Stable Flows in Transportation Networks

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Abstract

In this paper we develop a new theory of static equilibrium in congested transportation networks. Our considerations are based on a physical meaning of the flows rather than on an artificially chosen model of travel time functions. We introduce a concept of the stable equilibrium and prove the existence theorems.

1 Introduction

During a long time already, the theory of static equilibrium in congested transportation networks looks rather comprehensive [1, 2, 4, 5, 6]. Recall, that this theory deals with a transportation network $\mathcal{R}$ composed by the set of nodes $\mathcal{N}$ and the set of directed arcs $\mathcal{A}$. For this network we define the set of origin-destination pairs (OD-pairs):

$$\mathcal{OD} = \{(i,j),\ i,j \in \mathcal{N},\ i \neq j\}.$$

Each OD-pair (i,j) generates a demand $d_{(i,j)}$. This demand is traditionally considered as an average flow of drivers, which need to travel from some node i to another node j; so the demand is a non-negative real number. In the simplest case, in order to describe the travel cost of a trip we introduce some cost functions (*travel time functions*)

$$c_\alpha(f_\alpha), \quad f_\alpha \geq 0, \quad \alpha = 1,\dots,m \ \equiv \ |\mathcal{A}|,$$

which can be specific for each arc. In this case we assume that the cost function is *non-decreasing in the flow f_α on the arc α*. In a more general case we can assume that the vector function $c(f) = (c_1(f),\dots,c_m(f))$ with $f \in R^m_+$ depends on all flows in the network and that is a monotone operator in f.

Further, for each OD-pair (i,j) let us define the set of the routes connecting i with j:

$$\left\{a^{(r)}_{(i,j)} \in R^m,\ r = 1,\dots r_{(i,j)}\right\},$$

where the α-component of the vector $a^{(r)}_{(i,j)}$ is equal to one if the arc α is included in this route; otherwise this component is zero. For the sake of simplicity, we denote $A_{(i,j)}$ the matrix with the columns $a^{(r)}_{(i,j)}$:

$$A_{(i,j)} = \left(a^{(1)}_{(i,j)},\dots,a^{(r_{(i,j)})}_{(i,j)}\right).$$

Finally, for each OD-pair (i,j) we introduce the set of feasible flows

$$\Delta_{(i,j)} = \{F_{(i,j)} \in R^{r_{(i,j)}}_+ :\ \sum_{r=1}^{r_{(i,j)}} F^{(r)}_{(i,j)} = d_{(i,j)}\}.$$

Now, for any choice of the route flows

$$F = \{F_{(i,j)}\}_{(i,j)\in\mathcal{OD}} \quad \in \quad \Delta \ \equiv \prod_{(i,j)\in\mathcal{OD}} \Delta_{(i,j)}$$

we can compute the vector of arc flows in the network:

$$f = \sum_{(i,j)\in\mathcal{OD}} A_{(i,j)}F_{(i,j)} \equiv AF.$$

Therefore, for each OD-pair (i,j) the choice of the vector F results in the following cost of the routes:

$$C_{(i,j)}(F) = A^T_{(i,j)}c(AF).$$

Thus, we can pose the well known *static equilibrium problem*:

Find a flow pattern $F \in \Delta$ such that:

$$F^{(k)}_{(i,j)} > 0 \quad \Rightarrow \quad C^{(k)}_{(i,j)}(F) = \min_r C^{(r)}_{(i,j)}(F).$$

It is known that this problem can be written in the form of variational inequality [4]:

$$\text{find } F^* \in \Delta : \quad \langle C(F^*), F - F^* \rangle \geq 0 \quad \forall F \in \Delta, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in the corresponding space. In the case when the cost function of the arc depends only on the flow of this arc, the variational inequality (1) is equivalent to a minimization problem, which is known as Beckmann model:

$$\min_{f,F} \left\{ \sum_{\alpha \in \mathcal{A}} \int_0^{f_\alpha} c_\alpha(\tau) d\tau \ : \ f = AF, \ F \in \Delta \right\}. \tag{2}$$

If all $c_\alpha(\cdot)$ are non-decreasing functions, the problem (2) is convex, so we can guarantee the existence of the solution.

The approach we have described is completely classical and during many years there was no attempt to revise it. Moreover, all existing software packages (see, for example, [3]) are using this theory for analyzing the real urban networks. However, in this paper we are trying to develop a kind of alternative theory. The main motivation for our research is that the traditional travel time model describes a situation which hardly happens in reality. Indeed, the intuition confirms that a large flow corresponds to a fast movement. Then, the travel time cannot be too big. On the contrary, if a route is congested, then the flow is small and the travel time is very big. These simple observations show that the assumption that the travel time is an *increasing* function of flow is quite artificial. On the other hand, it is clear that the flow on the arc cannot be used as the main modeling parameter. If we say that the flow is small, we should also explain somehow why it is so: Either the route is congested or there is nobody on this street.

Another possibility to describe a congested arc is given by the *queering model.* In this model we assume that the travel time on an arc is given by the function

$$tt_\alpha(n_\alpha) = \min\{ \bar{t}_\alpha, n_\alpha \ / \bar{f}_\alpha \}, \tag{3}$$

where n_α is the number of drivers currently located at the arc, $\bar{t}_\alpha$ is the free traffic travel time on the arc, and $\bar{f}_\alpha$ is the capacity of the bottleneck. However, as far as we know, the theory of the queering models was never developed for general networks. The main problem with this is that in the queering model the flows are absent. That makes difficult to ensure the flow conservation lows at the nodes.

In this paper we present a new approach, in which we use together the main tools of the above two approaches, the flows and the loading of the arcs. In fact, the main results we describe below are not based on a particular model of the travel time functions. They follow directly from the physical meaning of the flows.

The paper is organized as follows. In Section 2 we introduce the notions of the stable state of the network and the stable equilibrium routes. The main result of this section is a characterization theorem which relates these objects. In the next Section

3 we prove the existence theorem for a network with a given loading. In Section 4 we prove that our approach can be extended onto the model with a priori fixed OD-flows. We compare the output of our model with that of the Beckmann model and show that in the latter case we always underestimate the level of congestion in the network. In Section 5 we show that the stable equilibrium for the model with fixed OD-flows can be obtained from a minimal cost multi-commodity linear problem with bounded arc capacities. Section 6 concludes the paper with a brief discussion of the results.

