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Mixing Mixed-Integer Inequalities

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Abstract

Mixed-integer rounding (MIR) inequalities play a central role in the development of strong cutting planes for mixed-integer programs. In this paper, we investigate how known MIR inequalities can be combined in order to generate new strong valid inequalities.

Given a mixed-integer region S and a collection of valid "base" mixed-integer inequalities, we develop a procedure for generating new valid inequalities for S. The starting point of our procedure is to consider the MIR inequalities related with the base inequalities. For any subset of these MIR inequalities, we generate two new inequalities by combining or "mixing" them. We show that the new inequalities are strong in the sense that they fully describe the convex hull of a mixed-integer region associated with the base inequalities.

We also study some extensions of this mixing procedure, and discuss how it can be used to obtain new classes of strong valid inequalities for various mixed-integer programming problems. In particular, we present examples for production planning, capacitated facility location, capacitated network design, and multiple knapsack problems.

Keywords: mixed integer programming, mixed integer rounding, Gomory mixed integer cuts.

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1 Introduction.

In the design phase of a cutting-plane (or branch-and-cut) algorithm for solving a mixed-integer programming problem, $z = min\{cx \ : \ x \in S\}$, one very important step is to define classes of strong valid inequalities for the mixed-integer set S. In these algorithms, the linear programming relaxation of the problem is tightened by adding a selected subset of these inequalities, so-called cuts, to the formulation (see Nemhauser and Wolsey [16] for a general description of these algorithms). Among other critical factors, the success of the algorithm depends on the quality of the tightened formulation as an approximation of the convex hull of S.

One way to generate such inequalities is to use general purpose cutting planes which do not require or exploit any a priori knowledge about the structure of the problem at hand. The Gomory mixed-integer cuts (see Gomory [11]) which have been successfully used in a branch and cut framework (see Balas, Ceria, Cornuéjols and Natraj [4]), or the disjunctive cutting planes (see Balas [2]) revisited and implemented as the equivalent lift and project cuts (see Balas, Ceria, Cornuéjols [3]) fall into this category.

Another possibility is to generate special purpose cutting planes that are based on the polyhedral analysis of the problem formulation. Since it is difficult to capture the whole structure of the optimization problem in these inequalities, the polyhedral analysis is usually performed on simple mathematical structures that are embedded in the problem formulation, or possibly, on relaxations of the problem. This approach was first used to solve pure 0-1 programs by considering the constraints of the formulation separately and using facet inducing valid inequalities for the knapsack problems associated with each constraint (see Crowder, Johnson and Padberg [10]) .

One important example of this approach is the mixed-integer rounding (MIR) inequality (see Nemhauser and Wolsey [15], and (3) below), which is derived from a single mixed-integer constraint. The MIR inequality can also be conseidered as a generalization of Chvatal's integer rounding inequality to the mixed-integer case (see Chvátal [8]). Based on the MIR inequality, and motivated by the Gomory mixed-integer cuts, an MIR procedure is

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derived for generating valid inequalities for any mixed-integer program. By applying this procedure a finite number of times (see Nemhauser and Wolsey [15]), one can generate all facet inducing valid inequalities for any mixed 0-1 integer program. This result has been extended to general mixed-integer programs by using a different recursion based on the so-called split cuts (see Cook, Kannan and Schrijver [9]).

The MIR inequality also provides a unifying framework for some of the facet inducing inequalities for well-known mixed-integer programs. For example, the basic flow cover inequality (see Pagberg, Van Roy and Wolsey [17]), which is defined on a single flow conservation constraint with variable upper bounds on the flows, can be obtained as an MIR inequality from an appropriate relaxation of the mixed-integer set. Similarly, basic continuous cover inequalities for the 0-1 knapsack problem with a single continuous variable can be obtained as MIR inequalities (see Marchand and Wolsey [13]). For many other linear mixed-integer models, simple MIR inequalities have been derived to produce strong valid inequalities which have proven to be computationally very effective (see [6], [7], [14], [18] and [20]).

Our primary objective in this paper is to contribute to the development of (general) techniques that can be used to generate new classes of strong valid inequalities for mixed-integer programs. More specifically, we investigate how to obtain new classes of valid inequalities by using or extending some known classes of valid inequalities for a mixed-integer problem. One such technique is the well-known lifting procedure which takes valid inequalities for a polyhedral set , and extends them to valid inequalities for a superset in a higher dimensional space (see Wolsey [21], Zemel [22], Gu, Nemhauser and Savelsbergh [12]).

