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# A model of corporate bond pricing with liquidity and marketability risk

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*We develop a corporate bond valuation model that takes into account both the risk of early default and the risk generated by lack of liquidity and marketability. Randomly matched investors who have heterogeneous prior beliefs about the value of the firm in bankruptcy bargain for the price of the asset in a secondary market. The liquidity and marketability risk is shown to be a function of the heterogeneity of investors' valuations, the average belief about the cost of bankruptcy and the bargaining power of bondholders. The model also captures the fact that, soon after the issue, a bond is relatively liquid and later becomes relatively illiquid depending on the underlying asset value.*

## 1 INTRODUCTION

In an early paper Fisher (1959) suggested that risk premia on corporate bonds depend on the "marketability" of the bond. Much academic interest in the pricing of corporate bonds has, however, focused solely on determining the influence of interest rate and default risks.[1] The risk that the bondholder will be unable to sell the instrument at no cost has surprisingly received little attention in the corporate bond pricing literature. This liquidity–marketability (L–M) risk has two components. The first component is the *lack of marketability*, which refers to an asset that cannot be sold immediately at any price, at any cost. The second is the *lack of liquidity*, which refers to an asset for which the asset holder must pay more to sell it immediately. In the present paper we explicitly incorporate the lack of

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[1] Papers include those by Kim, Ramaswamy and Sundaresan (1993), Longstaff and Schwartz (1995) and Leland and Toft (1996).

both marketability and liquidity in the pricing of contingent claims. Moreover, we suggest the first approach that incorporates market microstructure considerations in a valuation setting.

Jones, Mason and Rosenfeld (1984) have shown that classical Merton-type models are unable to generate the yield spreads observed in the market for sufficiently realistic values of the underlying parameters such as the volatility of the value of the firm. Kim, Ramaswamy and Sundaresan (1993) included bankruptcy costs and a stochastic term structure of interest rates, but they too were unable to obtain realistic spreads except for unreasonable values of the cost of bankruptcy. Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997) introduced game theoretic bargaining among debt- and equity-holders in standard valuation models and they showed that concessions extracted by equity-holders from debt-holders go some way towards explaining observed yield spreads. But these models do not take into account illiquidity and are, for instance, unable to explain the flat term structure of spreads observed in high-grade corporate bonds (see, for example, Duffee, 1999).

There are, however, strong reasons for believing that corporate bond markets are relatively illiquid. Altman (1991) has claimed that the lack of liquidity and marketability is an important determinant of the value of distressed securities. Shulman, Bayless and Price (1993) provided some empirical evidence that investors require an additional premium for marketability risk as measured by trading frequency. Moreover, the empirical studies of Anderson and Sundaresan (2000) and Duffee (1999) also suggest that a liquidity factor may be missing in models of corporate debt pricing. Despite this evidence, illiquidity has not been incorporated into models of corporate bond pricing except in a very reduced way. Reduced-form models such as Duffie and Singleton's (1999) are able to include liquidity as a component of risk premium. In reduced-form models it is, however, difficult to separate the effect of liquidity from the credit risk embedded in corporate bonds. Enabling a separation of these two different risks could be very useful for many risk management purposes.

Empirical evidence has also shown that bankruptcy is costly. However, several authors have reported very different magnitudes for the cost of bankruptcy when direct and indirect costs are taken into account.[2] The indirect costs of financial distress arise from a firm's inability to conduct business as usual and are particularly difficult to assess. The study by Alderson and Betker (1996) threw some light on the difficulty of explaining the variations in liquidation costs by means of the classical accounting variables which are used to proxy these costs.

Various empirical and theoretical studies have argued and documented that investment returns on less liquid assets should be higher than those on their identical liquid counterparts. On the empirical side, Amihud and Mendelson (1991) studied the effects of the liquidity of capital assets on the prices of those assets.

[2] Warner (1977) found direct bankruptcy costs to be about 5.3% of a firm's market value, while Alderson and Betker (1995) have reported mean total liquidation costs of 36.5% (ranging from 12.8% to 61.8%). See also Altman (1984).

To be precise, they examined the effects of illiquidity on the yields of finite-maturity securities with identical cashflows: US Treasury bills and notes with maturities of less than six months. For these maturities, both securities are effectively discount bonds and should be equivalent. Their liquidity, however, is different: the cost of transacting bills is lower than the cost of transacting notes. This observation enabled them to study the relation between asset yields and liquidity without the need to control for other factors. Their results confirm the existence of a liquidity effect in asset pricing: the yield-to-maturity is higher on notes, which have lower liquidity. Garbade and Silber (1976) investigated the characteristics of price dispersion in the US Treasury securities market. Empirical evidence showed that dispersion is influenced by characteristics of securities such as volume and maturity and by market supply–demand conditions as reflected in price level changes. It was also shown that the cost of liquidity services in a competitive market is determined by price dispersion and is not equal to the bid–ask spread. Perraudin and Taylor (2002) have shown that defaultable debt spreads reflect not only expected losses, risk premiums and taxes, but also substantial liquidity premiums. Houweling, Mentink and Vorst (2005) have provided empirical evidence on the importance of liquidity risk in corporate bond markets. They tested whether liquidity is priced in the euro-denominated corporate bond market. They used the adaptive portfolio training (APT) framework to control for other sources of risk, and yields were used to measure the expected returns on bonds. They approximated the liquidity by four indirect measures: the amount of the issue, its age, the number of quotes, and the dispersion of the quotes. The results they obtained show that pricing anomalies due to liquidity do exist in the euro-denominated corporate bond market.[3]

On the theoretical side, Grinblatt (1994) modeled liquidity as an exogenous state variable that follows an Ornstein–Uhlenbeck process in order to find a closed-form solution for interest rate swap spreads. Reduced-form models like Duffie and Singleton's (1999) have attempted to replicate observed yield spreads by an exogenous stochastic process that may include potential liquidity or credit risks. In our model, we are able to isolate the influence of these two components without assuming any exogenous stochastic process, which has proved to be very difficult to justify. Other related work is that by Boudhouck and Whitelaw (1993), who studied the issue of liquidity in Japanese government bond markets by considering a model in which price differentials are explained in terms of heterogeneous investors in their endowments with short-sales restrictions that limit their trading strategies. Browne, Milevsky and Salisbury (2003) have developed a method for benchmarking the returns from fixed annuity products that contain

[3] The yield premium between liquid and illiquid euro-denominated bonds ranges from 0.2 to 47 basis points, depending on which liquidity indicator is used. Indeed, the size of the yield difference between liquid and illiquid bond portfolios depends on the liquidity indicator employed. The authors found a negligible liquidity premium for the issued amount indicator (0.2bps), a substantial liquidity impact for age (14bps) and for number of quotes (27bps), and a large effect for the dispersion indicator (47bps).

liquidity restrictions. They computed the yield needed to compensate for the utility loss induced by the inability to continuously rebalance and maintain an optimal portfolio. They found a negative relationship between a greater level of individual risk-aversion and the demanded liquidity premium. The required liquidity premium was an increasing function of the holding period restriction and the subjective return from the market, and was quite sensitive to the individual's endowed portfolio. A related study is that of Longstaff (1995). He developed an option-pricing approach to determine an upper bound that compensates for the lack of marketability. He assumed an agent with perfect foresight and computed the monetary loss from being unable to sell at a maximum. Recently, Ericsson and Renault (2000) introduced liquidity risk into a debt-pricing model through heterogeneous liquidity shocks.

In this paper we propose a model of contingent claims pricing which explicitly takes into consideration, first, the determinants of liquidity of the secondary market for claims; and, second, the fact that bankruptcy costs cannot be known via investors' heterogeneous expectations or beliefs about the loss they will face in the event of bankruptcy. Hence, our model gives an explicit form for the L–M risk in the valuation of defaultable bonds. This model considers a coupon-paying debt contract with default risk in a dynamic setting. For simplicity, the underlying asset value is assumed to follow a binomial process. Potential investors in the market are assumed to have heterogeneous prior beliefs about the value of the firm in the event of default. During each period the current bondholder is matched in the secondary market with an investor who is chosen at random from the population of market participants.[4] Depending on the valuation of collateral value a secondary market transaction may or may not result.[5] By comparing the resulting price in this illiquid market with the price resulting from transacting in a liquid market, we are able to study the influence of the L–M risk in the pricing of risky debt contracts. This influence is shown to be function of the heterogeneity of investors' valuations, the average belief about the cost of bankruptcy, and the bondholder's bargaining power. In particular, even though the average belief about the cost of bankruptcy may be that it is relatively low, the heterogeneity of beliefs can generate relatively large spreads. Furthermore, this model can generate a large variety of shapes for the term structure of yield spreads.

Heterogeneity in the investor base allows gains from trade under certain circumstances. Different values that can be extracted in the event of default and the impossibility of trading at every instant induce an imperfection in the secondary market for the asset prior to this event. The result of these two imperfections is an

[4] A closely related article is that by Williams (1995), who has studied the valuation of housing properties in a model consisting of costly search and bargaining. But Williams did not explicitly study liquidity issues and was not concerned with the valuation of corporate bonds.
[5] This is similar to models of trading volume such as Karpoff's (1986), where potential traders are randomly paired and a transaction occurs if and only if there are gains to be made from trading. It is shown that the frictions generated by the secondary market can account for a substantial portion of the observed premia.

additional premium which investors require on top of the pure credit premium. Hence, combining a lack of marketability (from the low frequency of matching opportunities) and taking into account the heterogeneity of investors results in the relative illiquidity of the market for risky assets.

A number of testable implications flow from our model. For example, we provide a clear explanation of the relationship between liquidity and the credit event: the closer the underlying firm is to bankruptcy, the greater the liquidity premium. Our model also predicts a decreasing liquidity of a corporate debt issue with age. Those two predictions have already been tested by Chakravarty and Sarkar (1999). Other testable implications are the importance of the dispersion or distribution of quotes (see also Houweling, Mentink and Vorst, 2005), the diversification of investors and their relative bargaining power. Changes in the perception of credit events and their consequences for the liquidity of risky debt markets could also be tested empirically.

Our model has also the feature that a bond is liquid following the initial public offering (IPO) and becomes more illiquid afterwards. Intuitively, a bond is liquid following the IPO simply because there is more chance of finding an investor placing a greater value on the bond just after the IPO. As time passes it becomes more and more difficult to sell the bond as only investors who make the higher valuation own the bond. This is not independent of the heterogeneity of potential investors as a greater heterogeneity of investors will result in a higher probability of finding a match where a transaction will occur.

The transaction price is determined by bargaining between the bondholder and the buyer (matched investor). So another novelty of our model is that we are able to link the L–M risk with the bargaining power the bondholder has. It is shown that the role of this bargaining power in explaining the L–M risk is not innocuous. The more powerful the bondholder, the higher is the liquidity premium. As the bargaining power of the bondholder approaches zero, so the liquidity premium vanishes.

The paper is organized as follows. In Section 2 we present the model and its general solution. In Section 3 we report different numerical solutions of the implied term structure of credit spreads for different distributions of the heterogeneous bankruptcy valuations and for different trading frequencies, and we show their influence on the term structure of risky spreads as well as on the liquidity premium itself. Section 4 concludes.