2 Stable equilibrium flows

Let at some network $\mathcal{R}$ we observe the vector of arc flows f, the vector of arc travel time

$$t = (t_1, \dots, t_m) \in R_+^m,$$

and the *arc loading* of the network

$$n = (n_1, \dots, n_m) \in R_+^m.$$

Each value n_α corresponds to an average number of drivers at arc α; we assume that this is a non-negative real value. We call the object $\mathcal{S} = (f, t, n)$ the *state* of the network $\mathcal{R}$. Note, that we do not need the flow balance at each node equal to zero. If that is not so, the corresponding node is either source or sink.

Definition 1 *The state $\mathcal{S}$ is called* stable *if the relation*

$$n_\alpha = t_\alpha \cdot f_\alpha \tag{4}$$

holds at any arc $\alpha \in \mathcal{A}$.

Thus, the stable states of the network have an important feature that they can survive in time. For right interpretation of the objects we deal in this section it is convenient to see them as constant functions of time.

Further, let the state $\mathcal{S}$ of the network $\mathcal{R}$ be a result of traffic flows between different OD-pairs. Instead of fixing the demand flows for the origin-destination pairs, let us assume that we know the average number of drivers $N_{(i,j)}$, which are currently in travel between their origin node i and the destination node j; these numbers must be non-negative and real. From now on we assume that such a loading

$$N = \{N_{(i,j)}\}_{(i,j)\in\mathcal{OD}}$$

is given and fixed. We call the vector N the *OD-loading* of the network $\mathcal{R}$. Denote $L = \sum\limits_{(i,j)\in\mathcal{OD}} N_{(i,j)}$, the total loading of the network.

Let us denote $M_{(i,j)}^{(r)}$ the number of drivers of the OD-pair (i,j), which travel along the route r; so that we can define the *route loading of the OD-pair* (i,j):

$$M_{(i,j)} = (M_{(i,j)}^{(1)}, \dots, M_{(i,j)}^{r_{(i,j)}}).$$

The vector $M = \{M_{(i,j)}\}_{(i,j)\in\mathcal{OD}}$ is called the *route loading of the network*. Clearly, we must have

$$N_{(i,j)} = \sum_{r=1}^{r_{(i,j)}} M_{(i,j)}^{(r)}, \quad (i,j) \in \mathcal{OD}. \tag{5}$$

Note that the travel time for the drivers of route r is given by the value

$$T_{(i,j)}^{(r)} = \langle t, a_{(i,j)}^{(r)} \rangle.$$

In vector notation, for $T_{(i,j)} = (T_{(i,j)}^{(1)}, \ldots, T_{(i,j)}^{r_{(i,j)}})$ we have $T_{(i,j)} = A_{(i,j)}^T t$. Therefore, for the whole route time vector $T = \{T_{(i,j)}\}_{(i,j)\in\mathcal{OD}}$ we have

$$T = A^T t.$$

Denote $m_{(i,j)}^{(r)} \in R^m$ the *state* of the route r. The αth component of this vector corresponds to the number of drivers of OD-pair (i,j), which travel along the route r and which are currently present on the arc α.

Definition 2 *We call the route r* stable *(with respect to the loading $M_{(i,j)}^{(r)}$ and the travel time vector t) if its state satisfies the following relation:*

$$m_{(i,j)}^{(r)} = \frac{M_{(i,j)}^{(r)}}{T_{(i,j)}^{(r)}} D(t) a_{(i,j)}^{(r)}, \tag{6}$$

where $D(t)$ is a diagonal matrix with the elements t_α.

Thus, if all routes of the OD-pair (i,j) are stable, the total impact of this OD-pair in the arc loading of the network is as follows:

$$m_{(i,j)} = \sum_{r=1}^{r_{(i,j)}} m_{(i,j)}^{(r)} = D(t) A_{(i,j)} D^{-1}(T_{(i,j)}) M_{(i,j)}.$$

Therefore, if all routes of the network are stable, the arc loading of the network $\mathcal{R}$ imposed by the route loading M is

$$n = D(t) A D^{-1}(T) M. \tag{7}$$

Note that in this case the imposed arc flows can be computed as follows:

$$f = A D^{-1}(T) M. \tag{8}$$

Note that the stable route states $m_{(i,j)}^{(r)}$ in the network are uniquely defined by the route loading M and the travel time vector t. Therefore we call $S = (f, t, n)$ with f from (8) and n from (7), the state, *generated by the stable route loading M*.

Lemma 1 *The state of the network, generated by a stable route loading, is stable.*

Proof:
Indeed, in view of (7) and (8) we have $n = D(t)AD^{-1}(T)M = D(t)f$, and that is exactly (4). $\square$

We need one more definition.

Definition 3 *We call M the* equilibrium route loading *(with respect to the travel time vector t) if*

$$M^{(k)}_{(i,j)} > 0 \quad \Rightarrow \quad T^{(k)}_{(i,j)} = \min_r T^{(r)}_{(i,j)}$$

for any $k = 1, \ldots, r_{(i,j)}$ and any $(i,j) \in \mathcal{OD}$.

This definition corresponds to so-called *Wardrop principle* of the route choice [6].

The main goal of this section is to characterize the stable state of the network, which can be obtained from a stable equilibrium route loading. To this end we need to introduce a special potential function. Denote by $T_{(i,j)}(t)$ the shortest path distance between the nodes i and j, computed with respect to the travel time vector t. Consider the following function

$$\phi(t) = \sum_{(i,j)\in\mathcal{OD}} N_{(i,j)} \ln T_{(i,j)}(t).$$

This function can be seen as a kind of *social cost* of the transportation flows in $\mathcal{R}$, generated by the travel time pattern t and the loading N.

