1_+ : s + \sum_{j \in N_1} x_j - \sum_{j \in N_2} x_j \ge b, x_j \le y_j \text{ for } j \in N\}$$

is of the form

$$s + f \sum_{j \in L_1} y_j + \sum_{j \in R_1} x_j \ge f\lceil b \rceil + \sum_{j \in L_2} [x_j - (1-f) y_j]$$

where (L_1, R_1) is a partition of N_1, and $L_2 \subseteq N_2$.

A point (s,x,y) satsifies all these inequalities if and only if

$$s + \sum_{j \in N_1} \min[fy_j, x_j] \geq f\lceil b \rceil + \sum_{j \in N_2} \max[0, x_j - (1-f)y_j].$$

Introducing variables π_j to represent $\min[fy_j, x_j]$ for $j \in N_1$, and π_j to represent $\max[0, x_j - (1-f)y_j]$ for $j \in N_2$, we obtain an extended formulation for $X_{\geq}^{SNF-CC}$.

Proposition 9. *The polyhedron*

$$\begin{gathered} s + \sum_{j \in N_1} x_j - \sum_{j \in N_2} x_j \geq b \\ s + \sum_{j \in N_1} \pi_j - \sum_{j \in N_2} \pi_j \geq f\lceil b \rceil \\ \pi_j \leq x_j, \pi_j \leq fy_j \text{ for } j \in N_1 \\ \pi_j \geq 0, \pi_j \geq x_j - (1-f)y_j \text{ for } j \in N_2 \\ x_j \geq 0, y_j \leq 1, x_j \leq y_j \text{ for } j \in N_1 \cup N_2 \end{gathered}$$

provides an extended formulation for $\text{conv}(X_{\geq}^{SNF-CC})$.

4.2 Single Node Inflow with Divisible Capacities

The set $X^{SNF-DIV}$ has still to be studied explicitly, but the corresponding knapsack problems with integer and bounded integer variables obtained by setting $x_j = a_j y_j$ for all $j \in N_1$ have been studied in [49] and [46] respectively. We present one of the basic results.

Proposition 10. *[49] Every valid inequality for the set*

$$\{(s,y) \in \mathbb{R}^1_+ \times \mathbb{Z}^n_+ : s + \sum_j a_j y_j \geq b\}$$

with $1 = a_1 | a_2 | \ldots | a_n$, *is of the form*

$$s + \sum_{t=1}^{p} \big(\prod_{s=1}^{t-1} \kappa_s\big) \sum_{j=i_t}^{j_t} \min[C_j / C_{i_t}, \kappa_t] y_j \geq \prod_{t=1}^{p} \kappa_t,$$

where $(\{i_1, \ldots, j_1\}, \{i_2, \ldots, j_2\}, \ldots, \{i_p, \ldots, j_p\})$ *is a partition of* $\{1, \ldots, n\}$, $i_1 = 1, j_p = n, i_t = j_{t-1} + 1$ *for* $t = 2, \ldots, p$, *and the parameters are defined as follows:* $\beta_p = b, \kappa_t = \lceil \beta_t / C_{i_t} \rceil, \mu_t = (\kappa_t - 1) C_{i_t}$ *and* $\beta_{t-1} = \beta_t - \mu_t$.

Other results include compact extended formulations and fast separation algorithms.

4.3 Single Node Inflow with Two Bound Values

The continuous integer knapsack set with two variables

$$X_2^{IK} = \{(y,s) \in \mathbb{Z}_+^2 \times \mathbb{R}_+^1 : a_1y_1 + a_2y_2 \leq b + s\},$$

the integer single node flow set with just two arcs

$$X_2^{SNF} = \{(x,y,s) \in \mathbb{R}_+^2 \times \mathbb{Z}_+^2 \times \mathbb{R}_+^1 : x_1 + x_2 \leq b + s, x_j \leq a_jy_j \text{ for } j = 1,2\},$$

and the single node inflow with constant variable lower and upper bounds

$$X^{SNF-CC-LB} = \{(x,y,s) \in \mathbb{R}_+^{n_1} \times \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^1 : \sum_{j \in N_1} x_j \leq b + s, \\ y_j \leq x_j \leq ay_j \text{ for } j \in N_1\}$$

have been studied in a series of papers [1, 2, 3]. For each set a complete description of the convex hull is given, as well as extended formulations based on a study of the two variable knapsack sets $X_{\leq}^K = \{y \in \mathbb{Z}_+^2 : a_1y_1 + a_2y_2 \leq b\}, X_{\geq}^K = \{y \in \mathbb{Z}_+^2 : a_1y_1 + a_2y_2 \geq b\}$ and $X_{bet}^K = \{y \in \mathbb{Z}_+^2 : b - \beta \leq a_1y_1 + a_2y_2 \leq b\}$. These sets were studied earlier algorithmically in [28] and from a polyhedral view in [59].

Earlier the set $X^{SNF-CC-LB}$ with constant lower bound and a very large upper bound $a = M$ was studied in [14]. One of the results is presented below. The set

$$\tilde{X}_{\geq}^{SNF-CC-LB} = \{(w,y) \in \mathbb{R}_+^{n_1} \times \{0,1\}^{n_1} : \sum_{j \in N_1} (w_j + y_j) \geq b, w_j \leq My_j \text{ for } j \in N_1\}$$

is obtained from $X_{\geq}^{SNF-CC-LB}$ by taking $w_j = x_j - y_j \geq 0$ for $j \in N_1$.