## 2 THE MODEL

### 2.1 Economic environment

A firm wants to finance a project by issuing a debt contract calling for the payment of a coupon, $c$, in each period, $t$, until the repayment of principal, $P$, at the maturity date, $T$. Let $r$ be the risk-free interest rate. Once issued, the bond will be traded on a secondary market consisting of many potential risk-neutral investors or traders.

The ongoing project is represented as a stochastic process. The value $V_t$ is assumed to follow a simple binomial process, where $V_t$ may be interpreted as the cumulative dividend value of the firm were it to be financed entirely by equity. This is also the value of assets of the firm under alternative financing arrangements. This specification allows us to nest as a special case of Merton's (1974) analysis of zero coupon debt. More precisely, we assume that the present value $V_t$ follows a simple binomial process with probabilities that are time- and state-invariant: $q$ is the probability of an up move, while $1 - q$ is the probability of a down move.

The firm is assumed to go bankrupt as soon as $V_t$ falls below a time-independent constant $\underline{V}$. This refers, for instance, to a debt protected by a net-worth covenant where bankruptcy is triggered as soon as $V_t$ falls below the value of the principal, $P$; ie, $\underline{V} = P$. In this case, the bankruptcy point, $\underline{V}$, is an absorbing state. We choose this bankruptcy scenario for simplicity in order to concentrate on modeling the liquidity of secondary trading. However, we note that other default triggers are possible: bankruptcy can be triggered by a strict liquidity constraint as in Kim, Ramaswamy and Sundaresan (1993), where bankruptcy occurs as soon as firm's cashflows are not sufficient to cover coupon payments or at the equity-holders' optimal abandonment point, as in Leland (1994). Our model could also be easily adapted for the case where there is a regime change as soon as the underlying variable crosses some specific point. One may think, for example, of a reorganization regime in which heterogeneous investors place different values on the reorganization process (see Franks and Torous (1989, 1994) or Anderson and Sundaresan (1996)).

In the case of bankruptcy debt-holders take possession of the firm only at a proportional cost, $\alpha$. This proportional cost may in part take a direct form, such as legal costs, that reflects a costly verification of collateral values. In part this may reflect a loss of project-specific human capital: after the transfer of control it takes time and effort for the creditor to find another management team that will be able to run the project at full efficiency. In what follows we therefore assume that the costs of bankruptcy are all summarized in a proportional liquidation cost of $\alpha$, so the collateral value obtained by debt-holders in the case of bankruptcy is simply $(1 - \alpha V)$. This collateral value is assumed to be transferred immediately to debt-holders in the case of bankruptcy. We therefore assume that default triggers an immediate lump-sum settlement.[6] Hence, the coefficient $\alpha$ also implicitly incorporates delays in the lump-sum settlement subsequent to default.[7]

Investors are partitioned into a finite number, $I$, of internally homogeneous classes with different beliefs about the cost of bankruptcy. All investors have access

[6] In the context of measuring duration for bonds with default risk, Fooladi, Roberts and Skinner (1997) have considered the delay between the occurrence of default and the final default payoff.

[7] In our model set-up the timing of the lump-sum settlement is not related to the issue of marketability. However, the fact that investors are heterogeneous in their valuation of $\alpha$ leads to liquidity issues in the secondary market before bankruptcy.

to the same substantive economic information, although members of different classes arrive at different assessments of the value they will extract on the basis of that information. Let $\alpha_i$ be the expected cost of bankruptcy of an investor belonging to class $i$. We will refer to $\alpha_i$ as the type of investor in class $i$, $i = 1,\ldots, I$. The classes are ranked in a way such that $\alpha_1 > \alpha_2 > \ldots > \alpha_I$. We denote by $\gamma(i)$ the probability of being of type $i$. Let $\#(i)$ be the number of investors of type $i$ among the global population of investors. In particular, we assume that investors have heterogeneous prior beliefs that are not explained by differences in information about the cost of bankruptcy. This is similar to Harrison and Kreps (1978), who considered a dynamic model where traders are risk-neutral, have heterogeneous prior beliefs (not explained by differences in information) about the dividend process of a risky asset, and are short sales constrained in that asset. They observed that the price of an asset is typically more than any trader's fundamental valuation of the asset because of the option value of being able to sell the asset to some other trader with a higher valuation in the future.[8]

An explanation for the heterogeneity of beliefs about the cost of bankruptcy is the different abilities that investors may have to value the bankrupt firm or to extract value from it. For example, there exist specialized firms that deal solely with bankrupt firms, whereas fund managers may be constrained to sell defaulted debt simply for reasons of window-dressing. Corporations in the same sector as the bankrupt firm may be able to extract more value than a private individual. Therefore, the fact that investors in corporate bonds are of different types (private individuals and institutional investors, including insurance companies, pension funds, investment trusts, investment banks and corporations) gives scope for different valuations of the bankrupt firm.[9] It has also been observed (see Bagwell, 1991) that investors can be heterogeneous along several dimensions, including different tax rules, different transaction costs, different information or different ways of analyzing the same information, and different tradable risks, and, finally, investors may differ for idiosyncratic or psychological reasons.[10] In the present paper, the heterogeneity of beliefs can be interpreted as follows. Either investors

[8] Morris (1996) examined a version of the Harrison and Kreps model where, although traders start out with heterogeneous prior beliefs, they are able to learn the true dividend process through time. A resale premium nonetheless arises reflecting the divergence of opinion before learning has occurred. Finally, note that heterogeneous prior beliefs are not inconsistent with rationality (see Allen and Morris, 2001).

[9] Reorganization is one of the two routes that a corporation in bankruptcy may take. When a corporation becomes insolvent and bankruptcy proceedings commence, the corporation will either be liquidated or reorganized. Bebchuk (2000) pointed out that a major problem faced by the rules of reorganization concerns valuation – that is, the difficulty inherent in estimating the value of the reorganized company. Bebchuk also explained why a bargaining-based procedure, such as that provided by Chapter 11 of US law, cannot serve well the goals of *ex-post* and *ex-ante* efficiency.

[10] Bernardo and Cornell (1997) have shown clearly that sophisticated investors are heterogeneous in the value they place on fixed-income securities. Their analysis implies that these differences in valuation are due to asymmetric information or differing valuation methodologies.

have different abilities to extract value or they use a different methodology to assess the value of the bankrupt firm (ie, they have different ways of analyzing the same information). The precise way in which the heterogeneity of beliefs emerges is left unmodeled, and heterogeneity of beliefs among the investors is simply summarized by the different $\alpha_i$, $i = 1,\ldots, I$.[11]

## 2.2 The game

In period zero the firm is matched randomly with one of the potential investors and makes a take-it-or-leave-it offer. This can be justified by search costs, which may prevent the firm from obtaining the best deal at issuance, or by the need for immediacy. Also, the underwriting business is often dominated by major investment banking firms, which generally practice some degree of underpricing.[12] This kind of simplification does not preclude the case where the bond is issued by means of first-price auctions where only a subset of potential investors are bidding. The probability of being matched with an investor of type $i$ is given by $\gamma(i)$. For simplicity, we assume a large population. This simply means that the probabilities $\gamma(i)$ do not change during the life of the contract, ie,

$$\gamma(i) = \frac{\#(i)}{\sum_{j=1}^{I} \#(j)} \simeq \frac{\#(i)-1}{\left(\sum_{j=1}^{I} \#(j)\right)-1}$$

This assumption could be relaxed to study small markets. However, the algebra would be slightly more complicated.

Thereafter, in each subsequent period, $t$, or node the debt contract can be traded on the market according to the following three stages:

**Stage 1** The bondholder receives the coupon payment of period $t$.

**Stage 2** The bondholder matches a potential investor randomly. Random matching means that the role of the matchmaker is left unmodeled. Also, random matching may be interpreted as a rough way of describing the activity of brokerage. We consider two different matching processes, representing two potential extreme forms of the organization of the market, in order to characterize a liquidity premium which will be the difference between these two benchmark processes.

In the *illiquid* matching process, the bondholder is matched with the whole population of investors. As a consequence, the bondholder cannot avoid the

[11] The potential investors use the same valuation methodology for bond valuation, but they use different inputs, $\alpha_i$, as a consequence of their heterogeneity in the event of bankruptcy.

[12] This modeling argument is also motivated by the empirical evidence on the underpricing of initial public offerings (see the survey of the literature on equity IPOs by Smith (1986)). Datta, Iskandar-Datta and Patel (1997) have provided some empirical evidence on underpricing of corporate straight debt. They documented that the degree of underpricing for bond IPOs, like stock IPOs, is inversely related to the reputation of the investment bank. Specifically, they found that low-quality or junk issues tend to be significantly underpriced whereas investment-

likelihood of matching with a potential investor with whom no gain from trade is possible and which results in no transaction. So, in each period the probability of the bondholder matching with an investor of type $i$ is $\gamma(i)$, $i = 1,\ldots, I$. In the *liquid* matching process the bondholder is matched with the subset of the population of investors that always results in a transaction. That is, the bondholder and a potential investor are matched in a way such that a gain from trade is always possible. In each period the probability that a bondholder of type $k$ will match with an investor of type $i \geq k$ is

$$\frac{\gamma(i)}{\sum_{j=k}^{I} \gamma(j)}$$

and the probability of matching with an investor of type $i < k$ is zero, $i = 1,\ldots, I$.

**Stage 3** The bondholder and the matched investor bargain over the selling price of the debt contract whenever gain from trade is possible.

The possibility of gain from trade in our model comes from the assumption that the potential investors have heterogeneous beliefs about the cost of bankruptcy. The definition of gain from trade which we use is the classical one: when a seller and a buyer meet, we say that there is gain from trade if both can agree on a price such that the exchange occurs and the seller makes a gain, positive or nil. Note that if the heterogeneity disappears, then strictly positive gains from trade vanish. Two remarks should be made. First, if a firm is matched with the investor from which it can get the best deal, then afterwards no strictly positive gain from trade is possible. Second, if the heterogeneity among investors disappears (ie, lack of liquidity vanishes and only lack of marketability remains), the illiquid and the liquid matching processes are identical and strictly positive gains from trade disappear.