Clearly, $\phi(t)$ is concave in t. Therefore its superdifferential $\partial\phi(t)$ is well defined at any interior point of its domain. Recall that for $t_0 \in \operatorname{dom}\phi$ we have

$$\partial\phi(t_0) = \{g : \ \phi(t) \le \phi(t_0) + \langle g, t - t_0\rangle \ \forall t \in \operatorname{dom}\phi\}.$$

If $\phi(\cdot)$ is differentiable at t then $\partial\phi(t) = \{\phi'(t)\}$.

Theorem 1 *The stable state $\mathcal{S} = (f, t, n)$ of the network $\mathcal{R}$ is generated by a stable equilibrium route loading if and only if*

$$f \in \partial\phi(t). \tag{9}$$

Proof:
Note that

$$\phi(t) = \sum_{(i,j)\in\mathcal{OD}} N_{(i,j)} \ln T_{(i,j)}(t),$$

with

$$T_{(i,j)}(t) = \min_{1\le r\le r_{(i,j)}} \quad \langle t, a^{(r)}_{(i,j)}\rangle.$$

Thus, the function $T_{(i,j)}(t)$ is concave in t and its superdifferential is as follows:

$$\begin{aligned}
\partial T_{(i,j)}(t) \quad &= \{G_{(i,j)} : \ G_{(i,j)} = A_{(i,j)}\lambda_{(i,j)}, \ \lambda_{(i,j)} \in \Delta^*_{(i,j)}(t)\},\\
\Delta^*_{(i,j)}(t) \quad &= \left\{\lambda_{(i,j)} \in R^{r_{(i,j)}}_+ : \ \sum_{r=1}^{r_{(i,j)}} \lambda^{(r)}_{(i,j)} = 1, \ \left(\langle t, a^{(r)}_{(i,j)}\rangle > T_{(i,j)}(t) \ \Rightarrow \ \lambda^{(r)}_{(i,j)} = 0\right)\right\}.
\end{aligned}$$

Therefore, the function $\phi_{(i,j)}(t) = N_{(i,j)} \ln T_{(i,j)}(t)$ is concave and its superdifferential is

$$\partial \phi_{(i,j)}(t) = \left\{ F_{(i,j)} \in R^m_+ : \; F_{(i,j)} = \tfrac{N_{(i,j)}}{T_{(i,j)}(t)} A_{(i,j)} \lambda_{(i,j)}, \; \lambda_{(i,j)} \in \Delta^*_{(i,j)}(t) \right\}.$$

In other words, the set $\partial \phi_{(i,j)}(t)$ is a conic hull of all equilibrium flows of the OD-pair (i,j).

Let the state $\mathcal{S} = (f,t,n)$ be generated by a stable equilibrium route loading M. Let us define

$$\lambda_{(i,j)} = \tfrac{1}{N_{(i,j)}} M_{(i,j)}.$$

Clearly, $\lambda_{(i,j)} \in \Delta^*_{(i,j)}(t)$. On the other hand, in view of (8)

$$\begin{aligned} f &= AD^{-1}(T(t))M = \sum_{(i,j)\in\mathcal{OD}} \tfrac{1}{T_{(i,j)}(t)} A_{(i,j)} M_{(i,j)} = \sum_{(i,j)\in\mathcal{OD}} \tfrac{N_{(i,j)}}{T_{(i,j)}(t)} A_{(i,j)} \lambda_{(i,j)} \\ &\in \sum_{(i,j)\in\mathcal{OD}} \partial \phi_{(i,j)}(t) = \partial \phi(t). \end{aligned}$$

This state is stable in view of Lemma 1.

To prove a converse statement, let us assume that (9) holds for some stable state $\mathcal{S} = (f,t,n)$. In this case

$$n_\alpha = t_\alpha \cdot f_\alpha.$$

Since $f \in \partial\phi(t)$, it can be represented as

$$f = \sum_{(i,j)\in\mathcal{OD}} F_{(i,j)},$$

where

$$F_{(i,j)} = \tfrac{N_{(i,j)}}{T_{(i,j)}(t)} A_{(i,j)} \lambda_{(i,j)}, \quad (i,j) \in \mathcal{OD},$$

and $\lambda_{(i,j)} \in \Delta^*_{(i,j)}(t)$. Let us choose

$$M_{(i,j)} = N_{(i,j)} \lambda_{(i,j)} \in R^{r(i,j)}_+,$$

and assign the drivers at each arc of the routes in accordance with (6). Note that in this case we get an equilibrium stable route loading M. It remains to verify that it gives the required arc flow f and the arc loading n. Indeed,

$$\begin{aligned} f &= \sum_{(i,j)\in\mathcal{OD}} F_{(i,j)} = \sum_{(i,j)\in\mathcal{OD}} \tfrac{N_{(i,j)}}{T_{(i,j)}(t)} A_{(i,j)} \lambda_{(i,j)} \\ &= \sum_{(i,j)\in\mathcal{OD}} \tfrac{1}{T_{(i,j)}(t)} A_{(i,j)} M_{(i,j)} = AD^{-1}(T(t))M. \end{aligned}$$

Therefore $n = D(t)f = D(t)AD^{-1}(T(t))M$. □

Definition 4 *Any state $\mathcal{S}$, which satisfies (9), we call the* stable equilibrium *of the network $\mathcal{R}$.*

The above theorem shows that in defining a stable equilibrium $\mathcal{S}$ we have only one degree of freedom. Indeed, if we choose a travel time pattern t, then the possible flows f are restricted by the set $\partial\phi(t)$ (which may consist of a single point). Moreover, if we choose some $f \in \partial\phi(t)$, then the arc loading is uniquely defined by (4). Let us show how to define a stable equilibrium starting from a vector of flows.

Lemma 2 *A vector of arc flows $\bar{f}$ can define a stable equilibrium if and only if there exists a solution $\bar{t}$ to the following maximization problem*

$$\max_t \, [\phi(t) - \langle \bar{f}, t \rangle]. \tag{10}$$

Then the state $S = (\bar{f}, \bar{t}, \bar{n})$, with $\bar{n}$ given by (4), is a stable equilibrium in $\mathcal{R}$. Moreover,

$$\langle \bar{f}, \bar{t} \rangle = L. \tag{11}$$

Proof:
The proof follows from Theorem 1 since the condition (9) is an optimality condition for the maximization problem (10). Since the problem (10) is concave, this condition is necessary and sufficient.