Proposition 11. *[14] For the set* $\text{conv}(\tilde{X}_{\geq}^{SNF-CC-LB})$ *with* $b > 2$*, every nontrivial facet-defining inequality is either of the form*

$$\sum_{j \in S_1 \cup S_2} w_j + \sum_{j \in S_2} y_j \geq (b - |S_1|)(1 - \sum_{j \in S_3} y_j)$$

where (S_1, S_2, S_3) *is a partition of* N_1*, or of the form*

$$\frac{1}{f} \sum_{j \in S_1} w_j + \sum_{j \in S_2} [\frac{1}{f} w_j + y_j] + \sum_{j \in S_3} [w_j + (1-f)y_j] \geq (\lceil b \rceil - |S_1|)(1 - \sum_{j \in S_4} y_j)$$

where (S_1, S_2, S_3, S_4) *is a partition of* N_1*.*

Descriptions of the convex hull and polynomial separation algorithms for the models with $\leq$ constraint and/or with integer variables are also presented.

5 Generalizing the Two Variable MIP Set

Here we consider three sets that have arisen in practice as subproblems of network design and production planning problems. Each is an obvious generalization of the two variable MIP set X^{MI} introduced in Section 2.3.

5.1 The Single Arc Set

The set

$$X_K^{ARC} = \{(x,z) \in \mathbb{R}_+^K \times \mathbb{Z}^1 : \sum_k x_k \leq d + z, x_k \leq a_k \text{ for } k = 1, \ldots, K\}$$

was first studied by in [37] where it arose as a single arc subproblem in multi-commodity network design. It can be rewritten in the form

$$\tilde{X}_K^{ARC} = \{(s,y) \in \mathbb{R}_+^K \times \mathbb{Z}^1 : \sum_k s_k + y \geq b, s_k \leq a_k \text{ for } k = 1, \ldots, K\},$$

which is an obvious generalization of the two variable MIP set.

In [37] it is shown that all the facet-defining inequalities are MIR inequalities.

Proposition 12. *Every non-trivial facet-defining inequality for* $\text{conv}(\tilde{X}_K^{ARC})$ *is of the form*

$$\sum_{k \in T} s_k \geq f_T(\lceil b_T \rceil - y) \tag{4}$$

for $T \subseteq \{1, \ldots, K\}$ *with* $b_T = b - \sum_{k \notin T} a_k > 0$ *and* $f_T = b_T - \lfloor b_T \rfloor > 0$.

Recently in [7] the same set X_K^{ARC} is called a splittable flow arc set, and a polynomial separation algorithm for the inequalities (4) is presented.

5.2 The Multiple MIP Set

Here we consider the set

$$X_K^{MI} = \{(s,y) \in \mathbb{R}_+^K \times \mathbb{Z}^1 : s_k + y \geq b_k \text{ for } k = 1, \ldots, K\}.$$

which arises in studying piecewise linear convex functions of the form $g(y) = \max_k [a_k(b_k - y)]$ and in discrete lot-sizing. Again it suffices to just add a simple MIR inequality for each of the K sets of the form X^{MI} to get the convex hull. What is more, this is still true when several such sets intersect, and have special constraints linking the integer variables. Note that this is an instance where a basic result, a description of a family of inequalities giving the convex hull, still holds when intersected with a special polyhedron (question 7 of Section 2.2). As before, we define $f_k = b_k - \lfloor b_k \rfloor$ for all k.

Proposition 13. *[40]*
i) The polyhedron

$$\begin{aligned} s_k + y &\geq b_k \text{ for } k = 1, \ldots, K \\ s_k + f_k y &\geq f_k \lceil b_k \rceil \text{ for } k = 1, \ldots, K \\ s &\geq 0 \end{aligned}$$

describes the convex hull of X_K^{MI}.

ii) Let $X^i = \{(s^i, y^i) \in \mathbb{R}_+^{K_i} \times \mathbb{Z}^1 : s_k^i + y^i \geq b_k^i \text{ for } k = 1, \ldots, K_i\}$ *for* $i = 1, \ldots, m$, *let* $y = (y^1, \ldots, y^m) \in \mathbb{Z}^m$, *and consider the set*

$$\bigcap_{i=1}^{m} X^i \cap \{y : By \leq d\} \subseteq \mathbb{R}_+^{K_1} \times \ldots \times \mathbb{R}_+^{K_m} \times \mathbb{Z}^m.$$

The polyhedron

$$\begin{aligned} s_k^i + y^i &\geq b_k^i \quad \text{for } k = 1, \ldots, K_i,\ i = 1, \ldots, m \\ s_k^i + f_k^i y &\geq f_k^i \lceil b_k^i \rceil \text{ for } k = 1, \ldots, K_i,\ i = 1, \ldots, m \\ By &\leq d \\ s &\geq 0 \end{aligned}$$

is integral if B *is a totally unimodular matrix and* d *is integer.*

5.3 The Mixed MIP Set

Here we consider the set

$$X_K^{MIX} = \{(s, y) \in \mathbb{R}_+^1 \times \mathbb{Z}^K : s + y_k \geq b_k \text{ for } k = 1, \ldots, K\}.$$

This set was studied in [24] as an abstraction of earlier results [47] for the constant capacity lot-sizing model with Wagner-Whitin costs (see Definition 3 below). Here the K MIR inequalities $s + f_k y_k \geq f_k \lceil b_k \rceil$ do not suffice to give the convex hull when $K > 1$.

Proposition 14. *[24] Let $T \subseteq \{1, \ldots, K\}$ with $|T| = t$, and suppose that $i_1, \ldots i_t$ is an ordering of T such that $0 = f_{i_0} \leq f_{i_1} \leq f_{i_2} \leq \ldots \leq f_{i_t} < 1$. Then the* mixing inequalities

$$s \geq \sum_{\tau=1}^{t} (f_{i_\tau} - f_{i_{\tau-1}})(\lceil b_{i_\tau} \rceil - y_{i_\tau}) \tag{5}$$

and

$$s \geq \sum_{\tau=1}^{t} (f_{i_\tau} - f_{i_{\tau-1}})(\lceil b_{i_\tau} \rceil - y_{i_\tau}) + (1 - f_{i_t})(\lceil b_{i_1} \rceil - 1 - y_{i_1}) \tag{6}$$

are valid for X_K^{MIX}.