We denote a given node of the value tree by $(t, m)$, where $t$ is the time period and $m$ is the number of up moves needed to reach this node. The bondholder's value from selling the debt contract at node $(t, m)$ is denoted by $\Pi_i(t, m)$, and $\Pi_i^0(t, m)$ is the bondholder's value from not selling the debt contract at node $(t, m)$. Let $U_j(t, m)$ be the investor's value from buying the debt contract at the negotiated price at node $(t, m)$, and let $U_j^0(t, m)$ be the investor's status-quo value. At each node $(t, m)$ a bondholder of type $i$ and an investor of type $j$ will agree to

grade issues are significantly overpriced. Kang and Lee (1996) provided empirical evidence on the underpricing of convertible debt offerings, documenting a significant initial underpricing for such offerings. Using a sample of 91 convertible debt offerings over the period 1988–92, they found a mean initial excess return of 1.11%, with nearly 70% of excess returns positive. Compared to the initial returns for other classes of security offerings, the average initial return for convertible debt offerings fits in the middle ground between those of IPOs and seasoned equity but above that of straight debt offerings. Finally, note that theoretical models predict no underpricing when investors are homogeneous.

trade if and only if a gain from trade is possible – that is, if and only if

$$\Pi_i(t,m) \geq \Pi_i^0(t,m) \text{ and } U_j(t,m) \geq U_j^0(t,m)$$

with

$$\begin{aligned}
\Pi_i(t,m) &= p_{t,m}^{i,j} \\
\Pi_i^0(t,m) &= q\left(\frac{1}{1+r}\right)\left[c + B_{t+1,m+1}^i\right] + (1-q)\left(\frac{1}{1+r}\right)\left[c + B_{t+1,m}^i\right] \\
U_j(t,m) &= q\left(\frac{1}{1+r}\right)\left[c + B_{t+1,m+1}^j\right] + (1-q)\left(\frac{1}{1+r}\right)\left[c + B_{t+1,m}^j\right] - p_{t,m}^{i,j} \\
U_j^0(t,m) &= 0
\end{aligned}$$

where $p_{t,m}^{i,j}$ is the selling price and $B_{t+1,m+1}^i$ is the expected price of a bond held by an investor of type $i$ at node $(t+1, m+1)$.

To predict the outcome to the bargaining we use the asymmetric Nash bargaining solution (see Binmore, Rubinstein and Wolinsky, 1986), in which the bargaining power of the bondholder facing an investor is equal to $\lambda \in (0, 1)$. This solution concept allows us to capture the fact that the current asset-holder may have more or less bargaining power than the potential investor. For example, the ability of investors to continue searching for alternative investors in the future may give them relatively greater bargaining power. Also, previous bargaining experience might give the current bondholder the edge.

Therefore, at each node $(t, m)$ of the binomial tree, the matching will result either in a transaction at a price

$$p_{t,m}^{i,j} = \arg\max\left[\Pi_i(t,m) - \Pi_i^0(t,m)\right]^{\lambda}\left[U_j(t,m) - U_j^0(t,m)\right]^{1-\lambda} \quad \text{if } j \geq i$$

or in disagreement if $j < i$, in which case the bond is not traded. The above equation is the asymmetric Nash bargaining solution. It can also be reinterpreted as the limiting sub-game perfect equilibrium to an alternating-offer bargaining model with complete information where the interval between offers and counter-offers is short (see Binmore, Rubinstein and Wolinsky, 1986).[13] Finally, notice that take-it-or-leave-it solutions are limiting Nash bargaining solutions where either the bondholder or the potential investor has all the bargaining power ($\lambda = 1$ or $\lambda = 0$).

[13] For example, in the case of bargaining when there is a risk of breakdown of negotiations, $\lambda$ is derived from the parties' beliefs concerning the likelihood of a breakdown.

## 2.3 The general solution

At each node $(t, m)$ the (expected) price of the bond when the market is *illiquid* is denoted by $p_{t,m}^{i,j}$. For each $j \geq i$ where $\alpha_i \geq \alpha_j$, gain from trade between a bondholder of type $i$ and an investor of type $j$ is possible. Therefore, the selling price $p_{t,m}^{i,j}$ is given by the asymmetric Nash bargaining solution

$$p_{t,m}^{i,j} = (1-\lambda)\left(\frac{q}{1+r}\left(c+B_{t+1,\,m+1}^{i}\right)+\frac{1-q}{1+r}\left(c+B_{t+1,\,m}^{i}\right)\right) \\ +\lambda\left(\frac{q}{1+r}\left(c+B_{t+1,\,m+1}^{j}\right)+\frac{1-q}{1+r}\left(c+B_{t+1,\,m}^{j}\right)\right)$$

For each $j < i$ where $\alpha_i > \alpha_j$, no gain from trade is possible. Therefore, the bondholder of type $i$ will keep the bond and the price of the bond will be equal to the bondholder's reservation price:

$$p_{t,m}^{i,j} = p_{t,m}^{i,i} = \left(\frac{q}{1+r}\left(c+B_{t+1,\,m+1}^{i}\right)+\frac{1-q}{1+r}\left(c+B_{t+1,\,m}^{i}\right)\right)$$

Whenever a match always results in a transaction, the market is described as *liquid* and we denote by $pl_{t,m}^{i,j}$ the price of the bond. For each $j \geq i$ where $\alpha_i \geq \alpha_j$, gain from trade between a bondholder of type $i$ and and an investor of type $j$ is possible. Therefore, the selling price $pl_{t,m}^{i,j}$ is given by the asymmetric Nash bargaining solution

$$pl_{t,m}^{i,j} = (1-\lambda)\left(\frac{q}{1+r}\left(c+BL_{t+1,\,m+1}^{i}\right)+\frac{1-q}{1+r}\left(c+BL_{t+1,\,m}^{i}\right)\right) \\ +\lambda\left(\frac{q}{1+r}\left(c+BL_{t+1,\,m+1}^{j}\right)+\frac{1-q}{1+r}\left(c+BL_{t+1,\,m}^{j}\right)\right)$$

But when bankruptcy is imminent – ie, when probability that the value of the firm will fall below the bankruptcy point $\underline{V}$ is $(1 - q)$, the selling price $p_{t,m}^{i,j}$ resulting from bargaining is given by

$$p_{t,m}^{i,j} = (1-\lambda)\left(\frac{q}{1+r}\left(c+B_{t+1,\,m+1}^{i}\right)+\frac{1-q}{1+r}\left(1-\alpha_i\right)\underline{V}\right) \\ +\lambda\left(\frac{q}{1+r}\left(c+B_{t+1,\,m+1}^{j}\right)+\frac{1-q}{1+r}\left(1-\alpha_j\right)\underline{V}\right)$$

for each $j \geq i$ where $\alpha_i \geq \alpha_j$. Note that $((1 - \alpha_i) \times \underline{V})$ is the expected residual value that an investor of type $i$ places on the firm in default. Meanwhile, for each $j < i$ where $\alpha_i > \alpha_j$, no gain from trade is possible. Therefore the price of the bond will be equal to the bondholder's reservation price. That is,

$$p_{t,m}^{i,j} = p_{t,m}^{i,i} = \frac{q}{1+r}\left(c + B_{t+1,\,m+1}^{i}\right) + \frac{1-q}{1+r}\left(1-\alpha_i\right)\underline{V}$$

Whenever a match always results in a transaction, the selling price, $pl_{t,m}^{i,j}$, is given by

$$pl_{t,m}^{i,j} = \left(1-\lambda\right)\left(\frac{q}{1+r}\left(c + BL_{t+1,\,m+1}^{i}\right) + \frac{1-q}{1+r}\left(\left(1-\alpha_i\right)\underline{V}\right)\right) + \lambda\left(\frac{q}{1+r}\left(c + BL_{t+1,\,m+1}^{j}\right) + \frac{1-q}{1+r}\left(1-\alpha_j\right)\underline{V}\right)$$

To value the illiquid bond, $B_{t,m}^{i}$, for the bondholder of type $i$ at node $(t, m)$, we simply have to weight the different transaction prices by the probabilities associated with the different types of investor:

$$B_{t,m}^{i} = \sum_{j} p_{t,m}^{i,j} \times \gamma\left(j\right)$$

where $\gamma(j)$ is the probability of matching an investor of type $j$, which is common knowledge, and $\sum_{j=1}^{I}\gamma(j) = 1$. Similarly, to value the liquid bond, $BL_{t,m}^{i}$, for the bondholder of type $i$ at node $(t, m)$, we need to weight the effective transaction prices by their respective probabilities:

$$BL_{t,m}^{i} = \frac{\sum_{j=i}^{I} pl_{t,m}^{i,j} \times \gamma\left(j\right)}{\sum_{j=i}^{I} \gamma\left(j\right)}$$

To find the expected values of the illiquid bond, $B_{t,m}$, and the liquid bond, $BL_{t,m}$, at node $(t, m)$, we simply have to weight the different bond values by the probabilities associated with the different types of investor:

$$B_{t,m} = \sum_{j} B_{t,m}^{j} \times \gamma\left(j\right)$$

and

$$BL_{t,m} = \sum_{j} BL_{t,m}^{j} \times \gamma(j)$$

We now define the liquidity–marketability (L–M) risk in the context of our model.[14] The L–M risk has two components. The first is the lack of marketability,

[14] We are interested primarily in the influence of liquidity risk in the pricing of corporate debt

which is the risk that one will not be able to sell the bond for a reasonable price in a reasonable period of time.[15] The second is the lack of liquidity, which is the risk that one will have to sell one's bond with loss of principal.[16] In our model the lack of marketability is reflected through the restriction on the frequency of matching opportunities. That is, opportunities to trade only occur at given intervals of time. As the frequency of matching opportunities increases (ie, as the time between two periods decreases), the lack of marketability goes down. The second component, ie, the lack of liquidity, is modeled by the problems of matching investors with heterogeneous beliefs. This induces us to characterize the L–M risk by the difference between the value of the bond in a liquid market and the value of the same debt contract in a market where one can experience matching problems. So, our definition of the L–M risk leads to the following definition of the liquidity premium, $LP_{t,m}$:

$$LP_{t,m} \equiv BL_{t,m} - B_{t,m}$$

As the lack of marketability disappears (ie, as the time interval between matching opportunities becomes very small), the L–M risk vanishes with it. Also, as the lack of liquidity disappears (ie, as the heterogeneity among investors disappears), the L–M risk vanishes. One should note that the lack of marketability (defined as the time restriction on trading) remains as time goes on; meanwhile, the lack of liquidity increases as time goes on since it becomes less and less likely that one will be matched with an investor with whom gain from trade is possible.

At each node of the tree we need to calculate different expected values for each type of investor. While this may appear to be costly in terms of computer time, we have developed a procedure that can considerably speed up calculations.

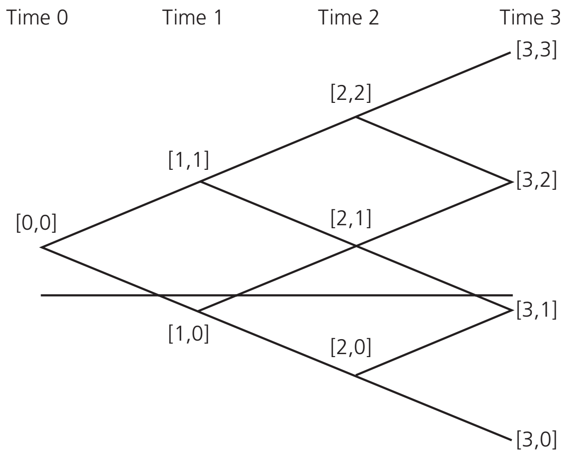
---

subject to default risk. We have not included all market microstructure factors that explain market liquidity, such as trade execution systems, trading commissions, disclosure of contracted prices and volume information, or market regulations. The market for corporate debt is a dealer, or quote-driven, market. We have not introduced explicit transaction costs, including commissions for trade and transaction taxes, but we may easily predict that this would reduce the liquidity of the market. Market transparency refers to the ability of market participants to observe the information in the trading process (O'Hara, 1995). Introducing information asymmetries would complicate the analysis without adding much to the relationship between credit and liquidity risks.

