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**SUBGAME PERFECTION IN A “DIVIDED GOVERNMENT”  
MODEL**

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**Abstract**

Purely non-cooperative principles, as iterated dominance and backward induction, explain divided government.

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## 1 Introduction

The Alesina and Rosenthal [1] model describes national policies as the result of institutional complexity and strategic behavior of the voters. Institutional complexity is captured by the existence of two decision branches of the State: the executive, elected under plurality rule, and the legislative, elected under proportional rule. Parties announce their policies and, then, voters vote. The strategic voting of citizens is described through coalition-proof behavior. The main implication of this model is that “divided government” can be explained through the behavior of voters with intermediate (between parties’ announced positions) preferences, who take advantage of the institutional structure above, by balancing the plurality of the winning party in the executive voting for the opposite party in the legislative election.

In this note we show that it is possible to analyze this kind of voting game, without any coalitional argument, as the coalition-proof Nash equilibrium chosen by [1]: purely non-cooperative rationality criterion explains results perfectly consistent with the results in Alesina and Rosenthal [1].

The model we analyze is a two-stage game where the executive branch is elected first and then the legislative.

We solve first the subgame where the result of the executive election is given. As the number of players increases, the proportion of players using mixed strategies tends to zero and basically only one Nash equilibrium in pure strategies survives. This unique Nash equilibrium is characterized by the fact that all the voters on the “right” of the corresponding outcome vote for the right party and all the voters on the “left” vote for the left party. Moreover it is possible to show, also in this context, the main result of Alesina and Rosenthal [1]: a party receives more votes in the legislative election if it has lost the executive election. We solve the game starting with a finite number of players and then increasing the number up to the continuum. The choice of a finite number of players captures the idea that a player is not negligible, and so dominance relations can be analyzed. Finally, in the limit case, it is possible to show that the iterative process of elimination of dominated strategies selects a single strategy for each player.

The proposed solution of the whole two-stage game respects iterated dominance and backward-induction criteria.

## 2 The Model

There are two parties  $k \in \{D, R\}$  and an arbitrary large number  $n$  of voters, indexed by  $i$ , each one characterized by the preferred policy point  $\theta_i \in [0, 1]$ , and by the utility function  $u_i = u(X, \theta_i)$  single peaked in the policy outcome  $X$ , continuous differentiable with  $\left| \frac{\partial u(X, \cdot)}{\partial X} \right| < M < \infty$ . We order the parties’

announced policies such that  $0 \leq \theta_D < \theta_R \leq 1$ <sup>1</sup>. The voters' preferred policy point is described, at the limit, through the cumulative distribution function  $H(\theta)$ , strictly increasing in the  $[0, 1]$  interval.

We describe the position of the legislature as a convex combination of the policy points of the two parties:

$$g(\theta_D, \theta_R, V_R) = \theta_D [1 - \gamma(V_R)] + \theta_R \gamma(V_R) \quad (1)$$

where  $V_R$  is the proportion of votes that party  $R$  gets in the legislative election and  $\gamma(\cdot)$  is continuous and strictly increasing from  $[0, 1]$  to itself with  $\left| \frac{\partial \gamma(\cdot)}{\partial V_R} \right| < M < \infty$ . If party  $k$  ( $k \in \{D, R\}$ ) wins the executive election, the policy outcome<sup>2</sup> will be:

$$X_k = (1 - \beta) \theta_k + \beta g(\theta_D, \theta_R, V_R), \quad 0 \leq \beta \leq 1 \quad (2)$$

Moreover, given the legislative result and with  $\beta < 1$ , we can easily find:

$$\theta_D \leq X_D < X_R \leq \theta_R \quad (3)$$

## 3 The Game

### 3.1 Legislative election

For a finite number of voters, given the presidential election, we state the following:

**Lemma 1** *Given  $k \in \{D, R\}$  elected president, let  $\alpha$  be a pure strategy combination. If voting for  $R$  is a best reply of player  $i$ , then  $X_k(\alpha_{-i}, D) < \theta_i$ . Mutatis mutandis if voting for  $D$  is a best reply of player  $j$  then  $X_k(\alpha_{-j}, R) > \theta_j$ .*