Theorem 15. *[24, 40]*
i) The constraints

$$s + y_k \geq b_k \text{ for } k = 1, \ldots, K, \;\; s \geq 0$$

and the mixing constraints (5),(6) give the convex hull of X_K^{MIX}.

ii) Let $X^i = \{(s^i, y^i) \in \mathbb{R}_+^1 \times \mathbb{Z}^{K_i} : s^i + y_k^i \geq b_k^i \text{ for } k = 1, \ldots, K_i\}$ for $i = 1, \ldots, m$, let $y = (y^1, \ldots, y^m) \in \mathbb{Z}^{K_1} \times \ldots \times \mathbb{Z}^{K_m}$, and consider the set

$$\cap_{i=1}^m X^i \cap \{y : By \leq d\} \subseteq \mathbb{R}_+^m \times \mathbb{Z}^{K_1} \times \ldots \times \mathbb{Z}^{K_m}.$$

The polyhedron

$$s^i + y_k^i \geq b_k^i \text{ for all } k, i$$
$$\text{the mixing inequalities } (5), (6) \text{ for all } k, i$$
$$By \leq d$$
$$s^i \geq 0 \text{ for all } i$$

is integral if B is the arc-node incidence matrix of a directed graph and d is integer, or if the polyhedron $\{z : Bz \leq d, l_j \leq z_j \leq h_j, l_{ij} \leq z_i - z_j \leq h_{ij}$ for $i, j \in \{1, \ldots, K\}\}$ is integral for all integral l_j, h_j, l_{ij}, h_{ij}.

An O($n \log n$) separation algorithm for the inequalities (5),(6) is also known, as well as a very compact extended formulation.

Theorem 16. *[40] An extended formulation for conv*(X_K^{MIX}) *is*

$$s = \sum_{j=0}^{K} f_j \delta_j + \mu$$
$$y_k \geq \sum_{j=0}^{K} \lceil b_k - f_j \rceil \delta_j - \mu \text{ for } k = 1, \ldots, K$$
$$\sum_{j=0}^{K} \delta_j = 1$$
$$\mu \geq 0, \delta_j \geq 0 \text{ for } j = 0, 1, \ldots, K,$$

with $f_0 = 0$.

Further generalizations of the K-mixing set are studied in [40], including reformulations for separable piecewise convex objective functions over integer variables. Many further results on tight formulations for lot-sizing models can be found in the survey [48] and in recent papers [56, 64].

6 Lot-Sizing Sets

The multiple and mixed MIR sets arose from the study of constant capacity lot-sizing sets. Here we consider new results and insights for uncapacitated (easy) and varying capacity (hard) sets.

In the sets below,the data and variables have the following interpretation:

- d_t represents the demand for an item in period t with $d_{kt} \equiv \sum_{u=k}^{t} d_u$,
- C_t represents the maximum amount of the item that can be produced in the period t
- x_t, a variable, is the amount of the item produced in period t
- s_t, a variable, is the stock of the item at the end of period t
- y_t, a 0-1 set-up variable, takes value 1 if production can occur in period t

We consider three sets, the first associated with the uncapacitated problem, and the latter two with the varying capacity problem.

1. *The Uncapacitated Lot-Sizing Set with Backlogging and Wagner-Whitin Costs*

$$X^{WW-U-B} = \{(s,r,y) \in \mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n} \times \{0,1\}^n : s_{k-1} + M \sum_{u=k}^{t} y_u + r_t \geq d_{kt} \text{ for } 1 \leq k \leq t \leq n\}$$

   where M is a large positive value.

2. *The Dynamic Knapsack Set*

$$X^{DK} = \{(s_0, y) \in \mathbb{R}_+^1 \times \{0,1\}^n : s_0 + \sum_{u=1}^{t} a_u y_u \geq d_{1t} \text{ for } t = 1, \ldots, n\}$$

3. *The Varying Capacity Lot-Sizing Set*

$$X^{LS-C} = \{(x,s,y) \in \mathbb{R}_+^{n} \times \mathbb{R}_+^{n+1} \times \{0,1\}^n : s_{t-1} + x_t = d_t + s_t, x_t \leq a_t y_t \text{ for } t = 1, \ldots, n\}$$

The last two sets are related to generalizations of knapsack sets with a single continuous variable and of single node flow sets.

6.1 Wagner-Whitin Uncapacitated Lot-Sizing with Backlogging

For uncapacitated lot-sizing with backlogging, shortest path [16] and facility location [17] extended formulations were already known in 1989, whereas the problem of characterizing all the facet-defining inequalities had defeated several researchers. However, as we noted in the introduction, the hypothesis of Wagner-Whitin costs has permitted considerable progress to be made.

Definition 3. *A lot-sizing problem with unit production costs* $\{p_t\}_{t=1}^n$ *and unit storage costs* $\{h_t\}_{t=0}^n$ *has* Wagner-Whitin costs *if* $p_{t-1} + h_{t-1} \geq p_t$ *for* $t = 2, \ldots, n$.
If in addition backlogging is allowed with unit backlog costs $\{b_t\}_{t=1}^n$*, the problem has* Wagner-Whitin costs *if* $p_{t-1} + h_{t-1} \geq p_t$ *for* $t = 2, \ldots, n$*, and* $p_{t+1} + b_t \geq p_t$ *for* $t = 1, \ldots, n-1$.

For X^{WW-U-B}, it suffices to just add $2n$ additional variables to obtain a tight extended formulation for conv(X^{WW-U-B}) with only O(n^2) constraints.