In order to prove the relation (11), note that the function $\phi(t)$ is logarithmically homogeneous:

$$\phi(\tau t) = \phi(t) + L \ln \tau, \quad \tau > 0.$$

Since the differentiable function $\xi(\tau) = \phi(\tau \bar{t}) - \tau \langle \bar{f}, \bar{t} \rangle$ must achieve its maximum at $\tau = 1$, we get (11) from the condition $\xi'(1) = 0$. □

The relation (11) shows that the maximization problem (10) can be rewritten in the following form:

$$\max_t \{ \phi(t) : \langle \bar{f}, t \rangle = L \}. \tag{12}$$

Hence, in the networks with a priori fixed flows the egoistic behavior of the drivers leads to a stable equilibrium state, which corresponds to a maximal social cost.

Note that in this section we did not use in any way the transportation specifics of the network $\mathcal{R}$.

3 Controllable congestion

In real urban transportation network each road has several specific characteristics, which define its performance. In order to model such objects, let us introduce for each arc $\alpha \in \mathcal{A}$ the following parameters:

$\bar{f}_\alpha$ - the maximal output flow. For real roads it depends on the number of lanes, traffic light schedule, etc.

$\bar{t}_\alpha$ - the minimal travel time on the arc. It depends on the length of the road, maximal allowed speed, etc.

$\bar{n}_\alpha$ - the volume of the arc. This value bounds the number of cars which can be present on this arc at the same moment of time. Usually it depends on the length of the road, the number of lanes and the average size of the cars in the network $\mathcal{R}$.

In all reasonable networks we can always assume that the parameter vectors $\bar{f}$, $\bar{t}$ and $\bar{n}$ are positive.

Clearly, the stable equilibrium $\mathcal{S} = (f, t, n)$ of the network $\mathcal{R}$ has sense only if its patterns are compatible with the set of parameters $(\bar{f}, \bar{t}, \bar{n})$. In this section we show that in the networks with volume $\bar{n}$ sufficiently large (with respect to the OD-loading N), the stable equilibrium always exists. Moreover, its social cost can be controlled by the parameter $\bar{f}$. If the volume $\bar{n}$ is small, the network flows may loose stability and controllability; we discuss this situation in Section 6.

Let us assume now that $\bar{n}$ is infinitely large. Consider the following maximization problem:

$$\max_t \{ \phi(t) - \langle \bar{f}, t \rangle : \ t \geq \bar{t} \}. \tag{13}$$

Theorem 2 *The set of solutions $\mathcal{T}^*$ of the problem (13) is always nonempty. Any $t^* \in \mathcal{T}^*$ is feasible:*

$$t^* \geq \bar{t}.$$

There exists an arc flow $f^* \in \partial\phi(t^*)$, which is feasible:

$$0 \leq f^* \leq \bar{f}.$$

The state $S = (f^*, t^*, n^*)$ with $n^* = D(t^*) f^*$ is a stable equilibrium in $\mathcal{R}$. The loading n^* is correct:

$$\sum_{\alpha \in \mathcal{A}} n_\alpha = L. \tag{14}$$

Proof:
Since $\bar{t} > 0$ we have $\bar{t} \in \operatorname{int} \operatorname{dom} \phi$. Let us consider a non-zero displacement $d \in R^m_+$. Denote

$$D = \sum_{\alpha \in \mathcal{A}} d_\alpha, \quad \hat{t} = \min_{\alpha \in \mathcal{A}} \bar{t}_\alpha.$$

Then for any $\tau > 0$, any $(i,j) \in \mathcal{OD}$ and $r = 1, \dots, r_{(i,j)}$ we have

$$\ln \langle \bar{t} + \tau d, a^{(r)}_{(i,j)} \rangle = \ln \langle \bar{t}, a^{(r)}_{(i,j)} \rangle + \ln \left(1 + \tau \frac{\langle d, a^{(r)}_{(i,j)} \rangle}{\langle \bar{t}, a^{(r)}_{(i,j)} \rangle} \right) \leq \ln \langle \bar{t}, a^{(r)}_{(i,j)} \rangle + \ln(1 + C\tau),$$

where $C = D/\hat{t}$. Therefore

$$\phi(\bar{t} + \tau d) \leq \phi(\bar{t}) + L \ln(1 + C\tau).$$

Since $\bar{f} > 0$, this upper estimate implies that the level set

$$\mathcal{Q}(\bar{t}) = \{t : \ \phi(t) - \langle \bar{f}, t\rangle \geq \phi(\bar{t})\}$$

is bounded along any ray $\{t = \bar{t} + \tau d, \ \tau \geq 0\}$ with non-negative d. Hence, the set $\mathcal{Q}(\bar{t})$ is bounded. Therefore the problem (13) is solvable.

Let s be the dual multipliers of the problem (13) for the constraints $t \geq \bar{t}$. Then the Lagrangian for that problem can be written as follows:

$$\mathcal{L}(t, s) = \phi(t) - \langle \bar{f}, t\rangle + \langle s, t - \bar{t}\rangle.$$

From Kuhn-Tucker theorem, there exist t^*, $f^* \in \partial\phi(t^*)$ and $s^* \geq 0$, which satisfy the following relations:

$$\begin{aligned} t^* \quad & \geq \bar{t}, \\ f^* \quad & = \bar{f} - s^*, \\ s^*_\alpha(t^*_\alpha - \bar{t}_\alpha) \quad & = 0, \ \alpha \in \mathcal{A}. \end{aligned}$$

Note that for $t \in \operatorname{int}\operatorname{dom}\phi$ we have $\partial\phi(t) \subset R^m_+$. Hence $f^* \geq 0$. On the other hand, since $s^* \geq 0$ we have $f^* \leq \bar{f}$. Thus, in view of Theorem 1 the state $S = (f^*, t^*, n^*)$ with $n^* = D(t^*)f^*$ is a stable equilibrium in $\mathcal{R}$. The relation (14) follows from (11). □