**Proof.** If  $X_k(\alpha_{-i}, D) \geq \theta_i$  then, since  $X_k(\alpha_{-i}, R) > X_k(\alpha_{-i}, D)$  and  $u(\cdot, \theta_i)$  is single peaked at  $\theta_i$ ,  $R$  is not a best reply. The same argument shows that  $X_k(\alpha_{-j}, R) > \theta_j$ . ■

If voting for  $R$  is best reply of player  $i$ , then the outcome, when  $i$  votes  $D$ , must be on the left of the voter. For a large number of voters, except for a small subset of voters around the outcome (whose length is inversely related with the number of voters  $n$ ), Lemma 1 implies that in any pure strategy Nash equilibrium, voters on the left of the equilibrium outcome vote  $D$  and voters on the right vote  $R$ <sup>3</sup>.

The following proposition shows that this is true also for mixed strategies, since the variance of the outcome tends to zero as the number of voters tends

<sup>1</sup>Nothing changes considering, as support of policy outcome, any other closed interval  $[\underline{\theta}, \bar{\theta}]$ .

<sup>2</sup>Given  $k$  elected president, when the opposite party gets no seats in the legislative election ( $V_{-k} = 0$ ) the policy outcome is  $X_k = \theta_k$ . When the opposite party gets all the seats in the legislative election ( $V_{-k} = 1$ ) the policy outcome is  $X_k = (1 - \beta) \theta_k + \beta \theta_{-k}$ .

<sup>3</sup>This result is explicitly proved in our Ph.D. thesis [2].

to infinity. As usual we define  $\sigma$  (that is the probability of voting for the right party) as a mixed strategy combination while  $BR_i(\sigma)$  denotes the best reply correspondence of player  $i$ .

**Proposition 1** *If  $k \in \{D, R\}$  is elected president,  $\forall \varepsilon > 0$ ,  $\exists n_0$  such that  $\forall n \geq n_0$ , if  $\theta_j - \theta_i > \varepsilon$  then  $\forall \sigma$  either  $R \notin BR_i(\sigma)$  or  $D \notin BR_j(\sigma)$ .*

**Proof.** See Appendix. ■

If voting for the right party is the best reply of player  $j$  we have shown that, in equilibrium, there is no voter on his right (of more than  $\varepsilon$ ) who votes with positive probability for the left party.

We remark that the results so obtained are completely independent, at the limit, of  $H(\theta)$ .

The game with  $n$  players is characterized, through the discrete distribution function  $H_n(\cdot)$  corresponding to the game with  $n$  players, by the following system which has a unique solution ( $g(\cdot)$  is continuous and strictly monotone and  $H_n(\cdot)$  is an increasing step function), say  $\tilde{\theta}_k^n = X_k$ :

$$\begin{cases} X_k = (1 - \beta)\theta_k + \beta[\theta_D(1 - \gamma(V_R)) + \theta_R\gamma(V_R)] \\ V_R = 1 - H_n(X_k) \end{cases} \quad (4)$$

It is natural, at this point, to state the following proposition, which follows from the previous one:

**Proposition 2** *If  $k \in \{D, R\}$  is elected president  $\forall \eta > 0$ ,  $\exists n_0$  such that  $\forall n \geq n_0$  in any Nash equilibrium of the corresponding game with  $n$  voters, if  $\theta_i \leq \tilde{\theta}_k^n - \eta$  then  $i$  votes  $D$  and if  $\theta_i \geq \tilde{\theta}_k^n + \eta$  then  $i$  votes  $R$ .*

**Proof.** It directly follows from proposition 1. ■

Given the previous results, we can analyze the game with a continuum of players as the limit game of the game with a discrete number of players. So each voter behaves as if he could be decisive. The cutpoint strategy characterizing the limit case is so deducible from the following remark:

**Remark 1** *At the limit, given the continuity and the strict monotonicity of  $g(\cdot)$  and of  $H(\cdot)$ , the point  $\tilde{\theta}_k = X_k$  is the unique solution of the system*

$$\begin{cases} X_k = (1 - \beta)\theta_k + \beta[\theta_D(1 - \gamma(V_R)) + \theta_R\gamma(V_R)] \\ V_R = 1 - H(X_k) \end{cases} \quad (5)$$

We will denote the strategy combination that respects the previous remark as the point  $\tilde{\theta}_k$ . It is clear that in the game with a continuum of players the results stated in games with a discrete number of players hold in a easier and direct way. Nevertheless, we have shown, through the analysis of games with a finite number of players, that the voting behavior of the voters can be explained simply with a purely non-cooperative framework.