Proposition 17. *[47] The polyhedron*

$$\alpha_t + y_t + \beta_t = 1 \text{ for all } \ t \text{ with } d_t > 0$$
$$s_{k-1} \geq \sum_{l=k}^{t} d_l(\alpha_l - \sum_{u=k}^{l-1} y_u) \text{ for all } \ k,t \text{ with } k \leq t$$
$$r_k \geq \sum_{l=t}^{k} (\beta_l - \sum_{u=l+1}^{k} y_u) \text{ for all } \ k,t \text{ with } k \geq t$$
$$s,r,\alpha,\beta \in \mathbb{R}^n_+, y \in [0,1]^n.$$

provides an extended formulation for $\text{conv}(X^{WW-U-B})$.

In this formulation, the additional variables have the interpretation

- $\alpha_t = 1$ if the demand $d_t > 0$ is satisfied from stock, and
- $\beta_t = 1$ if the demand $d_t > 0$ is satisfied from backlog.

It is possible to project out the α and β variables, so as to obtain a characterization of the valid inequalities in the original space.

Proposition 18. *[47] Every facet-defining inequality of conv*(X^{WW-U-B}) *is of the form*

$$\sum_{k=1}^{n}\sum_{t=k}^{n} v_{kt}(s_{k-1} + \sum_{l=k}^{t} d_l \sum_{j=k}^{l-1} y_j) + \sum_{k=1}^{n}\sum_{t=1}^{k} w_{kt}(r_k + \sum_{l=t}^{k} d_l \sum_{j=l+1}^{k} y_j)$$
$$\geq \sum_{l=1}^{n} u_l d_l (1 - y_l)$$

where (v,w) *is the characteristic vector of an elementary cycle in the digraph* $D = (V,A)$ *with* $V = \{1,\ldots,n+1\}$, *forward arcs* $(k,t+1)$ *corresponding to* v_{kt} *for* $1 \leq k \leq t \leq n$, *and backward arcs* $(k+1,t)$ *corresponding to* w_{kt} *for* $1 \leq t \leq k \leq n$, *and* $u_l = \sum_{k:k\leq l}\sum_{t:t\geq l} v_{kt}$.

For problems with general costs the shortest path reformulation has been generalized to handle sales and piecewise concave production costs [65].

6.2 Dynamic Knapsack Sets

Observe first that the dynamic knapsack set X^{DK} can be rewritten as a varying capacity discrete lot-sizing set

$$X^{DLS-C} = \{(s,y) \in \mathbb{R}_+^{n+1} \times \{0,1\}^n : s_{t-1} + a_t y_t = d_t + s_t \text{ for } t = 1, \dots, n\}.$$

Recently a family of strong valid inequalities has been derived for such sets. We suppose without loss of generality that $d_t \geq 0$ for $t = 1, \dots, n-1$, and $d_n > 0$.

Proposition 19. *[33] The inequality*

$$s_0 + \sum_{j=1}^{n} \min[a_j, d_{jn}] y_j \geq d_{1n}$$

is valid and facet-defining for conv(X^{DLS-C}).

Now suppose that we fix $y_j = 1$ for $j \in T \subset N = \{1, \dots, n\}$, and let $\{\tilde{d}_t\}$ be the resulting "reduced" demands. Now starting from the valid inequality

$$s_0 + \sum_{j \in N \setminus T} \min[a_j, \tilde{d}_{jn}] y_j \geq \tilde{d}_{1n},$$

we need to lift back in the variables $j \in T$ that have been set to 1. Even though the corresponding lifting function $\psi_{N \setminus T}$ is not superadditive, it is shown in [33] that a simultaneous lifting result holds and there is a facet-defining inequality for X^{DLS-C} of the form

$$s_0 + \sum_{j \in N \setminus T} \min[a_j, \tilde{d}_{jn}] y_j \geq \tilde{d}_{1n} + \sum_{j \in T} \psi_{N \setminus T}(a_j e_j)(1 - y_j)$$

where e_j denotes the j^{th} unit vector.

The latter inequalities can also be used to derive strong valid inequalities for X^{LS-C} by replacing any term $\min[a_j, \tilde{d}_{jn}] y_j$ by x_j for any $j \in N \setminus T$.

6.3 Varying Capacity Lot-Sizing Sets

Recently it has been observed in [6] that the alternative formulation

$$\tilde{X}^{LS-C} = \{(x, s_n, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^1 \times \{0,1\}^n : \sum_{u=t}^{n} x_u \leq d_{tn} + s_n, x_t \leq a_t y_t \text{ for } t = 1, \dots, n\}$$

of X^{LS-C} is very useful. Let v be the set function defined as

$$v(T) = \max\{\sum_{j\in T} x_j : (x, s_n, y) \in \tilde{X}^{LS-C},\ y_t = 0 \text{ for } j \in N \setminus T, s_n = 0\}$$

for all $T \subseteq N$.

Proposition 20. *The submodular inequality*

$$\sum_{j\in T} x_j \le v(T) - \sum_{j\in T}[v(T) - v(T \setminus \{j\})](1 - y_j) + s_n \tag{7}$$

is valid for $\tilde{X}^{LS-C}$.

It is shown that these inequalities are strong in most circumstances. In addition we see that their coefficients are easily calculated.

Observation 1.

$$\begin{aligned} v(N) &= \max\{\sum_{u=1}^{n} x_u : (x, s_n, y) \in \tilde{X}^{LS-C}, y_t = 1 \text{ for all } t, s_n = 0\} \\ &= \min_{t=0,1,\dots,n}[\sum_{u=1}^{t} a_u + d_{t+1,n}]. \end{aligned}$$

Observation 2. *The set function* $v : \mathcal{P}(N) \to \mathbb{R}^1$ *is submodular and nondecreasing.*

From Observation 2, the validity of the inequalities (7) follows immediately using results on submodular valid inequalities from [61].