[15] Longstaff (1995) used option-pricing theory to derive an upper bound on the value of marketability by imposing a restriction on selling the security prior to a fixed date. He showed that discounts for lack of marketability can potentially be large even when the illiquidity period is very short. Furthermore, his analysis provides new insights into the relation between discounts for lack of marketability and the length of the marketability restriction. To be precise, it allows us to assess directly whether empirical estimates of discounts for lack of marketability are consistent with rational market pricing.

[16] Amihud and Mendelson (1986) modeled and tested the lack of liquidity using the bid–ask spread. The quoted ask (offer) price included a premium for immediate buying, and the bid price similarly reflected a concession required for immediate sale. They showed that expected asset returns increase in the relative bid–ask spread.

**FIGURE 1** Binomial tree.




In Appendix A we show how to simplify the calculations by looking only at nodes where bargaining takes place. The values of the bond at the other nodes are known: for sufficiently high values of the underlying variable, the value of the bond is equal to its riskless value. This also shows that strict gains from trade will be possible only for sufficiently low values of the expected variable, ie, when market participants expect the event of default.[17] The model therefore captures the fact that soon after the issue a bond is relatively liquid but later may become relatively illiquid.

## 2.4 An example: three periods and three types

We consider now a simple version of our pricing model with three periods. Figure 1 gives us the binomial tree. The market consists of three classes or types of potential investor. These investors have heterogeneous beliefs about the value of the cost of bankruptcy. The bankruptcy cost for an investor of type $i$ is $\alpha_i \in [0,1]$, and let $\alpha_1 > \alpha_2 > \alpha_3$. Without too much loss of generality, let $(\gamma_1, \gamma_2, \gamma_3)$ be the probability distribution over types $(\alpha_1, \alpha_2, \alpha_3)$, with $\sum_{j=1}^{3}\gamma_j = 1$. Then a high $\gamma_3$ corresponds to a large proportion of investors having a low expectation of the cost of bankruptcy. Remember that $B_{0,0}$ and $BL_{0,0}$ are the expected prices of the bond in period zero with the illiquid and liquid matching processes, respectively. These prices are solved backwards by beginning at the end of the binomial tree. See Appendix B for a detailed derivation of the results obtained for this example. The expected price of the bond with the illiquid matching process is given by

[17] Shulman, Bayless and Price (1993) have shown in an empirical study that the bond market's anticipation of default results in greater trading frequency.

$$
\begin{aligned}
B_{0,0} &= q(2+r)c(1+r)^{-2} + (2-q)q^2(c+P)(1+r)^{-3} \\
&+ \left[(1-q)(1+r)^{-1} + q(1-q)^2(1+r)^{-3}\right] \\
&\times\left[\gamma_1(1-\alpha_1)+\gamma_2(1-\alpha_2)+\gamma_3(1-\alpha_3)\right]\underline{V} + q(1-q)^2(1+r)^{-3} \\
&\times\left[\gamma_1\gamma_2\lambda(2-\lambda(1-\gamma_1))(\alpha_1-\alpha_2)+\gamma_1\gamma_3\lambda(2-\lambda(1-\gamma_1))(\alpha_1-\alpha_3)\right. \\
&\left.+\gamma_2\gamma_3\lambda(\gamma_1\lambda-\gamma_3\lambda+2)(\alpha_2-\alpha_3)\right]\underline{V}
\end{aligned}
$$

and the expected price of the bond with the liquid matching process is given by

$$
\begin{aligned}
BL_{0,0} &= q(2+r)c(1+r)^{-2} + (2-q)q^2(c+P)(1+r)^{-3} \\
&+ \left[(1-q)(1+r)^{-1} + (1-q)^2 q(1+r)^{-3}\right] \\
&\times\left[\gamma_1(1-\alpha_1)+\gamma_2(1-\alpha_2)+\gamma_3(1-\alpha_3)\right]\underline{V} + q(1-q)^2(1+r)^{-3} \\
&\times\gamma_1\lambda(2-\lambda(1-\gamma_1))(\gamma_2[\alpha_1-\alpha_2]+\gamma_3[\alpha_1-\alpha_3])\underline{V} \\
&+ q(1-q)^2(1+r)^{-3}(1-\gamma_1)^{-1} \\
&\times\gamma_2\gamma_3\lambda\left[\lambda\gamma_1+(1-\gamma_1)^{-1}(\gamma_2+(1-\gamma_1)+\gamma_3-\gamma_3\lambda)\right](\alpha_2-\alpha_3)\underline{V}
\end{aligned}
$$

Then we obtain the liquidity premium, which is

$$
\begin{aligned}
LP_{0,0} &= q(1-q)^2(1+r)^{-3}(\gamma_2+\gamma_3)^{-2}\gamma_2\gamma_3\lambda\left[(1-\gamma_2-\gamma_3)\lambda(\gamma_2+\gamma_3)\right. \\
&\left.+2(\gamma_2+\gamma_3)-\gamma_3\lambda+\left(-(1-\gamma_2-\gamma_3)\lambda-2+\gamma_3\lambda\right)(\gamma_2+\gamma_3)^2\right] \\
&\times(\alpha_2-\alpha_3)\underline{V}
\end{aligned}
$$

Since $\sum_{j=1}^{3}\gamma_j = 1$ and $\lambda \in (0,1)$, we have $LP_{0,0} \geq 0$. From comparative static results we observe that $\partial LP_{0,0}/\partial\gamma_2 > 0$, $\partial LP_{0,0}/\partial[\alpha_2 - \alpha_3] > 0$ and $\partial LP_{0,0}/\partial\lambda > 0$. Our first proposition summarizes these results.

Proposition 1 *Consider the debt-pricing game with three periods and three types where* $(\gamma_1, \gamma_2, \gamma_3)$ *is the probability distribution over types* $(\alpha_1, \alpha_2, \alpha_3)$ *and* $\sum_{j=1}^{3}\gamma_j = 1$. *First, the more powerful the bondholder, the higher is the value of the debt contract,* $B_{0,0}$. *Second, as the bargaining power of the bondholder approaches zero, the liquidity premium,* $LP_{0,0}$, *vanishes. Third, the more powerful the bondholder, the higher is the liquidity premium,* $LP_{0,0}$. *Fourth, the larger the value of* $\gamma_2$, *the higher is the liquidity premium,* $LP_{0,0}$. *Fifth, the greater* $[\alpha_2 - \alpha_3]$, *the higher is the liquidity premium,* $LP_{0,0}$.

The intuition behind the results in Proposition 1 follows. First, the more bargaining power the bondholder has, the higher the selling price he will obtain from a transaction, and therefore the higher the expected value of the debt contract. Second, the less bargaining power the bondholder has, the lower the difference between the values of the bond in a liquid and in an illiquid market, and therefore the lower the liquidity premium. As soon as the bargaining power of the bondholder vanishes, he will obtain at most his reservation value whatever the next matches may be. Therefore, the value of the bond in the liquid and illiquid markets will converge towards the same value. Third, and conversely, the more bargaining power the seller has, the higher the liquidity premium. Fourth, the greater the proportion of investors with a high expectation of the bankruptcy cost, the higher the likelihood of having a match that will not result in a transaction, and thus the higher the liquidity premium. Fifth, the greater the divergence in beliefs about the cost of bankruptcy, the higher will be the liquidity premium. Indeed, if all investors are homogeneous and hold the same belief about the cost of bankruptcy, the liquidity premium will be zero as all investors will agree on the same prices. When they have heterogeneous beliefs, they know that the expected values of the different investors will be even more different the more heterogeneous are the beliefs about the cost of bankruptcy. This leads to the prediction that, even though the average belief about the cost of bankruptcy may be relatively low, uncertainty about it can generate relatively large spreads.

Our model also captures the fact that the bond may be relatively liquid soon after issue and then become relatively illiquid. In particular, in the bargaining zone (points $(0, 0)$, $(1, 1)$ and $(2, 1)$ in Figure 1) where the value of the asset is affected by the risk of default, the probability of having a transaction decreases with the time to maturity. In the first period, the probability of having a transaction at node $(1, 1)$ is equal to $\gamma_1 + \gamma_2 \times [\gamma_2 + \gamma_3] + (\gamma_3)^2$; in the next period, at node $(2, 1)$, the probability of having a transaction falls to $(\gamma_1)^2 + \gamma_2 \times [1 + 2 \times \gamma_1] \times [\gamma_2 + \gamma_3] + (\gamma_3)^2[1 + \gamma_1 + \gamma_2]$. So, as the time-to-maturity increases, the probability of having a transaction converges towards $\gamma_3$, ie, the probability of being of the type with the higher value in default.

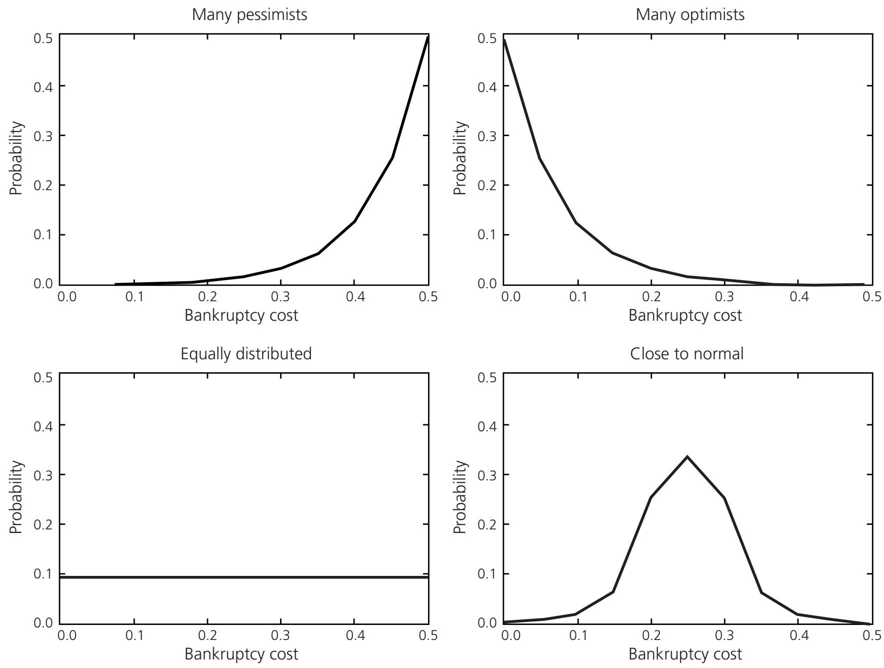
PROPOSITION 2 *In our debt-pricing game, the probability of having a transaction decreases with the time-to-maturity of the debt contract.*

When investors expect the event of default, since a transaction occurs only if gain from trade is possible, then, as time passes, the bond will be in the hands of an investor with a lower cost of bankruptcy who will find it more and more difficult to sell his asset.