Let us present an interpretation of the above result in terms of the arc performance. If $t^*_\alpha > \bar{t}_\alpha$, then the arc α is congested. In this case the dual multiplier $s^*_\alpha = 0$. Therefore the equilibrium flow at that arc is the maximal possible:

$$f^*_\alpha = \bar{f}_\alpha.$$

This means that the travel time, the flow and the loading on a congested arc are related as follows:

$$t^*_\alpha = n^*_\alpha / \bar{f}_\alpha.$$

For non-congested arcs we have $t^*_\alpha = \bar{t}_\alpha$. Then the flow f^*_α does not reach the maximal value $\bar{f}_\alpha$. In this case the loading of this arc is defined as

$$n^*_\alpha = \bar{t}_\alpha \cdot f^*_\alpha.$$

Thus, our optimal solutions satisfy the relations of the queering model (3).

Theorem 2 proves that in any network with positive $\bar{f}$ and $\bar{t}$ and unlimited volume $\bar{n}$ there exist a stable equilibrium. However, we have seen that such an equilibrium tends to maximize the social cost. Therefore it is reasonable to try to decrease this cost using the flow bounds $\bar{f}$ as the control variables.

More precisely, let $\mathcal{F}$ be the set of arc flow bounds we can impose in the network $\mathcal{R}$. We assume that the set $\mathcal{F}$ is convex. Then the control problem we are going to solve is as follows:

$$\min_f \{\Psi(f) : \ f \in \mathcal{F}\}, \tag{15}$$

where

$$\Psi(f) = \max_t \{\phi(t) - \langle f, t\rangle : \ t \geq \bar{t}\}.$$

Note that $\Psi(f)$ is convex in f. Therefore (15) is a convex minimization problem, for which we can guarantee the existence of the solution under very mild assumptions. Namely, it is enough to assume that $\mathcal{F}$ is bounded and

$$\mathcal{F} \bigcap \mathrm{dom}\, \Psi \neq \emptyset.$$

Note that the problem (15) can be rewritten in a dual form. Namely, let us define a support function of the set $\mathcal{F}$:

$$\xi(t) = \max_f \{\langle f, t\rangle : \ f \in \mathcal{F}\}.$$

If $\mathcal{F}$ is bounded, this function is well defined for any $t \in R^m$. Then, in view of Duality Theorem

$$\begin{aligned}
\min_{f\in\mathcal{F}} \Psi(f) \quad &= \min_{f\in\mathcal{F}} \max_{t\geq\bar{t}} [\phi(t) - \langle f, t\rangle] \\
&= \max_{t\geq\bar{t}} \min_{f\in\mathcal{F}} [\phi(t) - \langle f, t\rangle] \\
&= \max_t \{\phi(t) - \xi(t) : \ t \geq \bar{t}\}.
\end{aligned}$$

Thus, we get the following dual form of the problem (15)

$$\max_t \{\phi(t) - \xi(t) : \ t \geq \bar{t}\}. \tag{16}$$

Note that this is a concave maximization problem, which can be efficiently solved provided that we can compute the function $\xi(t)$.

Let us give several examples of the situations, in which the function $\xi(t)$ is easily computable.

1. Traffic light regulation. Let us assume that at each intersection $\nu \in \mathcal{N}$ we have a traffic light, which allows each incoming arc $\alpha \in \mathcal{I}(\nu)$ to be functioning during λ_α portion of its cycle. Clearly,

$$\sum_{\alpha\in\mathcal{I}(\nu)} \lambda_\alpha = 1.$$

Then the real bound on the arc flow f_α becomes $\lambda_\alpha \bar{f}_\alpha$. In this case we can easily compute the support function $\xi(t)$:

$$\xi(t) = \max_f \{\langle f, t\rangle : \ f_\alpha \leq \lambda_\alpha \bar{f}_\alpha, \ \alpha \in \mathcal{A}; \ \sum_{\alpha\in\mathcal{I}(\nu)} \lambda_\alpha = 1, \ \nu \in \mathcal{N}\} = \sum_{\nu\in\mathcal{N}} \max_{\alpha\in\mathcal{I}(\nu)} \{ \ \bar{f}_\alpha \cdot t_\alpha \ \}.$$

In a more realistic case we can have some lower bounds $\lambda_\alpha \geq \bar{\lambda}_\alpha > 0$. Then the form of the function $\xi(t)$ becomes more complicated, but it still can be given by an explicit expression.

2. Reversing the lane direction. Sometimes in the transportation networks we have roads with two-directional traffic, in which we can change the direction of the lanes. Let such a road be presented in our model of the network by two arcs α and β. Then the bounds on the flows at these arcs can be written as follows:

$$\begin{aligned} f_\alpha &\leq \lambda(\bar{f}_\alpha + \bar{f}_\beta), \\ f_\beta &\leq (1-\lambda)(\bar{f}_\alpha + \bar{f}_\beta), \end{aligned}$$

where $\lambda \in [0,1]$ is the portion of the whole surface of the road assigned to the direction α. Then the corresponding support function $\xi_{\alpha,\beta}$ is as follows:

$$\xi_{\alpha,\beta}(t_\alpha, t_\beta) = (\bar{f}_\alpha + \bar{f}_\beta)\max\{t_\alpha, t_\beta\}.$$

Again, we can easily incorporate in this model some bounds on λ, which separate it from zero and one.

4 Comparing with Beckmann model

It is interesting to compare the output we have from the model (13) with that of the Beckmann model (2). In order to do that, we need first to rewrite the Beckmann model in a more tractable form. Let us define

$$\bar{c}_\alpha(f_\alpha) = \int\limits_0^{f_\alpha} c_\alpha(\tau)d\tau, \quad \operatorname{dom}\bar{c}_\alpha = \{f_\alpha : \ f_\alpha \geq 0\}, \quad \bar{c}^*_\alpha(t_\alpha) = \max_{f_\alpha \geq 0}[f_\alpha t_\alpha - \bar{c}_\alpha(f_\alpha)].$$

Note that $\bar{c}_\alpha(f_\alpha) = \max\limits_{t_\alpha}[f_\alpha t_\alpha - \bar{c}^*_\alpha(t_\alpha)]$.