In [6] it is also shown how to lift in the variables in $N \setminus T$ so as to strengthen the submodular inequalities. Now work is under way to use the submodularity to tackle extensions of these sets involving outflows and backlogging.

7 Using Extended Formulations

Considering some of the most interesting developments of the last few years, one is the Integral Basis Method [26, 27], and a second consists of the many generalizations of the disjunctive approach of Balas, including the lift-and–project algorithm [9], the semi-definite approach [36], the linearization reformulation approach [54], and more recent extensions in [12, 30]. A third development is the possibility of tackling significant practical production planning problems using extended formulations [64]. This in turn has shown up the importance of reducing the size of certain extended formulations, and thus of developing smaller approximations to the convex hulls of the corresponding sets. Here we briefly introduce the first and third of these approaches, viewing them as new ways of deriving and working with extended formulations. The second also suggests many new algorithmic possibilities for reformulation and/or cutting planes, but probably merits a survey to itself.

7.1 Discrete Extended Formulations

The Integral Basis Method Approach [26, 27].

Here one works with a primal feasible all-integer tableau. Considering a relaxation consisting of the objective function and one row, and let N denote the set of non-basic variables with $n = |N|$. One has

$$\max \bar{a}_{00} + \textstyle\sum_j \bar{a}_{0j} y_j \tag{8}$$

$$z_u + \textstyle\sum_{j \in N} \bar{a}_{uj} y_j = \bar{a}_{u0} \tag{9}$$

$$z_u \in \mathbb{Z}^1_+, y \in \mathbb{Z}^n_+ \tag{10}$$

with $\bar{a}_{uj} \in \mathbb{Z}^1$ for $j \in N$ and $\bar{a}_{u0} \in \mathbb{Z}^1_+$. Given some variable y_k and some row u for which $\bar{a}_{0k} > 0$ and $0 < \bar{a}_{u0} < \bar{a}_{uk}$, one wishes to augment y_k, but augmenting as far as $y_k = 1$, while keeping $y_j = 0$ for $j \in N \setminus \{k\}$, is not feasible.

Rather than generating primal all-integer Gomory cutting planes, see Young [67], the idea is to use an extended formulation for the set Y, where

$$Y = \{y \in \mathbb{Z}^n_+ : \sum_{j \in N} \bar{a}_{uj} y_j \le \bar{a}_{u0}\}.$$

Suppose that $Y'_k \supseteq Y_k = Y \cap \{y : y_k \ge 1\}$ and that $Y'_k = \{y : y = C\lambda, \lambda \in \mathbb{Z}^r_+\}$ is a discrete reformulation of the set Y'_k.

Now every point in Y can be expressed in the form

$$y = C\lambda + \sum_{j \in N \setminus \{k\}} \tilde{y}_j e_j \text{ with } \lambda \in \mathbb{Z}^r_+, \tilde{y} \in \mathbb{Z}^{|N \setminus \{k\}|}_+.$$

Using this to eliminate the y variables in (8)-(10) leads to the reformulation

$$\max \bar{a}_{00} + \bar{a}_0^T C\lambda + \sum_{j\neq k} \bar{a}_{0j}\tilde{y}_j$$
$$z_u + \bar{a}_u^T C\lambda + \sum_{j\in N\setminus\{k\}} \bar{a}_{uj}\tilde{y}_j = \bar{a}_{u0}$$
$$z_u \in \mathbb{Z}_+^1, \lambda \in \mathbb{Z}_+^r, \tilde{y} \in \mathbb{Z}_+^{|N\setminus\{k\}|}.$$

When the number r of columns in the reformulation for Y_k' is too large, one looks at supersets or relaxations of Y or Y_k'. One such set is the relaxation obtained by weakening the Gomory integer cut

$$Y^* = \{y \in \mathbb{Z}_+^n : y_k + \sum_{j\in N:\bar{a}_{uj}<0} \lfloor \frac{\bar{a}_{uj}}{\bar{a}_{uk}} \rfloor y_j \leq 0\}$$

for which the set C is easily generated.

Other Variants

It appears interesting to try to apply this approach of using extended formulations dynamically during the optimization process in other situations. One possibility being investigated [34] is to work with a standard optimal LP tableau, and then find a discrete reformulation for the knapsack or group problem [20] obtained from one row of the tableau.

Another, that can be applied to mixed integer as well as integer programs, is to construct the extended formulation of a disjunction consisting of one or several rows of the optimal LP tableau, as described by Balas [8], and add it to the problem. The potential advantage is that here the additional variables do not need to be integer as for the all integer problems discussed above.

7.2 Approximate Extended Formulations

Tight extended formulations have been proposed for various uncapacitated and constant capacity lot-sizing problems, as well as fixed charge network flow problems. Though tight, these reformulations are often very large, with the result that the linear programs become difficult or impossible to solve. In [57], it is shown how many well-known formulations can be relaxed, and the strength of the resulting formulations can be analyzed.

To demonstrate, we approximate the well-known shortest path reformulation of the uncapacitated lot-sizing set X^{LS-U} (i.e. the set X^{LS-C} of Section 5 with a_t very large for all t).

We choose some parameter $k(\leq n)$ which determines the size and strength of the formulation. For simplicity we assume that $d_t > 0$ for all $t = 1, \ldots, n$ and $s_0 = s_n = 0$. The variables are defined as follows:

- $z_{it} = 1$ if production takes place in period i, and the amount produced is d_{it} satisfying all the demands from periods i up to t, with $t < i + k$
- $u_i = 1$ for $1 \leq i \leq n - k$ if production takes place in period i, and the amount produced is d_{it} for some $t > n - k$
- $v_t = 1$ for $k + 1 \leq t \leq n$ if exactly d_{it} is produced in some period $i \leq t - k$.
- $w_t = 1$ for $t = 2, \ldots, n - k$ if demand for periods $t - 1, \ldots, t + k$ is all produced simultaneously in some period $i \leq t - 1$.