## 3 THE TERM STRUCTURE OF RISKY SPREADS

Before presenting numerical simulations on the term structure of risky spreads, it should be noted that our framework is rich enough to accommodate four different

**FIGURE 2** Distributions of investors' bankruptcy costs.

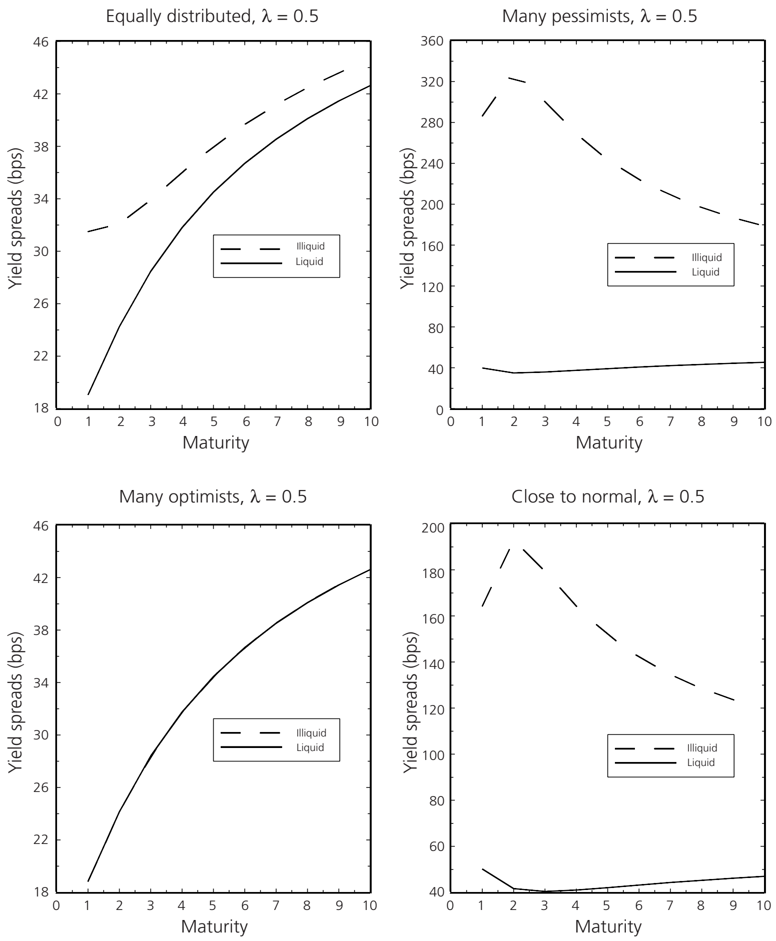



descriptions of the distribution of types of investor. In the following simulations we will consider four simple, different distributions:

1. Investors with different valuations of the cost of bankruptcy $\alpha_l \in [0.5, 0]$ are equally distributed. That is, $\gamma(l) = (I)^{-1}$.
2. Investors with different valuations of the cost of bankruptcy $\alpha_l \in [0.5, 0]$ are "pessimists". That is, the higher the investor's cost of bankruptcy, the higher the probability of encountering him: $\gamma(i) = \varepsilon^i$ with $i \in [1, I]$ such that $\varepsilon \in [0, 1]$ and $\sum_{i=1}^{I} \gamma(i) = 1$.
3. Investors with different valuations of the cost of bankruptcy $\alpha_i \in [0.5, 0]$ are "optimists". That is, the lower the investor's cost of bankruptcy, the higher the probability of encountering him: $\gamma(i) = \varepsilon^{I-i+1}$ with $i \in [1, I]$ such that $\varepsilon \in [0,1]$ and $\sum_{i=1}^{I} \gamma(i) = 1$.
4. Investors with different valuations of the cost of bankruptcy $\alpha_i \in [0.5, 0]$ are "close to normal". That is, the investors' bankruptcy costs are distributed around the mean bankruptcy cost of the range of possible bankruptcy costs: $\gamma(i) = \varepsilon^{(I-1)/2-i+1}$ with $i \in [1, I]$ such that $\varepsilon \in [0, 1]$ and $\sum_{i=1}^{I} \gamma(i) = 1$.

These different distributions can be interpreted as a proxy for market conditions or market participants. These distributions are plotted in Figure 2.

**FIGURE 3** Term structure of credit spreads for different distributions of investor types and for λ = 0.5. The other parameters are those of the base-case example.




In Figures 3, 4 and 5 we present the term structure of credit spreads, ie, the risk premium of debt as a function of the maturity under a variety of assumptions about the distribution of investor types and the bargaining power of the seller for the illiquid and liquid cases. Throughout our numerical simulations we have adopted the following values for the other parameters: $V_0 = 100$, $P = 50$, $\sigma = 0.15$, $\alpha \in [0, 0.5]$, $c = 0.066$, $r = 0.06$, and the number of types of investor, $I = 10$. Given some empirical evidence, we have chosen as a trading frequency or lack of marketability a time step of 1/52, representing one transaction a week.[18]

[18] Shulman, Bayless and Price (1993) gave some statistics on frequency distributions of bond trading which showed that 32% of the bonds in their sample were traded less than 25% of the time.

In Figure 3 we have plotted the term structures of credit spreads for the four different cases considered with $\lambda = 0.5$. It will be observed that our framework can generate yield spreads close to those seen in the market. Indeed, Kim, Ramaswamy and Sundaresan (1993) reported 77bps as the average spread for investment-grade corporate bonds, and Litterman and Iben (1991) reported historical ranges for par spreads of between 20 and 130 bps. Finally, Duffee (1999) has computed mean yield spreads on Baa bonds of between 115 and 198bps.

The difference between the two curves can be seen as a proxy for the L–M risk. Observe that the magnitude as well as the shape is substantially different depending on the set of assumptions used to describe the market environment. This could prove very useful if we want to fit observed yield spreads in the markets. The term structure of credit spreads is monotone increasing for both the "equally distributed" and "optimist" cases, but it is hump-shaped for those cases where the market is "pessimist" and in the case where investors' bankruptcy costs are distributed around an average value. The hump-shaped curve can be explained by the fact that the longer the time-to-maturity, the higher the chance of finding a match, and hence the lower the discount for L–M risk. This effect tends therefore to lower the influence of default risk, which increases with time-to-maturity.

Not surprisingly, the magnitude of risk premia is higher in a market where many investors place a low value on the firm in default. This is consistent with what one may observe in reality. When agents are pessimists – in a period of recession for instance – they tend to demand a higher risk premium. This higher risk premium is due not only to a more pessimistic expectation of the event of default but also to a less liquid market because of an expectation of problems in matching buyers and sellers.[19]

Figures 4 and 5 present the term structures of credit spreads for two different values of the bargaining power of the seller. Observe that both the magnitude and the shape of the term structure depend on $\lambda$. Thus there is a clear influence of the relative power of the seller on the term structure of risk premia. The risk premium is higher for small values of $\lambda$. Indeed, when the bondholder has less bargaining power, the resulting price from the transaction tends to be lower, which induces a higher risk premium. Observe also that in the equally distributed case the shape of the term structure can also be different according to $\lambda$. It is decreasing for $\lambda = 0.1$ and increasing for $\lambda = 0.9$.

[19] Collin-Dufresne, Goldstein and Martin (2001) have investigated the determinants of credit spread changes. They found that variables that should in theory determine credit spread changes had rather limited explanatory power. They suggested that an aggregate factor driving liquidity in the corporate bond market could explain their result. In our setting this could be explained by changes in the shape of the investors' distribution that could be driven by external events. For example, in periods of recession investors are rather more pessimistic about future events. This results in a higher proportion of investors with a lower value to be extracted in case of bankruptcy. In our model this would induce a lower liquidity of the market for corporate bonds.

**FIGURE 4** Term structure of credit spreads for different distributions of investor types and for different values of λ. Other parameters as for base-case example.

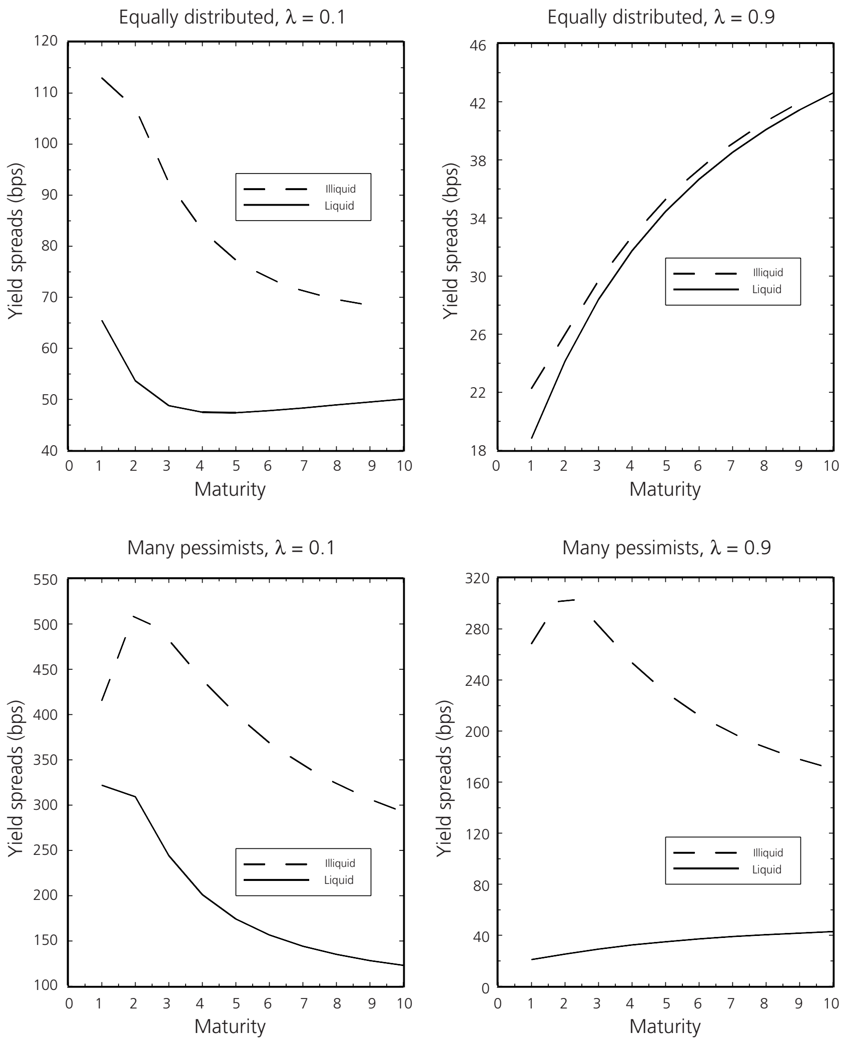
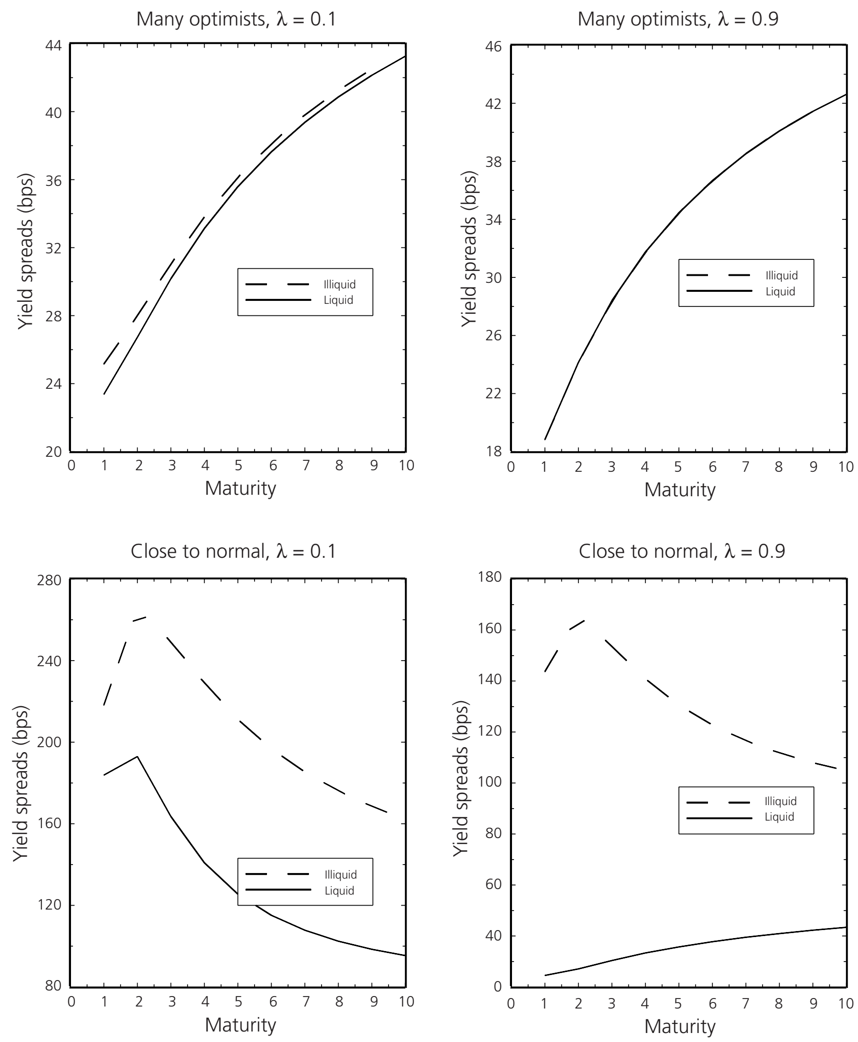