Lemma 3 *The problem dual to (2) can be written as follows:*

$$\max_t \left[\sum_{(i,j)\in\mathcal{OD}} d_{(i,j)} T_{(i,j)}(t) - \sum_{\alpha\in\mathcal{A}} \bar{c}^*_\alpha(t_\alpha) \right]. \tag{17}$$

Proof:
Indeed, for the Beckmann model we have:

$$\begin{aligned} &\min_{f,F}\{ \sum_{\alpha\in\mathcal{A}} \bar{c}_\alpha(f_\alpha) \ : \ f = AF, \ F \in \Delta\} \\ &= \min_{f,F}\{ \sum_{\alpha\in\mathcal{A}} \max_{t_\alpha}[f_\alpha t_\alpha - \bar{c}^*_\alpha(t_\alpha)] \ : \ f = AF, \ F \in \Delta\} \\ &= \max_t \min_{f,F}\{\langle f, t\rangle - \sum_{\alpha\in\mathcal{A}} \bar{c}^*_\alpha(t_\alpha) \ : \ f = AF, \ F \in \Delta\} \\ &= \max_t \left[\min_{f,F}\{\langle f, t\rangle : \ f = AF, \ F \in \Delta\} \ - \sum_{\alpha\in\mathcal{A}} \bar{c}^*_\alpha(t_\alpha) \right]. \end{aligned}$$

It remains to note that

$$\langle f, t\rangle = \langle AF, t\rangle = \langle F, A^T t\rangle = \sum_{(i,j)\in\mathcal{OD}} \langle F_{(i,j)}, A^T_{(i,j)} t\rangle.$$

Hence,

$$\min_{f,F}\{\langle f,t\rangle:\ f=AF,\ F\in\Delta\} = \sum_{(i,j)\in\mathcal{OD}} \min\{\langle F_{(i,j)}, A^T_{(i,j)}t\rangle:\ F_{(i,j)}\in\Delta_{(i,j)}\}$$
$$= \sum_{(i,j)\in\mathcal{OD}} d_{(i,j)}T_{(i,j)}(t).$$

□

We have seen that the output of the model (13) resembles the relations of the queering model (3). Therefore let us check what kind of results we can get from the Beckmann model, which uses the following queering-type travel time function

$$c_\alpha(f_\alpha) = \max\left\{\bar{t}_\alpha,\ \frac{f_\alpha}{\gamma_\alpha}\right\}, \tag{18}$$

where γ_α is a positive parameter.

Lemma 4 *For the travel time function $c_\alpha(\cdot)$ defined by (18) the conjugate function*

$$\bar{c}^*_\alpha(t_\alpha) = 0, \quad t_\alpha \le \bar{t}_\alpha.$$

For $t_\alpha \ge \bar{t}_\alpha$ this function has the following form:

$$\bar{c}^*_\alpha(t_\alpha) = \tfrac{1}{2\gamma_\alpha}[t^2_\alpha - \bar{t}^2_\alpha].$$

Proof:
Denote $(a)_+ = \max\{0,a\}$. Then, for $c_\alpha(\cdot)$ defined by (18) the simple integration gives

$$\bar{c}_\alpha(f_\alpha) = \bar{t}_\alpha\cdot f_\alpha + \tfrac{\gamma_\alpha}{2}\left(f_\alpha - \tfrac{\bar{t}_\alpha}{\gamma_\alpha}\right)^2_+.$$

Thus,

$$\bar{c}^*_\alpha(t_\alpha) = \max_{f_\alpha\ge 0}\left[f_\alpha(t_\alpha-\bar{t}_\alpha) - \tfrac{\gamma_\alpha}{2}\left(f_\alpha - \tfrac{\bar{t}_\alpha}{\gamma_\alpha}\right)^2_+\right].$$

Hence, for $t_\alpha \le \bar{t}_\alpha$ this function equal to zero. For $t_\alpha > \bar{t}_\alpha$ the maximum in the above expression is achieved at $f_\alpha = t_\alpha/\gamma_\alpha$. □

Since the shortest path functions $T_{(i,j)}(t)$ are non-decreasing in t, the dual Beckmann formulation (17), (18) can be written as follows:

$$\max_t\left[\sum_{(i,j)\in\mathcal{OD}} d_{(i,j)}T_{(i,j)}(t) - \sum_{\alpha\in\mathcal{A}} \tfrac{1}{2\gamma_\alpha}[t^2_\alpha - \bar{t}^2_\alpha]:\ t\ge\bar{t}\right]. \tag{19}$$

Note that the models (13) and (19) are based on the different data. Let us show how the model (13) can be rewritten in the form with a priori fixed demand flows. In order to do that, we need the following auxiliary result.

Lemma 5 *Let $f(x)$ be a concave function of x. Then the function*

$$\hat{f}(x,\tau) = \tau \ln f(x) - \tau \ln \tau$$

is concave on $\{(x,\tau) : \ f(x) > 0, \ \tau > 0\}$.

Proof:
Indeed, let

$$\tau_1 > 0, \quad f(x_1) > 0, \quad \tau_2 > 0, \quad f(x_2) > 0.$$

Then, for $\hat{\tau} = \frac{1}{2}(\tau_1 + \tau_2)$ and $\hat{x} = \frac{1}{2}(x_1 + x_2)$ we have:

$$\begin{aligned} \hat{f}(\hat{x},\hat{\tau}) \quad &= \hat{\tau} \ln \tfrac{f(\hat{x})}{\hat{\tau}} \geq \hat{\tau} \ln \tfrac{f(x_1)+f(x_2)}{\tau_1+\tau_2} = \hat{\tau} \ln \tfrac{\tau_1 \frac{f(x_1)}{\tau_1} + \tau_2 \frac{f(x_2)}{\tau_2}}{\tau_1+\tau_2} \\ &\geq \hat{\tau} \left[\tfrac{\tau_1}{\tau_1+\tau_2} \ln \tfrac{f(x_1)}{\tau_1} + \tfrac{\tau_2}{\tau_1+\tau_2} \ln \tfrac{f(x_2)}{\tau_2} \right] = \tfrac{1}{2}\hat{f}(x_1,\tau_1) + \tfrac{1}{2}\hat{f}(x_2,\tau_2). \end{aligned}$$

□

Let us assume now that the desired demand flow vector $d = \{d_{(i,j)}\}_{(i,j)\in\mathcal{A}}$ is a priori fixed. Consider the following problem

$$\max_t \left[\sum_{(i,j)\in\mathcal{OD}} d_{(i,j)} T_{(i,j)}(t) - \sum_{\alpha\in\mathcal{A}} \bar{f}_\alpha \cdot (t_\alpha - \bar{t}_\alpha) : \quad t \geq \bar{t} \right]. \tag{20}$$

Clearly, that is a concave maximization problem.