The resulting approximate formulation X_k^{SP} with O(nk) variables and O(n) constraints is

$$-\sum_{i=1}^{k} z_{1i} - u_1 = -1 \tag{11}$$

$$\sum_{i=\max[t-k+1,1]}^{t} z_{it} + v_t - \sum_{i=t+1}^{\min[t+k,n]} z_{t+1,i} - u_{t+1} = 0 \text{ for } t = 1, \ldots, n-1 \tag{12}$$

$$\sum_{i=n-k+1}^{n} z_{in} + v_n = 1 \tag{13}$$

$$u_t + w_t - v_{t+k} - w_{t+1} = 0 \text{ for } t = 1, \ldots, n-k \tag{14}$$

$$\sum_{i=t}^{\min[t+k-1,n]} z_{ti} + u_t \leq y_t \text{ for } t = 1, \ldots, n \tag{15}$$

$$x_t \geq \sum_{i=t}^{\min[t+k-1,n]} d_{ti} z_{ti} + d_{t,t+k} u_t \text{ for } t = 1, \ldots, n \tag{16}$$

$$s_{t-1} \geq \sum_{i=1}^{t-1} \sum_{j=t}^{n} d_{tj} z_{ij} + \sum_{i=t}^{t+k-1} d_{ti} v_i + d_{t+k} w_t \text{ for } t = 2, \ldots, n \tag{17}$$

$$z, u, v, y \geq 0, \; y \leq 1, \tag{18}$$

$$v_t = 0 \text{ for } t \leq k, u_t = w_t = 0 \text{ for } t \geq n-k+1, w_1 = 0, z_{it} = 0 \text{ for } t \geq i+k. \tag{19}$$

In Figure 1, we show a shortest path representation of X_k^{SP} for $n = 5$ and $k = 2$. Here the solution $y_1 = y_3 = y_5 = 1, x_1 = d_{12}, x_3 = d_{34}, x_5 = d_5$ is represented exactly by the path $1, 1', 3, 3', 5, 5', 6$, but the solution $y_1 = y_5 = 1, x_1 = d_{14}, x_5 = d_5$ is represented by the path $1, 1', 1", 2", 5, 5', 6$ and its cost is underestimated.

Figure 1: Approximate Shortest Path Formulation

Proposition 21. *[57] The polyhedron (11)-(19) is integral.*
If the linear program $\min\{px+hs+fy : (x, y, s)$ *satisfying* $(11)-(19)\}$ *has an optimal solution with* $w_t = 0$ *for all* t*, this solution is optimal for the uncapacitated lot-sizing problem* $\min\{px + hs + fy : (x, y, s) \in X^{LS-U}\}$.

It is also important to say something about the strength of such an approximation in comparison with a cutting plane approach. For $\text{conv}(X^{LS-U})$, it is well-known [10] that every nontrivial facet-defining inequality is a (t, l, S) inequality of the form $\sum_{j\in[1,t-1]\cup S} x_j + \sum_{j\in[t,l]\setminus S} d_{jl}y_j \geq d_{tl}$ with $1 \leq t \leq l \leq n$ and $S \subseteq [t + 1, l] = \{t + 1, \ldots, l\}$.

Proposition 22. *[57] The* (t, l, S) *inequality is valid for* X_k^{SP} *if and only if* $l \leq t+k$.

It follows that if the linear program obtained by adding the (t, l, S) inequalities with $l \leq t + k$ to the original formulation solves the lot-sizing problem, then the approximate shortest path formulation also solves it, and vice versa.

The above reformulation can be extended easily to include backlogging. In [57], other approximate formulations are given for Steiner trees, fixed charge network flows, and both constant capacity and uncapacitated single item lot-sizing sets with and without Wagner-Whitin costs.

8 Final Remarks

In Sections 3-6 we have largely concentrated on the areas that have seen significant progress, namely single node flow sets and sets arising from single item lot-sizing. In 1989 we cited machine scheduling as a promising area for polyhedral studies, but there progress has been limited, in spite of a variety of results developed in the 80s, see for example the survey [50]. On the positive side, the known polyhedral results have proved useful in developing and analyzing approximation algorithms for a variety of classical machine scheduling problems, see for instance [53, 25]. However there has been little progress in solving practical scheduling problems by mixed integer programming, whereas a considerable number of problems have been tackled with some success using constraint programming [31, 44]. This in turn leads to the wide open topic of finding effective ways to coordinate IP and CP [55, 29].

Linked to the semi-definite approach of Lovasz and Schrijver cited above, semi-definite convex optimization has been used successfully to provide approximation algorithms [18, 43] and useful dual bounds for the max cut and related graph partitioning problems, that can be viewed as quadratic 0-1 optimization problems with simple linear constraints. This raises many questions about the possibility of extending the strong cutting plane approach to nonlinear integer and mixed integer programs.

There are several computational questions. One of them, the need to solve very large LPs, often based on extended formulations, has led to new developments in the approximate solution of very large LPs, see [11] among others. Another relates to cutting plane separation algorithms for MIPs. We conjecture that a large majority of fractional points that are cut off by an MIP solver have only a small number of fractional variables in the support of the set used to generate the cut. This would provide strong encouragement to continue the study of valid inequalities and separation for simple MIP sets with a small number of variables, combined with lifting of the remaining variables.

References

[1] A. Agra and M. Constantino, Basic integer knapsack polyhedra, Working Paper 3/2000, Department of Mathematics, University of Lisbon, 2000.