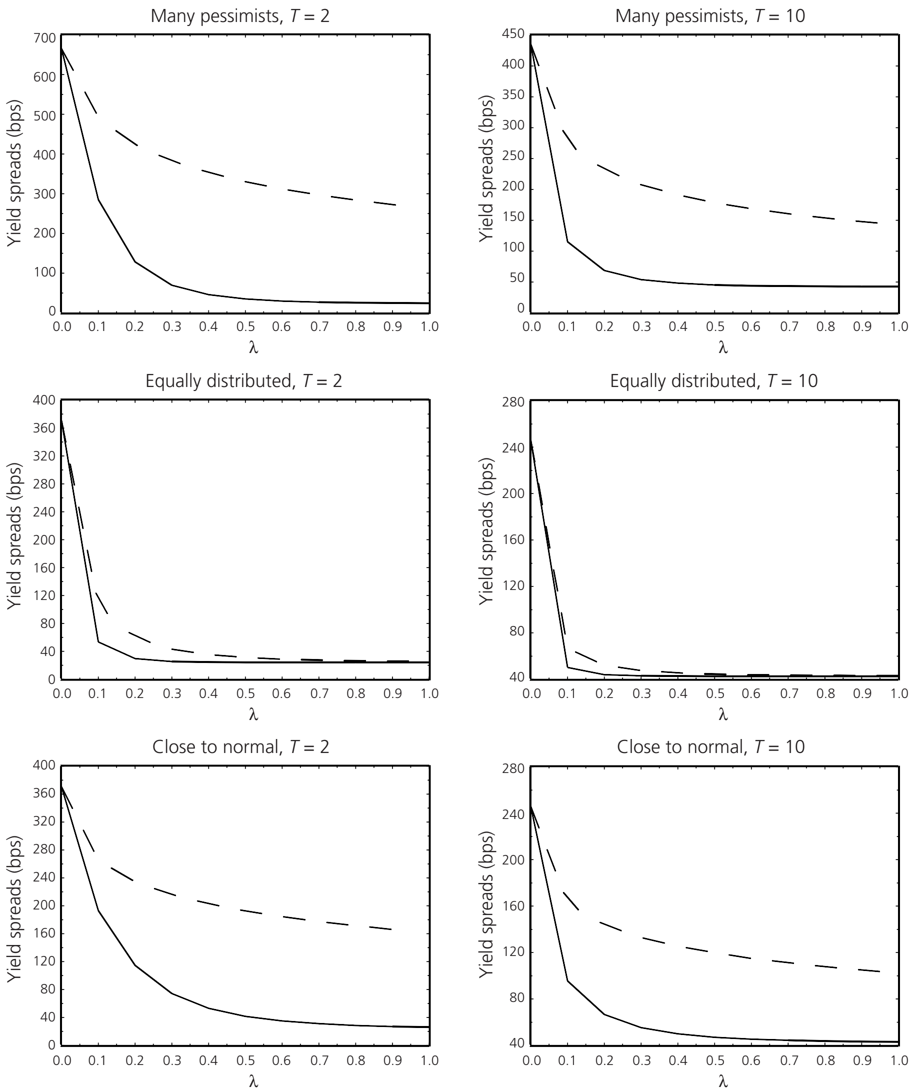

Figure 6 presents the risk premia function of the bargaining power, λ, for $T = 2$ and $T = 10$ for the assumed different distributions of beliefs. Risk premia show a sharply decrease with the bargaining power of the seller. This is consistent with the intuition that if the seller has all the bargaining power, the resulting price from the bargaining will be higher and this will induce a lower risk premium. Observe also that the risk premia in the illiquid and liquid markets tend to converge to the same value when λ approaches one. The convergence is less rapid in a pessimist market as in this case the probability of obtaining a match is more difficult.

**FIGURE 5** Term structure of credit spreads for different distributions of investor types and for different values of λ. Other parameters as for base-case example.




When we look at the premium generated by the lack of liquidity and the lack of marketability (Figure 7), we observe that the liquidity premium reaches a maximum at an intermediate value of λ. This is due to two effects that occur as λ increases (at $\lambda = 0$ the liquidity premium is zero). There is a direct effect that pushes up the price which the seller will obtain. Also, there is an indirect effect due to the continuation value of the bondholder. Indeed, when the market is liquid the bondholder will be able to sell the bond whatever the next matches, whereas he will face matching problems in the illiquid case. This implies that the bond price will increase less rapidly with λ in the illiquid case. Finally, and not

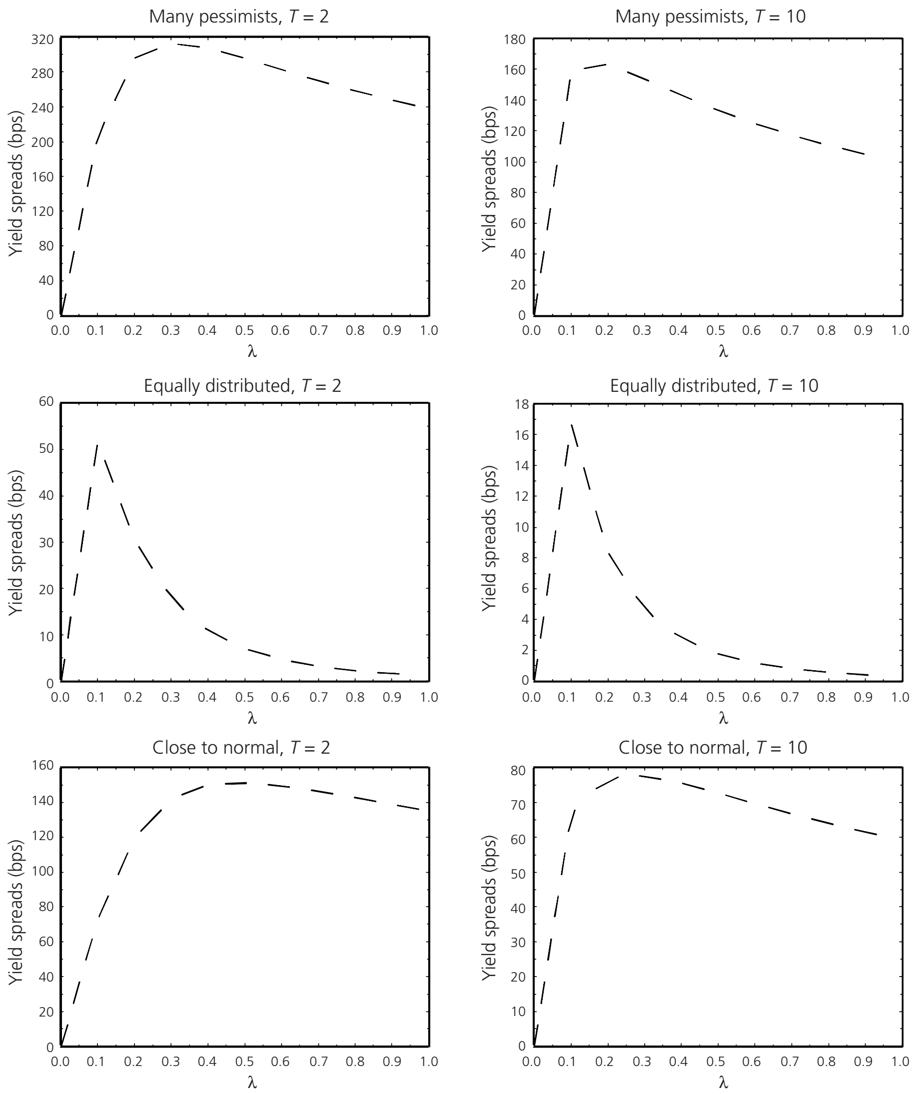
**FIGURE 6** Effect of λ on the risk premium for the illiquid and liquid cases.




surprisingly, as λ approaches unity one converges to the take-it-or-leave it solution whatever the liquidity of the market. Nevertheless, the convergence is faster in the liquid case. Indeed, if the buyer has less bargaining power, he is willing to concede more rapidly in the liquid case than in the illiquid case where he expects problems of matching.