Given the central role of the MIR inequalities in the development of strong cutting planes for mixed-integer programs, we investigate how to generate new valid inequalities by combining known MIR inequalities.

We next formalize our approach and introduce the notation used throughout the paper. Given a mixed-integer region $S \subseteq R^{m_1} \times Z^{m_2}$ and a collection of valid inequalities

$$f^i(x) + Bg^i(x) \geq \pi^i \qquad i \in \mathcal{I} = \{1, ..., m\}, \tag{1}$$

where $B \in R^1_+$ and $\pi^i \in R^1$, our purpose is to generate new valid inequalities for S. We note that f^i and g^i can be non-linear and π^i can be negative. In particular, we concentrate on the case where the starting valid inequalities (1) satisfy $f^i(x) \geq 0$ and $g^i \in Z$ (not necessarily positive) for all $x \in S$. In other words :

$$S \subseteq S^R = \{ x \in R^{m_1} \times Z^{m_2} \; : \; \begin{array}{l} f^i(x) + Bg^i(x) \geq \pi^i \quad i \in \mathcal{I}, \\ f^i(x) \geq 0, \; g^i(x) \in Z \quad i \in \mathcal{I} \end{array} \} \tag{2}$$

We call the valid inequalities of the form (1) *"base"* inequalities, and the new inequalities we generate are valid for the associated mixed-integer set S^R. The starting point of our procedure is to consider the simple MIR inequality (3) associated with each base inequality. We refer the reader to Nemhauser and Wolsey [15] and [16] for a more general definition of MIR inequalities. The simple MIR inequalities are derived as follows. For $i \in \mathcal{I}$, let $\tau^i = \lceil \pi^i / B \rceil$ and $\gamma^i = \pi^i - (\tau^i - 1)B$ and note that $\tau^i \in Z^1$, and $B \geq \gamma^i > 0$. We first re-write (1) as:

$$\begin{aligned} f^i(x) &\geq \pi^i - Bg^i(x) \\ &= \gamma^i + B(\tau^i - g^i(x) - 1) \end{aligned}$$

and then, using $f^i(x) \geq 0$ for $x \in S$, we obtain the valid inequality

$$f^i(x) \geq \gamma^i(\tau^i - g^i(x)). \tag{3}$$

In the remainder of the text inequality (3) is called the MIR inequality associated with the base inequality (1). We next present a simple example to demonstrate how these inequalities work.

Example 1.1 *Let $\sqrt{x_1} + 5x_2 \geq 6.7$ be a base inequality where $x \in R \times Z$. If we define $f(x) = \sqrt{x_1}$ and $g(x) = x_2$, then $\tau = 2$, $\gamma = 1.7$ and the related MIR inequality is: $\sqrt{x_1} \geq 1.7(2 - x_2)$. In Figure 1, the shaded area corresponds to the set of points in R^2 that satisfy the base inequality but violate the MIR inequality.*

The paper is organized as follows. In the next section, we present a "mixing" procedure that, for any subset of $\mathcal{I}$, combines the related MIR inequalities and generates two new inequalities. The MIR inequalities can be considered as a special case of this procedure (when the subset contains only one inequality). In Section 3 we demonstrate the application of the mixing procedure to various mixed-integer programming problems.

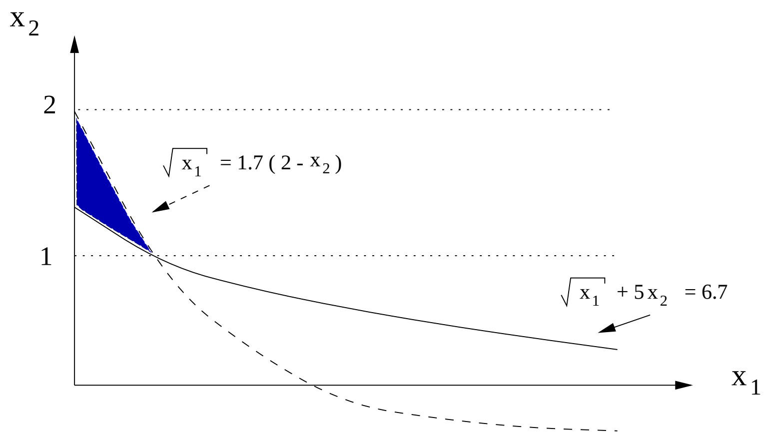

Figure 1: A non-linear MIR inequality.