[2] A. Agra and M. Constantino, Polyhedral description of basic knapsack problems with a continuous variable, Department of Mathematics, University of Lisbon, 2001.

[3] A. Agra and M. Constantino, Polyhedral description of a mixed integer problem with constant bounds, Working Paper 4/2001, Department of Mathematics, University of Lisbon, 2001.

[4] A. Atamtürk, On capacitated network design cut-set polyhedra, *Mathematical Programming* **92**, 425-438 (2002).

[5] A. Atamtürk, On the facets of single constraint mixed integer sets, Department of Industrial Engineering and Operations Research, University of California at Berkeley, March 2002, revised June 2002.

[6] A. Atamtürk and J.C. Munoz, A study of the lot-sizing polytope, Department of Industrial Engineering and Operations Research, University of California at Berkeley, September 2002.

[7] A. Atamtürk and D. Rajan, On splittable and unsplittable flow capacitated network design arc–set polyhedra, *Mathematical Programming* **92**, 315–333 (2002).

[8] E. Balas, Disjunctive programming, *Annals of Discrete Mathematics* **5**, 3 – 51 (1979).

[9] E. Balas, S. Ceria, and G. Cornuéjols, A lift-and-project cutting plane algorithm for mixed $0-1$ programs, *Mathematical Programming* **58**, 295 – 324 (1993).

[10] I. Barany, T.J. Van Roy and L.A. Wolsey, Uncapacitated lot sizing : the convex hull of solutions, *Mathematical Programming Study* **22**, 32-43 (1984).

[11] D. Bienstock, *Potential function methods for approximately solving linear programming problems: theory and practice*, CORE Lecture Series, Louvain-la-Neuve, Belgium (2002).

[12] D. Bienstock and M. Zuckerberg, Subset algebra lift operators for 0-1 integer programming, Columbia University, New York, CORC Report 2002-01 (2002).

[13] S. Ceria, C. Cordier, H. Marchand and L.A. Wolsey, Cutting Planes for Integer Programs with General Integer Variables, *Mathematical Programming* **81**, 201-214 (1998).

[14] M. Constantino, Lower bounds in lot-sizing models: a polyhedral study, *Mathematics of Operations Research* **23**, 101-118 (1998).

[15] ILOG Cplex 7.0 User's Manual, August 2000.

[16] G.D. Eppen and R.K. Martin, Solving multi-item lot-sizing problems using variable definition, *Operations Research* **35**, 832-848, 1987.

[17] J. Krarup and O. Bilde, Plant location, set covering and economic lot sizes: an $O(mn)$ algorithm for structured problems, in *Optimierung bei Graphentheoretischen und Ganzzahligen Probleme*, L. Collatz et al. eds, Birkhauser Verlag, Basel, 155-180 (1977).

[18] M.X. Goemans and D.P. Williamson, Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming, *Journal of the ACM* **6**, 1115-1145 (1995).

[19] R.E. Gomory, An algorithm for the mixed integer problem, RM-2597, The Rand Corporation (1960).

[20] R.E. Gomory, Some polyhedra related to combinatorial problem, *Linear Algebra and its Applications* **2**, 451-558 (1969).

[21] M. Grötschel, L. Lovasz and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, *Combinatorica* **1**, 169-197 (1981).

[22] Z. Gu, G.L. Nemhauser and M.W.P. Savelsbergh, Lifted flow cover inequalities for mixed 0-1 integer programs, *Mathematical Programming* **85**, 439-468 (1999).

[23] Z. Gu, G.L. Nemhauser and M.W.P. Savelsbergh, Sequence independent lifting in mixed integer programing, *Journal of Combinatorial Optimization*, **4**, 109-129 (2000).

[24] O. Günlük and Y. Pochet, Mixing mixed integer inequalities, *Mathematical Programming* **90**, 429-457 (2001).

[25] L.A. Hall, Approximation algorithms for scheduling, in D.S. Hochbaum (ed.) *Approximation algorithms for NP-hard problems*, PWS Publishing Company, Boston, 1-45 (1997).

[26] U-U. Haus, M. Köppe and R. Weismantel, The Integral Basis Method for integer programming, *Mathematical Methods of Operations Research* **53**, 353-361 (2001).

[27] U-U. Haus, M. Köppe and R. Weismantel, A primal all-integer algorithm based on irreducible solutions, *Mathematical Programming* (to appear).

[28] D.S. Hirschberg and C.K. Wong, A polynomial time algorithm for the knapsack problem with two variables, *Journal of ACM* **23**,147-154 (1976).

[29] V. Jain and I.E. Grossman, Algorithms for hybrid MILP/CP models for a class of optimization problems, *INFORMS Journal on Computing* **13**, 258-276 (2001).

[30] J.B. Lasserre, An explicit exact SDP relaxation for nonlinear 0-1 programs in: Integer Programming and Combinatorial Optimization, K. Aardal and A.M.H. Gerards (ed), Lecture Notes in Computer Science 293-303, 2001.

[31] C. Le Pape and Ph. Baptiste, Resource constraints for preemptive job-shop scheduling, *Constraints* **3**, 263-287 (1998).

[32] M. Loparic, Stronger mixed 01 models for lot-sizing problems, Ph.D. thesis, Faculty of Engineering, Université Catholique de Louvain, Belgium (2001).

[33] M. Loparic, H. Marchand and L.A. Wolsey, Dynamic knapsack sets and capacitated lot-sizing, *Mathematical Programming B* **95**, 53-69 (2003).

[34] Q. Louveaux, M. Köppe and R. Weismantel, Private communication (2003).