Given the previous results, we can state the following:

**Proposition 3** *If  $k \in \{D, R\}$  is elected president, then:*

$$\theta_D < \tilde{\theta}_D = X_D < \tilde{\theta}_R = X_R < \theta_R \quad \forall \beta \in (0, 1) \quad (6)$$

**Proof.** This easily follows from (3) and from the consideration that  $\tilde{\theta}_k$  is the solution of the system (5). ■

This is the “moderation” result also obtained by Alesina and Rosenthal [1]: a party receives more votes in the legislative election if it has lost the presidential election.

Giving more specification on the  $H(\cdot)$  function, precisely on its slope, we can state the following:

**Proposition 4** *If*

$$\left| -\frac{\partial \gamma(1 - H(\cdot))}{\partial X} \right| < \frac{1}{\beta(\theta_R - \theta_D)} \quad (7)$$

*holds, then  $\tilde{\theta}_k$  is the only strategy combination that survives the iterative process of elimination of dominated strategies.*

**Proof.** Suppose  $k = D$  is elected. The policy outcome is

$$X_D = (1 - \beta)\theta_D + \beta g(\theta_D, \theta_R, V_R) \quad (8)$$

First we notice that  $V_R \in [0, 1]$ . Let  $Dom_i$  be the dominance relation of player  $i$  and  $X_{D,0}^k$  the point, at zero iteration time given the election of  $D$  such that voting  $k$  is the dominant strategy, we can so easily find these relations:

$$\forall \theta_i \in [0, \theta_D] \quad DDom_i R \text{ and } X_{D,0}^D = \theta_D$$

$$\forall \theta_i \in (\theta_D, (1 - \beta)\theta_D + \beta\theta_R) \text{ there are no dominance relations}$$

$$\forall \theta_i \in [(1 - \beta)\theta_D + \beta\theta_R, 1] \quad RDom_i D \text{ and } X_{D,0}^R = (1 - \beta)\theta_D + \beta\theta_R$$

Eliminating these dominated strategies, the range of variation of the proportion of votes for party  $R$  is:

$$V_R \in (1 - H(X_{D,0}^R), 1 - H(X_{D,0}^D))$$

and so we have, at the first iteration,

$$\begin{cases} X_{D,1}^D = (1 - \beta)\theta_D + \beta g(1 - H(X_{D,0}^R)) \\ X_{D,1}^R = (1 - \beta)\theta_D + \beta g(1 - H(X_{D,0}^D)) \end{cases} \quad (9)$$

We can go through  $t$  iteration deducing the following discrete dynamical system:

$$\begin{cases} X_{D,t}^D = (1 - \beta)\theta_D + \beta g(1 - H(X_{D,t-1}^R)) \\ X_{D,t}^R = (1 - \beta)\theta_D + \beta g(1 - H(X_{D,t-1}^D)) \end{cases} \quad (10)$$

The system can be rewritten as:

$$\begin{cases} X_{D,t}^D = f(X_{D,t-1}^R) \\ X_{D,t}^R = f(X_{D,t-1}^D) \end{cases} \quad (11)$$

where  $f(\cdot)$  is a decreasing function of  $X$ , with  $f(0) = (1 - \beta)\theta_D + \beta\theta_R$ ,  $f(1) = \theta_D$  and slope less than one for condition (7). The statement directly follows from the consideration that (11) is a contraction mapping. The same argument holds also in the case when party  $R$  wins the executive election. ■

We remark the interpretation of the slope condition. It means that the variation of the policy outcome of the legislature for a given variation of the policy outcome  $X$  (when the voters shift their vote to the opposite party) normalized to the announced parties' position is less than the inverse of the relative power of the two branches of the state<sup>4</sup>.