Figure 8 presents the yield spreads in the illiquid market as a function of time-to-maturity for three smaller supports of the distribution of bankruptcy costs, keeping the lower bound constant ($\alpha \in [0, 0.4]$, $\alpha \in [0, 0.3]$, $\alpha \in [0, 0.2]$), and for

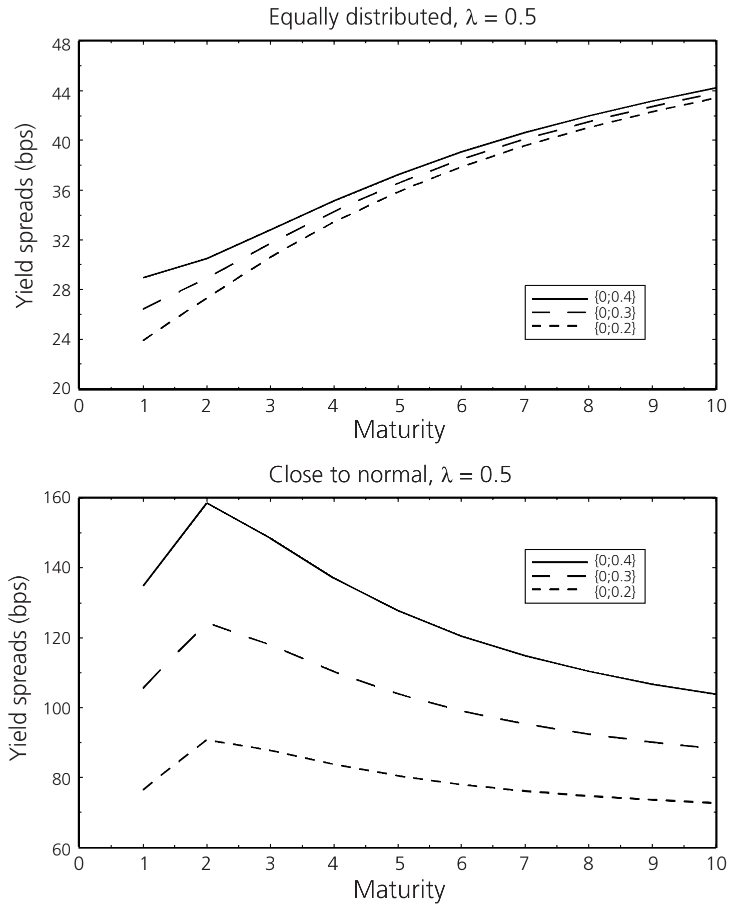
**FIGURE 7** Liquidity premium as a function of $\lambda$.




two different distributions of investor types ("Equally distributed" and "Close to normal"). In each case, the larger the support of the distribution, the larger are the risk premia.

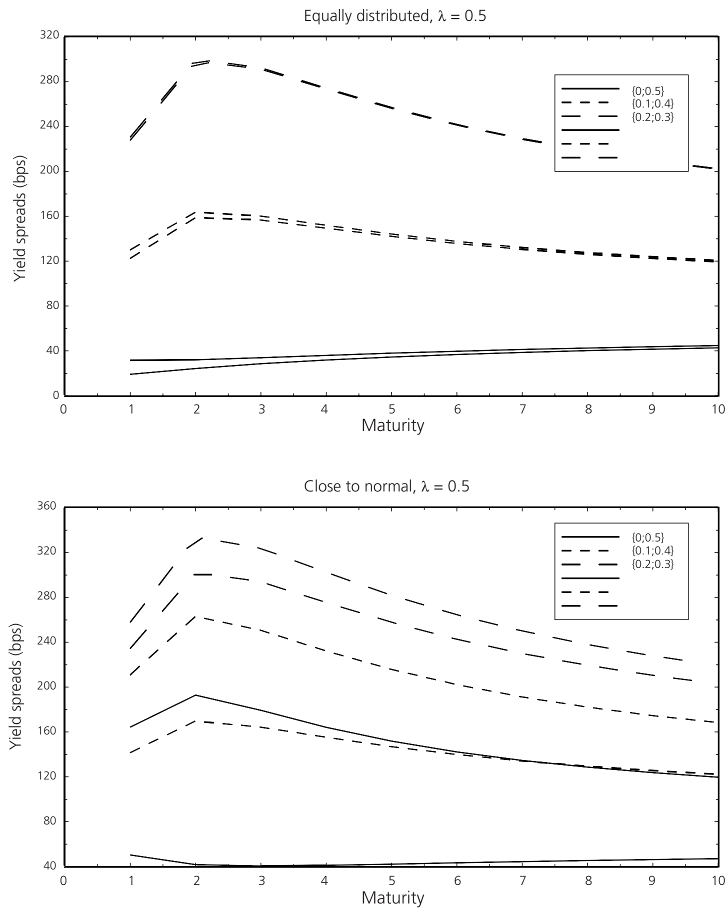
Figure 9 presents the term structure of credit spreads in the liquid and illiquid markets for different supports of the distribution, each with an average bankruptcy cost of 25% ($\alpha \in [0, 0.5]$, $\alpha \in [0.1, 0.4]$ and $\alpha \in [0.2, 0.3]$) for the "Equally distributed" and "Close to normal" cases. Observe than in both cases the higher the lower bound of the support of the distribution, the higher are the risk premia

**FIGURE 8** Term structure of credit spreads for different supports of the distribution of bankruptcy costs.



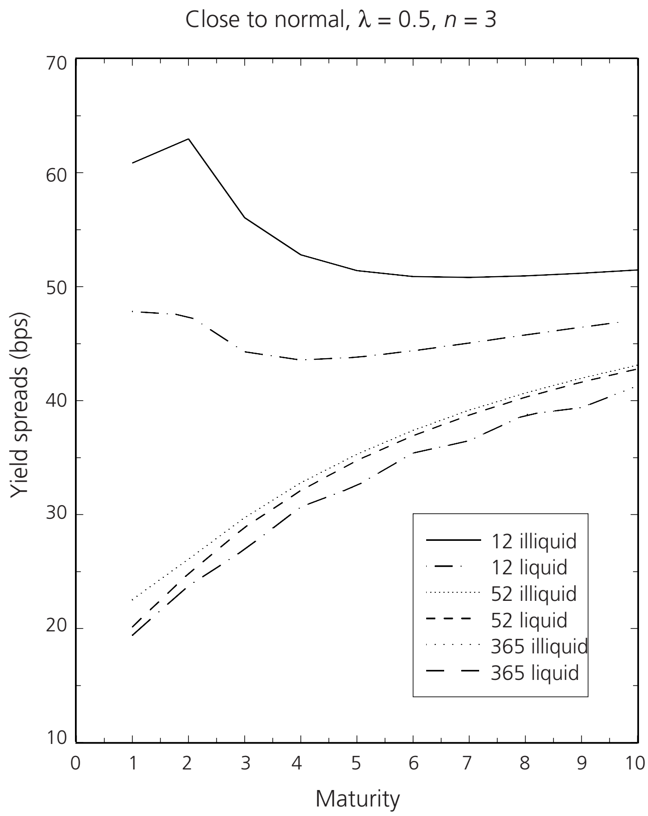

in the illiquid market. For instance, in the equally distributed case risk premia are above 200bps when $\alpha \in [0.2, 0.3]$, while they are about 40bps when $\alpha \in [0, 0.5]$. Hence, although in both cases the average cost of bankruptcy is 25%, the resulting risk premia are substantially different and are actually higher when the support of the distribution is smaller. This is because the lower bound is then larger, resulting in a higher risk premium. In the "Close to normal" case we also observe that the liquidity premium – and hence the difference between the spreads in the liquid and illiquid markets – increases with the support of the distribution. Therefore, the higher the lower bound of the support of the distribution, the higher are the risk premia, while the larger the support, the larger the premium due to the L–M risk. This confirms the results obtained in the simple case with three periods and three types. Note also that our model is able to generate a relatively flat term structure of yield spreads.

**FIGURE 9** Term structure of credit spreads for different supports of the distribution, average bankruptcy cost kept constant.




Finally, we look at the influence of the lack of marketability (ie, the trading frequency) on the term structure of credit spreads. In Figure 10 we plot yield spreads as a function of the maturity of the debt contract when there are three different types of investor and where trades are supposed to occur monthly, weekly and daily. Not surprisingly, risk premia decrease with trading frequency. Furthermore, the premium due to L–M risk also decreases with trading frequency, and we observe that this premium vanishes when trades occur daily. We also see that the shape of the term structure is dependent on trading frequency. Whereas the term structure is monotone increasing with weekly and daily trades, in the monthly case it is hump-shaped. The hump-shaped term structure can be explained by the trade-off between L–M risk and default risk. While default risk increases with the maturity of the contract, L–M risk decreases with the number of trades, which is a function of the maturity of the debt contract.

**FIGURE 10** Term structure of credit spreads for different trading frequencies (monthly: 12 trades a year; weekly: 52 trades a year; and daily: 365 trades a year) when there are three different types of investor in the "close to normal" case.




## 4 CONCLUDING COMMENTS

We have developed a corporate bond valuation model that takes into account both the risk of early default and the risk generated by lack of liquidity and marketability. Randomly matched investors who have heterogeneous prior beliefs about the value of the firm in bankruptcy bargain over the price of the asset in a secondary market. The liquidity–marketability (L–M) risk has been shown to be a function of the heterogeneity of investors' valuations, the average belief about the cost of bankruptcy, and the bondholders' bargaining power. Even though the average belief about the cost of bankruptcy may be that it is relatively low, the heterogeneity of beliefs can generate relatively large spreads. Furthermore, the model can generate a large variety of shapes for the term structure of yield spreads as well as spreads that are consistent with empirical evidence. Finally, the model also captures the fact that soon after the issue a bond is relatively liquid and later becomes relatively illiquid depending on the underlying asset value.

Our analysis has led to contributions not only at the theoretical level but also at the practical level. When using structural models of corporate debt pricing, it is important to note that such models do not incorporate an important aspect of the corporate debt market – namely, the relative illiquidity or lack of marketability with respect to the market for risk-free government issues. As a result, structural models have proved to be incapable of reproducing observed spreads in the corporate debt market. Willeman (2004) has recently shown that another reason why structural models are apparently unable to reproduce spreads is basically a calibration issue of the estimated models. Indeed, Willeman (2004) has implemented a calibration proposed by Collin-Dufresne and Goldstein (2001) to model bond prices. Using alternative specifications of a liquidity discount, he evaluated the model based on a fit to yield spreads and sensitivities to underlying fundamentals, producing a novel approach to evaluation of the fit of a credit risk model. When augmented with a liquidity discount, he obtained a very good fit and outperformed a two-factor reduced-form model. For investment-grade bonds the model attributed 40% of the yield spread of five years of maturity to credit risk, which is in some agreement with the results of Longstaff, Mithal and Neis (2004) but contrary to those of Huang and Huang (2003).

We have shown that the liquidity premium can be important in the total risk premium observed. However, using the information in credit default swaps to obtain direct measures of the magnitude of the default and non-default components in corporate spreads, Longstaff, Mithal and Neis (2004) have found that most of the corporate spread is due to default risk. This result holds for all rating categories and is robust to the definition of the riskless rate curve. Nevertheless, their results indicate that the default component does not account for the entire corporate credit spread. This non-default component ranges from about 20 to 100bps. They also found that the non-default component is time-varying and related to bond-specific illiquidity as well as to macroeconomic measures of bond-market illiquidity.

Finally, we have established a relationship between credit risk and liquidity risk. We have formalized the decreasing liquidity of a corporate debt issue with its age. Our analysis has also suggested that the dispersion of quotes, the diversification of investors and their respective bargaining power are important factors in the resulting liquidity premium.[20] Moreover, our framework is able to explain changes in liquidity that are not only asset-specific but which may be due to exterior events, such as a change in default perception that can be modeled by a change in the shape of the distribution of investors' bankruptcy costs.