[35] Q. Louveaux and L.A. Wolsey, Lifting, Superadditivity, Mixed Integer Rounding and Single Node Flow Sets Revisited, Core DP 2003-1, Université Catholique de Louvain, Louvain-la-Neuve (2003).

[36] L. Lovász and A. Schrijver, Cones of matrices and set-functions and $0-1$ optimization, *SIAM Journal on Optimization* **1**, 166 – 190 (1991).

[37] T.L. Magnanti, P. Mirchandani and R. Vachani, The convex hull of two core capacitated network design problems, *Mathematical Programming* **60**, 233–250 (1993).

[38] H. Marchand and L.A. Wolsey, The $0 - 1$ knapsack problem with a single continuous variable, *Mathematical Programming* **85**, 15-33 (1999).

[39] H. Marchand and L.A. Wolsey, Aggregation and mixed integer rounding to solve MIPs, *Operations Research* **49**, 363-371 (2001).

[40] A. Miller and L.A. Wolsey, Tight formulations for some simple MIPs and Convex Objective IPs, *Mathematical Programming B* (to appear).

[41] G.L. Nemhauser and L.A. Wolsey, *Integer and Combinatorial Optimization*,Wiley, New York (1988).

[42] G.L. Nemhauser and L.A. Wolsey, A recursive procedure for generating all cuts for 0-1 mixed integer programs, *Mathematical Programming* **46**, 379-390 (1990).

[43] Y. Nesterov, H. Wolkowicz and Y. Ye, Semidefinite programming relaxations of nonconvex quadratic optimization, in H. Wolkowicz, R. Saigal and L. Vandenberghe (eds.), *Handbook of Semidefinite Programming: Theory, Algorithms, and Applications*, Dordrecht, Kluwer Academic Press, 361-419, (2000).

[44] W.P.M. Nuijten and E.H.L. Aarts, A computational study of constraint satsfaction for multiple capacitated job shop scheduling, *European Journal of Operational Research* **90**, 269-284 (1996).

[45] M.W. Padberg, T.J. Van Roy and L.A. Wolsey, Valid inequalities for fixed charge problems, *Operations Research* **33**, 842-861 (1985).

[46] Y. Pochet and R. Weismantel, The sequential knapsack polytope, *SIAM Journal of Optimization* **8**, 248-264 (1998).

[47] Y. Pochet and L.A. Wolsey, Polyhedra for lot-sizing with Wagner-Whitin costs, *Mathematical Programming* **67**, 297-324 (1994).

[48] Y. Pochet and L.A. Wolsey, Algorithms and reformulations for lot-sizing problems, in *Combinatorial Optimization*, eds. W. Cook, L. Lovasz and P. Seymour, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol 20, American Mathematical Society 1995.

[49] Y. Pochet and L.A. Wolsey, Integer knapsacks and flow covers with divisible coefficients: polyhedra, optimization and separation, *Discrete Applied Mathematics* **59**, 57–74 (1995).

[50] M. Queyranne and A. Schulz, Polyhedral Approaches to Machine Scheduling, Working Paper, University of British Columbia, Vancouver, November 1994, last revised June 1997.

[51] J.-P.P. Richard, I.R. de Farias and G.L. Nemhauser, Lifted inequalities for 0-1 mixed integer programming: basic theory and algorithms, School of Industrial and Systems Engineering, 2002.

[52] J.-P.P. Richard, I.R. de Farias and G.L. Nemhauser, Lifted inequalities for 0-1 mixed integer programming: superlinear lifting, School of Industrial and Systems Engineering, Georgia Institute of Technology, 2002.

[53] P. Schuurman and G.J. Woeginger, Approximation schemes - a tutorial, September 2000.

[54] H. Sherali and W. Adams, A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems, *SIAM Journal of Discrete Mathematics* **3**, 411 – 430 (1990).

[55] P. Van Hentenryck, Constraint and Integer Programming in OPL, Working Paper, Brown University, Providence, R.I.

[56] M. Van Vyve, An integral extended formulation of size O(T^3) for the single item lot-sizing problem with constant capacities and backlogging, Internal Report, CORE, Université Catholique de Louvain, October 2001.

[57] M. Van Vyve, Hierarchies of extended formulations, Internal Report, CORE, Université Catholique de Louvain, October 2001.

[58] T.J. Van Roy and L.A. Wolsey, Valid inequalities for mixed $0-1$ programs, *Discrete Applied Mathematics* **14**, 199–213, (1986).

[59] R. Weismantel, Hilbert bases and the facets of special knapsack polytopes, *Mathematics of Operations Research* **21**, 886 - 904 (1996).

[60] L.A. Wolsey, Valid inequalities and superadditivity for 0-1 integer programs, *Mathematics of Operations Research* **2**, 66-77 (1977).

[61] L.A. Wolsey, Submodularity and valid inequalities in capacitated fixed charge networks, *Operations Research Letters* **8**, 119-124 (1989).

[62] L.A. Wolsey, Strong formulations for mixed integer programming: a survey, *Mathematical Programming* **45**, 173-191 (1989).

[63] L.A. Wolsey, Valid inequalities for mixed integer programs with generalised upper bound constraints, *Discrete Applied Mathematics* **25**, 251-261 (1990).

[64] L.A. Wolsey, Solving Multi-Item Lot-Sizing Problems with an MIP Solver using Classification and Reformulation, *Management Science* **48**, 1587-1602 (2002).

[65] L.A. Wolsey, Uncapacitated lot-sizing with buying, sales and backlogging, Internal report, CORE, Université Catholique de Louvain, March 2002.

[66] XPRESS-MP Optimiser, Dash Associates, Blisworth House, Blisworth, Northants NN73BX, UK (2001).

[67] R.D. Young, A primal (all-integer) algorithm, *J. Res Nat. Bur. Stds.* **69B**, 113-250 (1965).