### 3.2 The two-stage game

Considering the continuum case and the result of iterated elimination of dominated strategies before stated, we can solve the two-stage game:

**Proposition 5** *If*

$$\left| -\frac{\partial \gamma(1 - H(\cdot))}{\partial X} \right| < \frac{1}{\beta(\theta_R - \theta_D)}$$

*holds, then the two-stage game, except for players who are indifferent between  $\tilde{\theta}_D$  and  $\tilde{\theta}_R$ , is solvable by iterated dominance.*

**Proof.** The statement trivially follows from the previous proposition: after the iterative process of elimination of dominated strategies in the sub-game each voter has to choose between  $\tilde{\theta}_D$  and  $\tilde{\theta}_R$ . We solve, then, through iterated dominance the whole game except for players who are indifferent between  $\tilde{\theta}_D$  and  $\tilde{\theta}_R$ . ■

**Remark 2** *Under the assumption that the utility function is symmetric, the solution of the first stage implies that if:*

$$\begin{aligned} H(\hat{\theta}) &< \frac{1}{2} \rightarrow R \text{ is elected} \\ H(\hat{\theta}) &> \frac{1}{2} \rightarrow D \text{ is elected} \end{aligned} \tag{12}$$

where  $\hat{\theta} = \frac{\tilde{\theta}_D + \tilde{\theta}_R}{2}$ .

**Proof.** As underlined in the previous proposition voters basically have to choose between  $\tilde{\theta}_D$  and  $\tilde{\theta}_R$ . Given the symmetry of the utility function the following holds: every  $\theta_i \in \left[0, \frac{\tilde{\theta}_D + \tilde{\theta}_R}{2}\right)$  votes for  $D$  and every  $\theta_i \in \left(\frac{\tilde{\theta}_D + \tilde{\theta}_R}{2}, 1\right]$  votes for  $R$ . ■

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<sup>4</sup>Clearly if the function which describes the legislature position is linear, condition (7) is always satisfied (see [2]).

**Remark 3** *Also in the case where the slope condition (7) doesn't hold, we can characterize the solution of the two-stage game coherent with the rationality principles. In fact backward induction implies that the solution is such that in the two subgames the equilibria are  $\tilde{\theta}_D$  and  $\tilde{\theta}_R$ , and then in the first stage game, regulated by plurality rule, one can use dominance criterium in order to characterize the solution.*

## 4 Conclusion and comments

We have described the rational behavior of the voters in such a way that each voter is not negligible on the outcome. In the sub-game, where the result of the presidential election is known, we are able to prove, starting with a finite number of voters and then increasing it until the continuum case, the uniqueness of the equilibrium. The equilibrium is characterized by the fact that each voter on the left/right of the equilibrium outcome votes for the left/right party. Moreover the “moderation” result obtained by Alesina and Rosenthal [1] is true.

Furthermore we have shown, under a condition on the slope of the function which describes the legislative position, that it is possible to solve this kind of voting game simply through dominance conditions. Given the preferences of the voters on the left of the announced point of the left party and on the right of the announced point of the right party, it is possible to start the process of iterative elimination of dominated strategies. It is evident the “moderation” behavior and it is evident who are the “moderate” voters: they are all the players in the interval  $(\tilde{\theta}_D, \tilde{\theta}_R)$ , who will change their voting behavior in the legislative election as a function of the winner in the executive election. In fact, if the  $D$  party wins the presidency they will vote for the  $R$  party in the legislative; if the  $R$  party wins the presidency they will vote for  $D$  in the legislative.

If the function which describes the legislative position is linear, it is possible to show an interesting result: the presidential election is “independent” of the legislative result (see [2]).

## 5 Appendix

*Proof of Proposition 1*

**Proof.** We derive the proof in three steps. First we show that the limit distribution of the outcome is “degenerate”. The second step assures a necessary condition for  $R/D$  to be best reply of player  $i/j$ . We finally clarify the contradiction between these two results.

Step 1. The  $\sigma_i$ ,  $i = \{1, \dots, n\}$ , are independent random variables, with mean and variance respectively equal to  $\mu_i$  and  $v_i^2 \leq \frac{1}{4}$ <sup>5</sup>. Applying Chebychev's Theorem, since  $\lim_{n \rightarrow \infty} n^{-2} \sum_{i=1}^n v_i^2 \leq \frac{1}{4n} = 0$ , we get  $\forall \phi > 0, \forall \psi > 0, \exists N$  such that

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<sup>5</sup>In fact  $v_i^2 = (1 - \sigma_i)(0 - \sigma_i) + \sigma_i(1 - \sigma_i)$ .