Theorem 3 *1. The problem (20) is solvable if and only if the desired demand pattern d can be implemented by some stable equilibrium flows.*
2. Let the optimal set $\mathcal{T}^*$ of the problem (20) be non-empty. Then for any $t^* \in \mathcal{T}^*$ we have

$$t^* \geq \bar{t}.$$

3. For any $t^* \in \mathcal{T}^*$ there exists the arc flow vector

$$f^* = \sum_{(i,j)\in\mathcal{OD}} f^*_{(i,j)}, \quad 0 \leq f^* \leq \bar{f},$$

with $f^*_{(i,j)} \in d_{(i,j)} \partial T_{(i,j)}(t^*)$. The state $S = (f^*, t^*, n^*)$ with $n^* = D(t^*) f^*$ is a stable equilibrium in $\mathcal{R}$.
4. The loading pattern $N^* = \{N^*_{(i,j)}\}_{(i,j)\in\mathcal{OD}}$ is defined as

$$N^*_{(i,j)} = d_{(i,j)} T_{(i,j)}(t^*), \quad (i,j) \in \mathcal{OD}.$$

The total loading of the network is defined as follows:

$$L^* = \sum_{(i,j)\in\mathcal{OD}} d_{(i,j)} T_{(i,j)}(t^*) \leq \sum_{\alpha\in\mathcal{A}} \bar{f}_\alpha \cdot t^*_\alpha.$$

Proof:
Consider the following problem:

$$\max_{t,N} \left[\sum_{(i,j)\in\mathcal{OD}} N_{(i,j)} \left(\ln \frac{d_{(i,j)}T_{(i,j)}(t)}{N_{(i,j)}} + 1 \right) - \sum_{\alpha\in\mathcal{A}} \bar{f}_\alpha \cdot (t_\alpha - \bar{t}_\alpha) : \;\; t \geq \bar{t} \right]. \tag{21}$$

In view of Lemma 5 the objective function of this problem is jointly concave in N and t. At the same time, the part of the objective function in (21), dependent in t, is the same as in (13). Therefore, for any pair (N^*, t^*), optimal for (21), there exists a flow vector

$$f^* \in \partial_t \left(\sum_{(i,j)\in\mathcal{OD}} N^*_{(i,j)} \ln T_{(i,j)}(t) \right)_{t=t^*}, \tag{22}$$

which defines a stable equilibrium state in $\mathcal{R}$ with respect to the loading N^* (see Theorem 2). Note that the maximization in N in (21) can be done in a closed form:

$$N_{(i,j)} = d_{(i,j)} T_{(i,j)}(t). \tag{23}$$

This expression with $t = t^*$ gives us the statement of Item 4 of the theorem. Substituting (23) with $t = t^*$ in (22) we get the statement of Item 3. It remains to note that the problem (20) can be obtained from (21) using the relations (23). $\square$

Now we can compare the models (19) and (20). Indeed, the feasible sets in these models are the same. The only difference is in the second term of the objective function. Note that the most natural way to choose the coefficient γ_α in (18) is as follows:

$$\gamma_\alpha = \bar{f}_\alpha / \bar{t}_\alpha$$

(see [3]). However, in this case for $t_\alpha \geq \bar{t}_\alpha$ we have

$$\frac{1}{2\gamma_\alpha}[t_\alpha^2 - \bar{t}_\alpha^2] \geq \frac{1}{2\gamma_\alpha}[2\bar{t}_\alpha(t_\alpha - \bar{t}_\alpha)] = \bar{f}_\alpha(t_\alpha - \bar{t}_\alpha).$$

Thus, the objective function in (19) always underestimates the level of congestion as compared with (20). In the Beckmann model (19) the congestion is detected only when the value of the flow on the arc becomes greater than its upper physical bound.

It is easy to see that by playing with the coefficients $d_{(i,j)}$ and γ_α in (19) it is always possible to retrieve the solution of the problem (20). May be this explains the necessity of a sophisticated calibration technique in the majority of existing software packages based on the Beckmann formulation.

5 Computing the Stable Equilibrium

Let us show that the dual form of the problem (20) sometimes is more convenient for computations.

For the sake of simplicity let us introduce some structure for the demand flow vector d. Denote by $\mathcal{O}$ the set of source nodes in $\mathcal{N}$, $|\mathcal{N}| = p$. For each $i \in \mathcal{O}$ denote $d_i \in R^p$ the demand flow vector. In our old notation for $i \in \mathcal{O}$ we have

$$d_i^{(j)} = d_{(i,j)}, \ (i,j) \in \mathcal{OD}, \quad d_i^{(j)} = 0, \ (i,j) \notin \mathcal{OD}.$$

It is convenient to set

$$d_i^{(i)} = -\sum_{j \neq i} d_i^{(j)}. \tag{24}$$

Denote by E the incidence matrix of the network $\mathcal{R}$. Its column E_α corresponds to the arc α. If this arc goes from the node i to node j then $E_\alpha^{(i)} = -1$, $E_\alpha^{(j)} = 1$ and all other entries of that column are zeros.