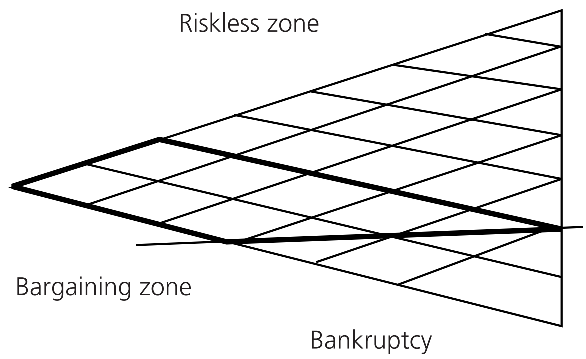
[20] The measurement of the heterogeneity of investors and of the bondholders' bargaining power, $\lambda$, may pose empirical problems since neither variable is observable (latent variables). One obvious way to get around this empirical problem is to use proxies – for example, variables that affect the bargaining power but neither the bondholder's value from selling the debt contract nor the investor's value from buying the debt contract. Indeed, as argued by Binmore, Rubinstein and Wolinsky (1986), $\lambda$ has to reflect asymmetries between bargainers that are not already captured by their utility functions and disagreement points (here, status-quo values). In Houweling, Mentink and Vorst (2005), the notion of heterogeneity of investor valuations is proxied with a yield dispersion statistic.

## APPENDIX A: SIMPLIFIED COMPUTATIONAL PROCEDURE

As in Cox and Rubinstein (1985), let $a$ stand for the minimum number of upward moves that the value of the firm must make over the next $m$ periods for the value of the bond to end up riskless. Thus, $a$ will be the smallest non-negative integer such that $u^a d^{m-a} V > P$. The problem therefore reduces to calculation of the expected values in the bargaining zone, which is of dimension $[a \times m]$. The value of the corporate bond in the riskless zone is simply the value of a riskless claim on a coupon and a principal repayment:

$$\sum_{j=1}^{m-j+1} \frac{c}{(1+r)^j} + \frac{P}{(1+r)^{m-j+1}}$$

**FIGURE A1** Simplified computational procedure.




## APPENDIX B: THE THREE-TYPES AND THREE-PERIODS CASE

The market consists of three types of player or potential investor. These investors place heterogeneous values on the cost of bankruptcy. The bankruptcy cost for an investor of type $i$ is $\alpha_i \in [0, 1]$, and let $\alpha_1 > \alpha_2 > \alpha_3$. Without too much loss of generality, let $(\gamma_1, \gamma_2, \gamma_3)$ be the probability distribution over types $(\alpha_1, \alpha_2, \alpha_3)$, with $\sum_{j=1}^{3} \gamma_j = 1$. Then a high $\gamma_3$ corresponds to a large proportion of investors with low bankruptcy costs. Remember that $B_{0,0}$ and $BL_{0,0}$ are the expected prices of the bond at period zero with the illiquid and liquid matching processes, respectively. To obtain the equilibrium prices we solve the entire game backwards from the end of the binomial tree. Then we obtain the following expressions:

$$p_{2,1}^{12} = \frac{1}{1+r}\left[q\left(c+P\right)+\left(1-q\right)\underline{V}\left[1-\alpha_1\left(1-\lambda\right)-\alpha_2\lambda\right]\right],$$

$$p_{2,1}^{13} = \frac{1}{1+r}\left[q\left(c+P\right)+\left(1-q\right)\underline{V}\left[1-\alpha_1\left(1-\lambda\right)-\alpha_3\lambda\right]\right],$$

$$p_{2,1}^{23}=\frac{1}{1+r}\left[q\left(c+P\right)+\left(1-q\right)\underline{V}\left[1-\alpha_2\left(1-\lambda\right)-\alpha_3\lambda\right]\right],$$

$$p_{2,1}^{22}=\frac{1}{1+r}\left[q\left(c+P\right)+\left(1-q\right)\underline{V}\left(1-\alpha_2\right)\right]=p_{2,1}^{21},$$

$$p_{2,1}^{11}=\frac{1}{1+r}\left[q\left(c+P\right)+\left(1-q\right)\underline{V}\left(1-\alpha_1\right)\right],$$

$$p_{2,1}^{33}=\frac{1}{1+r}\left[q\left(c+P\right)+\left(1-q\right)\underline{V}\left(1-\alpha_3\right)\right]=p_{2,1}^{31}=p_{2,1}^{32}$$

$$B_{2,1}^{1}=\frac{1}{1+r}\left[q\left(c+P\right)+\left(1-q\right)\underline{V}\left[\left(1-\alpha_1\right)+\lambda\gamma_2\left(\alpha_1-\alpha_2\right)+\lambda\gamma_3\left(\alpha_1-\alpha_3\right)\right]\right],$$

$$B_{2,1}^{2}=\frac{1}{1+r}\left[q\left(c+P\right)+\left(1-q\right)\underline{V}\left(1-\alpha_2\right)+\left(1-q\right)\underline{V}\lambda\gamma_3\left(\alpha_2-\alpha_3\right)\right],$$

$$B_{2,1}^{3}=\frac{1}{1+r}\left[q\left(c+P\right)+\left(1-q\right)\underline{V}\left(1-\alpha_3\right)\right]$$

$$p_{2,2}^{ij}=\frac{1}{1+r}\left[c+P\right]=B_{2,2}^{i}$$

$$p_{1,1}^{12}=q\left[\frac{c+B_{2,2}^{i}}{1+r}\right]+\left(1-q\right)\frac{c}{1+r}+\frac{1-q}{1+r}\left[\left(1-\lambda\right)B_{2,1}^{1}+\lambda B_{2,1}^{2}\right],$$

$$p_{1,1}^{13}=q\left[\frac{c+B_{2,2}^{i}}{1+r}\right]+\left(1-q\right)\frac{c}{1+r}+\frac{1-q}{1+r}\left[\left(1-\lambda\right)B_{2,1}^{1}+\lambda B_{2,1}^{3}\right],$$

$$p_{1,1}^{11}=q\left[\frac{c+B_{2,2}^{i}}{1+r}\right]+\left(1-q\right)\left[\frac{c+B_{2,1}^{1}}{1+r}\right],$$

$$p_{1,1}^{22}=q\left[\frac{c+B_{2,2}^{i}}{1+r}\right]+\left(1-q\right)\left[\frac{c+B_{2,1}^{2}}{1+r}\right]=p_{1,1}^{21},$$

$$p_{1,1}^{23}=q\left[\frac{c+B_{2,2}^{i}}{1+r}\right]+\left(1-q\right)\frac{c}{1+r}+\frac{1-q}{1+r}\left[\left(1-\lambda\right)B_{2,1}^{2}+\lambda B_{2,1}^{3}\right],$$

$$p_{1,1}^{33}=q\left[\frac{c+B_{2,2}^{i}}{1+r}\right]+\left(1-q\right)\left[\frac{c+B_{2,1}^{3}}{1+r}\right]=p_{1,1}^{32}=p_{1,1}^{31}$$

$$B_{0,0}^{1}=\frac{q}{1+r}\left[c+\gamma_1 p_{1,1}^{11}+\gamma_2 p_{1,1}^{12}+\gamma_3 p_{1,1}^{13}\right]+\frac{1-q}{1+r}\left(1-\alpha_1\right)\underline{V},$$
$$B_{0,0}^{2}=\frac{q}{1+r}\left[c+\left(1-\gamma_3\right)p_{1,1}^{22}+\gamma_3 p_{1,1}^{23}\right]+\frac{1-q}{1+r}\left(1-\alpha_2\right)\underline{V},$$
$$B_{0,0}^{3}=\frac{q}{1+r}\left[c+p_{1,1}^{33}\right]+\frac{1-q}{1+r}\left(1-\alpha_3\right)\underline{V}$$

$$B_{0,0}=\gamma_1 B_{0,0}^{1}+\gamma_2 B_{0,0}^{2}+\gamma_3 B_{0,0}^{3}$$

Now we turn to the case where at each node there will be transactions:

$$BL_{2,1}^{1}=\frac{1}{1+r}\left[q\left(c+P\right)+\left(1-q\right)\underline{V}\left[\left(1-\alpha_1\right)+\lambda\gamma_2\left(\alpha_1-\alpha_2\right)+\lambda\gamma_3\left(\alpha_1-\alpha_3\right)\right]\right],$$
$$BL_{2,1}^{2}=\frac{1}{1+r}\left[q\left(c+P\right)+\left(1-q\right)\underline{V}\left(1-\alpha_2\right)+\frac{\gamma_3}{1-\gamma_1}\left(1-q\right)\underline{V}\lambda\left(\alpha_2-\alpha_3\right)\right],$$
$$BL_{2,1}^{3}=\frac{1}{1+r}\left[q\left(c+P\right)+\left(1-q\right)\underline{V}\left(1-\alpha_3\right)\right]$$

$$BL_{2,2}^{i}=\frac{1}{1+r}\left[c+P\right]$$

$$p_{1,1}^{12}=q\left[\frac{c+BL_{2,2}^{i}}{1+r}\right]+\left(1-q\right)\frac{c}{1+r}+\frac{1-q}{1+r}\left[\left(1-\lambda\right)BL_{2,1}^{1}+\lambda BL_{2,1}^{2}\right],$$

$$p_{1,1}^{13}=q\left[\frac{c+BL_{2,2}^{i}}{1+r}\right]+\left(1-q\right)\frac{c}{1+r}+\frac{1-q}{1+r}\left[\left(1-\lambda\right)BL_{2,1}^{1}+\lambda BL_{2,1}^{3}\right],$$

$$p_{1,1}^{11}=q\left[\frac{c+BL_{2,2}^{i}}{1+r}\right]+\left(1-q\right)\left[\frac{c+BL_{2,1}^{1}}{1+r}\right],$$

$$p_{1,1}^{22}=q\left[\frac{c+BL_{2,2}^{i}}{1+r}\right]+\left(1-q\right)\left[\frac{c+BL_{2,1}^{2}}{1+r}\right]=p_{1,1}^{21},$$

$$p_{1,1}^{23}=q\left[\frac{c+BL_{2,2}^{i}}{1+r}\right]+\left(1-q\right)\frac{c}{1+r}+\frac{1-q}{1+r}\left[\left(1-\lambda\right)BL_{2,1}^{2}+\lambda BL_{2,1}^{3}\right],$$

$$p_{1,1}^{33}=q\left[\frac{c+BL_{2,2}^{i}}{1+r}\right]+\left(1-q\right)\left[\frac{c+BL_{2,1}^{3}}{1+r}\right]=p_{1,1}^{32}=p_{1,1}^{31}$$

$$BL^1_{0,0} = \frac{q}{1+r}\left[c + \gamma_1 p^{11}_{1,1} + \gamma_2 p^{12}_{1,1} + \gamma_3 p^{13}_{1,1}\right] + \frac{1-q}{1+r}\left(1-\alpha_1\right)\underline{V},$$

$$BL^2_{0,0} = \frac{q}{1+r}\left[c + \frac{\gamma_2}{1-\gamma_1} p^{22}_{1,1} + \frac{\gamma_3}{1-\gamma_1} p^{23}_{1,1}\right] + \frac{1-q}{1+r}\left(1-\alpha_2\right)\underline{V},$$

$$BL^3_{0,0} = \frac{q}{1+r}\left[c + p^{33}_{1,1}\right] + \frac{1-q}{1+r}\left(1-\alpha_3\right)\underline{V}$$

Let $BL$ be the price of the liquid bond at time zero. Then,

$$BL_{0,0} = \gamma_1 BL^1_{0,0} + \gamma_2 BL^2_{0,0} + \gamma_3 BL^3_{0,0}$$

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