In section 4 we analyse the strength of the mixing procedure. There we study the mixed-integer set

$$Q = \Big\{ y \in R_+ \times Z^m \ : \ y_0 + By_i \geq \pi^i \quad \text{for } i \in \{1, \ldots, m\} \Big\}$$

and show that our inequalities give a complete linear description of the convex hull of Q. We also study the separation problem associated to the new mixed MIR inequalities.

In Section 5 we propose ways to identify and modify the base inequalities if they do not exactly have the form presented above in (1) and (2). Lastly, in Section 6, we present an extension of the mixing procedure that produces stronger valid inequalities when the base inequalities have additional structure.

2 A Mixing Procedure.

Let S be a mixed-integer region and $\mathcal{I} = \{1, \ldots, m\}$ be the index set of the base inequalities as described above. Let I be a non-empty subset of $\mathcal{I}$. To simplify the notation, we assume that $I = \{1, \ldots, n\}$, and $\gamma^i \geq \gamma^{i-1}$ for all $n \geq i \geq 2$. Note that this can be done without loss of generality.

Given a real valued function $\bar{f}$ satisfying $\bar{f}(x) \geq f^i(x) \geq 0$ for all $x \in S$ and $i \in I$, it is possible to mix the right-hand-sides of the MIR inequalities and generate the following valid inequalities.

Theorem 2.1 *The following two inequalities*

$$\bar{f}(x) \geq \sum_{i=1}^{n} (\gamma^i - \gamma^{i-1})(\tau^i - g^i(x)) \tag{4}$$

and

$$\bar{f}(x) \geq \sum_{i=1}^{n} (\gamma^i - \gamma^{i-1})(\tau^i - g^i(x)) + (B - \gamma^n)(\tau^1 - g^1(x) - 1) \tag{5}$$

where $\gamma^0 = 0$*, are valid for* S*.*

Proof. For any fixed $\bar{x} \in S$, define $\beta = \max_{i \in I}\{\tau^i - g^i(\bar{x})\}$ and $v = \max\{i \in I : \beta = \tau^i - g^i(\bar{x})\}$. If $\beta \leq 0$ then the right-hand sides of (4) and (5) are at most zero and the inequalities are valid. Therefore we assume that $\beta = \tau^v - g^v(\bar{x}) \geq 1$.

Using $\beta \geq \tau^i - g^i(\bar{x})$ for all $i \leq v$ and $\beta \geq \tau^i - g^i(\bar{x}) + 1$ for $i > v$, we write

$$\begin{aligned}
\sum_{i=1}^{n} (\gamma^i - \gamma^{i-1})(\tau^i - g^i(\bar{x})) &\leq \sum_{i=1}^{v} (\gamma^i - \gamma^{i-1})(\beta) + \sum_{i=v+1}^{n} (\gamma^i - \gamma^{i-1})(\beta - 1) \\
&= (\gamma^v)(\beta) + (\gamma^n - \gamma^v)(\beta - 1) \\
&= \gamma^n(\beta - 1) + \gamma^v \\
&\leq B(\beta - 1) + \gamma^v \\
&\leq f^v(\bar{x}) \\
&\leq \bar{f}(\bar{x}).
\end{aligned}$$

Also note that the above argument is still valid when $v = n$. This proves the validity of (4). To show the validity of (5), we similarly write

$$\begin{aligned}
&\sum_{i=1}^{n} (\gamma^i - \gamma^{i-1})(\tau^i - g^i(\bar{x})) + (B - \gamma^n)(\tau^1 - g^1(\bar{x}) - 1) \\
&\leq \sum_{i=1}^{v} (\gamma^i - \gamma^{i-1})(\beta) + \sum_{i=v+1}^{n} (\gamma^i - \gamma^{i-1})(\beta - 1) + (B - \gamma^n)(\beta - 1) \\
&= (\gamma^v)(\beta) + (B - \gamma^v)(\beta - 1) \;=\; B(\beta - 1) + \gamma^v \\
&\leq \bar{f}(\bar{x}).
\end{aligned}$$

∎

Note that when $|I| = 1$, and $\bar{f} = f^1$, the related mixed inequality (4) gives the simple MIR inequality (3) and inequality (5) gives the original base inequality (1). We next give a small example to demonstrate how the mixing procedure works.