Thus, in our new notation, the stable equilibrium in the network $\mathcal{R}$ with respect to the fixed demand flows $\{d_i\}_{i \in \mathcal{O}}$ can be found from the solution of the following maximization problem:

$$\max_t \left\{ \sum_{i \in \mathcal{O}} \sum_{j \neq i} d_i^{(j)} T_{(i,j)}(t) - \langle \bar{f}, t - \bar{t} \rangle : \ t \geq \bar{t} \right\}. \tag{25}$$

In order to rewrite this problem in a dual form we need the following evident result.

Lemma 6

$$\sum_{j \neq i} d_i^{(j)} T_{(i,j)}(t) = \min_h \{ \langle h, t \rangle : \ Eh = d_i, \ h \geq 0 \in R^m \}.$$

(Indeed, if in the above minimization problem d_i satisfies (24), then the structure of the optimal vector h^* corresponds to the shortest path tree.)

Now we can rewrite the problem (25) using the Duality Theory:

$$\begin{aligned} &\max_t \left\{ \sum_{i \in \mathcal{O}} \sum_{j \neq i} d_i^{(j)} T_{(i,j)}(t) - \langle \bar{f}, t - \bar{t} \rangle : \ t \geq \bar{t} \right\} \\ &= \max_{t \geq \bar{t}} \left\{ -\langle \bar{f}, t - \bar{t} \rangle + \sum_{i \in \mathcal{O}} \min_{h_i} \{ \langle h_i, t \rangle : \ Eh_i = d_i, h_i \geq 0 \} \right\} \\ &= \min_{h_i, i \in \mathcal{O}} \left\{ \left(\max_{t \geq \bar{t}} \{ -\langle \bar{f}, t - \bar{t} \rangle + \langle \sum_{i \in \mathcal{O}} h_i, t \rangle \} \right) : \ Eh_i = d_i, h_i \geq 0 \right\}. \end{aligned}$$

Thus, our initial problem (25) can be rewritten in the following dual form:

$$\begin{aligned} &\min_{f, h_i} \quad \langle f, \bar{t} \rangle, \\ &\text{s.t.} \quad \sum_{i \in \mathcal{O}} h_i = f \leq \bar{f}, \\ &\qquad Eh_i = d_i, \ i \in \mathcal{O}, \\ &\qquad h_i \geq 0, \ i \in \mathcal{O}. \end{aligned} \tag{26}$$

Note that this formulation belongs to a well known subclass of Linear Programming; that is the *minimal cost multi-commodity transportation problem with bounded arc capacities*. For problems of that type there exists many efficient numerical schemes. However, for us it is important now to clarify the relations between the solution of the problem (26) and the stable equilibrium in the network $\mathcal{R}$.

Indeed, the analytical form of the problem (26) is quite surprising. Its objective function depends only on the free traffic travel time. Therefore we could expect that its solution will be close to the solution of a shortest path problem. However, note that the problem (26) itself does not give us any information about the travel time and the arc loading. It indicates only the set of reasonable arcs, which will be used by the drivers. (These arcs are given by non-zero components of the optimal flow vectors h_i^*.) Thus, up to now we can make only the following conclusion:

> *At stable equilibrium the arc distribution of the flows corresponds to the minimal social distance in the network.*

Of course, this conclusion is true only if the free traffic travel time is computed as the length of the arc divided by the maximal speed allowed in the network. Another interpretation of the objective function in (26) is the free traffic loading of the network.

In order to find the equilibrium travel time pattern, we need to recall the origin of the problem (26). It was obtained as a problem dual to (25). Therefore, the travel time on the arcs can be found from the *dual multipliers* for the constraints

$$\sum_{i \in \mathcal{O}} h_i \leq \bar{f}. \tag{27}$$

Indeed, denote these multipliers by Δ, $\Delta \geq 0 \in R^m$. Then, we have

$$\min_{\substack{h_i \\ Eh_i = d_i,\ i \in \mathcal{O}, \\ h_i \geq 0,\ i \in \mathcal{O}}} \quad \max_{\Delta \geq 0} \quad \left[\langle \sum_{i \in \mathcal{O}} h_i, \bar{t} \rangle + \langle \sum_{i \in \mathcal{O}} h_i - \bar{f}, \Delta \rangle \right]$$

$$= \max_{\Delta \geq 0} \quad \min_{\substack{h_i \\ Eh_i = d_i,\ i \in \mathcal{O}, \\ h_i \geq 0,\ i \in \mathcal{O}}} \quad \left[\sum_{i \in \mathcal{O}} \langle h_i, \bar{t} + \Delta \rangle - \langle \bar{f}, \Delta \rangle \right]$$

Now, in view of Lemma 6 we come to the following problem:

$$\max_{\Delta \geq 0} \left[\sum_{i \in \mathcal{O}} \sum_{j \neq i} d_i^{(j)} T_{(i,j)}(\bar{t} + \Delta) - \langle \bar{f}, \Delta \rangle \right].$$

Thus, we have proved the following statement.

Theorem 4 *Let f^* be an optimal arc flow in the problem (26) and Δ^* be the optimal dual multipliers for the constraints (27). Then the state $S = (f^*, t^*, n^*)$ with*

$$t^* = \bar{t} + \Delta^*, \quad n^* = f^* \cdot t^*,$$

is a stable equilibrium in the network $\mathcal{R}$.

6 Discussion

In the previous sections we have studied the stable equilibrium in a network with sufficiently large arc volume $\bar{n}$ or, with relatively small demand OD-flow d. In fact, in an urban network such an assumption usually is valid during the day period of time. Indeed, in the most cities the traffic between the morning and evening peak periods is congested, but not too much. The volume of the roads is sufficient for all drivers, which wish to use them at the same moment of time. At that period, the congestion is shown up as a queue at a traffic light, for which the length of the green is not enough to disappear. If the length of these queues remains more or less stable, we can expect that in this network there exists a stable equilibrium. For such networks, our model allows to compute the corresponding origin-destination flows. Moreover, we can use this model to decrease the level of congestion.

In the peak period the situation is quite different. The number of drivers on the roads very often achieves their physical volume. Mathematically, this leads to the loss of the stability of the flows. It seems that the static environment and our main assumption on the stability of the flows is not applicable to such situations at all.

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