$\forall n \geq N$ :

$$\Pr \{|\bar{\sigma}_n - \bar{\mu}_n| > \phi\} < \psi \quad (13)$$

where, as usual,  $\bar{\sigma}_n = \frac{1}{n} \sum_{i=1}^n \sigma_i$  and  $\bar{\mu}_n = \frac{1}{n} \sum_{i=1}^n \mu_i$ . Moreover the continuity of the outcome function implies:  $\forall \phi > 0, \forall \psi > 0, \exists N$  such that  $\forall n \geq N$ :

$$\Pr \{|X_k(\bar{\sigma}_n) - X_k(\bar{\mu}_n)| > \phi\} < \psi \quad (14)$$

Clearly also  $\text{plim } X_k(\bar{\sigma}_{-i}, D)$ ,  $\text{plim } X_k(\bar{\sigma}_{-i}, R)$  and  $\text{plim } X_k(\bar{\sigma}_{-i})$  are equal to  $X_k(\bar{\mu}_n)$ .

Step 2. By definition  $R \in BR_i(\sigma) \iff$ :

$$\sum_{\alpha_{-i} \in A_{-i}} \left[ \prod_{l \neq i} \sigma_l(\alpha_l) \right] [u(X_k(\alpha_{-i}, R), \theta_i) - u(X_k(\alpha_{-i}, D), \theta_i)] \geq 0 \quad (15)$$

Using the assumption of continuous differentiability and multiplying for  $n$  both sides of (15) we get:

$$\begin{aligned} n \sum_{m=1}^n \Pr \left( \bar{\sigma}_{-i} = \frac{m}{n} \right) \left[ u \left( X_k \left( \frac{m+1}{n} \right), \theta_i \right) - u \left( X_k \left( \frac{m}{n} \right), \theta_i \right) \right] = \\ \sum_{m=1}^n \Pr \left( \bar{\sigma}_{-i} = \frac{m}{n} \right) \left[ u' \left( X_k(\bar{\sigma}_n), \theta_i \right) + h \left( X_k(\cdot), \frac{1}{n} \right) \right] \geq 0 \end{aligned} \quad (16)$$

Necessary condition to assure (16) positive is  $u' \left( X_k(\bar{\sigma}_n), \theta_i \right)$  not negative. In fact from the previous step we know that the probability that  $\bar{\sigma}_{-i}$  takes value in a neighborhood, arbitrary small, of  $\frac{m}{n} = \bar{\mu}_n$  converges to one, as  $n \rightarrow \infty$ ; and moreover  $h \left( \cdot, \frac{1}{n} \right)$  is a continuous function with  $h(\cdot, 0) = 0$ .

Analogously for player  $j$  necessary condition for  $D \in BR_j(\sigma)$ :

$$D \in BR_j(\sigma) \iff \sum_{\alpha_{-j} \in A_{-j}} \left[ \prod_{l \neq j} \sigma_l(\alpha_l) \right] [u(X_k(\alpha_{-j}, D), \theta_j) - u(X_k(\alpha_{-j}, R), \theta_j)] \geq 0 \quad (17)$$

is  $u'(\bar{\sigma}_n, \theta_j)$  not positive<sup>6</sup>.

Step 3. Let's define  $\underline{\theta} = \frac{\theta_i + \theta_j}{2}$ . If  $X_k(\bar{\mu}_n) \geq \underline{\theta}$  then  $u'(X_k(\bar{\sigma}_n), \theta_i)$  is surely negative, while if  $X_k(\bar{\mu}_n) \leq \underline{\theta}$  then  $u'(X_k(\bar{\sigma}_n), \theta_j)$  is positive. Hence the contradiction.

We can so conclude that for  $n$  large enough  $D$  best reply of player  $j$  and  $R$  best reply of player  $i$  cannot occur with positive probability. ■

<sup>6</sup>The case  $X_k(\bar{\mu}_n) = \theta_l$ ,  $l = i, j$ , can occur only for one player because we have imposed a distance of  $\varepsilon$  between the two voters.

## References

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- [2] Iannantuoni G. (1997-1998) “Spatial Voting Games: a Non-Cooperative Approach”, Ph.D. Thesis, Bocconi University, Milano, Italy.