Example 2.2 *Let* $S = \{(x,y) \in R^1_+ \times Z^2 \ : (a) x_1 + 10y_1 \geq 3, (b) x_1 + 10y_2 \geq 5\}$ *and* $S_+ = S \cap \{(x,y) \in R^1_+ \times Z^2_+\}$. *If we write the MIR inequalities related with (a) and (b), we obtain:* $(a') : \ x_1 \geq 3(1-y_1)$ *and* $(b') : \ x_1 \geq 5(1-y_2)$ *and when we apply the mixing procedure with* $\bar{f} = f^1 = f^2 = x_1$, $B = 10$, *and* $g^i = y_i$, *the resulting inequality (4) is*

$$x_1 \geq 3(1-y_1) + 2(1-y_2). \qquad (ab')$$

A complete polyhedral description of S_+ *is given by* $conv(S_+) = \{(x,y) \in R^1_+ \times R^2_+ : (a'),\ (b'),\ (ab')\}$.

Similarly, the inequality (5) obtained by mixing the MIR inequalities (a') *and* (b') *is*

$$x_1 \geq 3(1-y_1) + 2(1-y_2) + 5(-y_1) \qquad (ab'')$$

and $conv(S) = \{(x,y) \in R^1_+ \times R^2 : (a),\ (b),\ (a'),\ (b'),\ (ab'),\ (ab'')\}$.

The strength of inequalities (4) and (5) clearly depends on how $\bar{f}$ is chosen. Note that $\bar{f}$ satisfies $\bar{f}(x) \geq f^*(x)$ for all $x \in S$, where $f^*(x) = \max_{i \in I}\{f^i(x)\}$. If it is practical, one should use $\bar{f} = f^*$ in (4) and (5), but this might be undesirable since in general f^* would not be a smooth function.

3 Examples.

In this section, we consider the application of the mixing procedure to some well known mixed-integer programming problems. We describe how to derive known classes of valid inequalities using the procedure. We also generate new valid inequalities for these problems.

In fixed charge capacitated network flow problems (such as lot-sizing, facility location and network design) a possible way to construct the base inequalities (1) is to aggregate flow balance constraints over a set of connected nodes in the underlying flow graph. This gives constraints of the form

$$\sum_{j \in N_1} x_j - \sum_{j \in N_2} x_j \quad = \quad d \ ,$$

which can be relaxed to

$$\sum_{j \in N_1} x_j \quad \geq \quad d \quad .$$

Now, partitioning N_1 into $F \cup G$ and using the variable upper bound constraints (or capacity constraints), $x_j \leq u_j y_j$ for $j \in G$, yields the base inequality

$$\sum_{j \in F} x_j + \sum_{j \in G} u_j y_j \quad \geq \quad d \quad .$$

This general principle for constructing base inequalities and the corresponding mixed inequalities (4) and (5) is now illustrated on several applications.

3.1 Production Planning.

The constant capacity single item lot-sizing problem (LCC) consists of a planning horizon n (i.e. a set of periods $\{1, \ldots, n\}$) with known demand d_t and production capacity C per period , $1 \leq t \leq n$. The objective is to find a production plan meeting the demands (without backlogging) and minimizing total production costs (unit production costs and setup costs) and inventory costs. Its feasible set X^{LCC} is defined by

$$X^{LCC} = \Big\{ (x, s, y) \in \quad R^n_+ \times R^{n+1}_+ \times B^n \quad : \\ \begin{array}{rcll} s_{t-1} + x_t & = & d_t + s_t & \text{for } 1 \leq t \leq n \\ x_t & \leq & C y_t & \text{for } 1 \leq t \leq n \\ s_0 & = & s_n = 0 & \end{array} \Big\}$$

where x_t, s_t, y_t represent respectively the production, inventory and binary setup variables in period t. Whenever there is a setup in period t (i.e. $y_t = 1$), the production capacity is C.

For a fixed time period $k \geq 1$, let $F \cup G$ be a partition of $N = \{k, \ldots, n\}$ such that $k \in G$. We can define a base inequality for each period $t \in N$ by aggregating the flow balance constraints over the periods k up to t. This gives

$$s_{k-1} + \sum_{j \in F \cap \{k, \ldots, t\}} x_j + \sum_{j \in G \cap \{k, \ldots, t\}} C y_j \;\geq\; \sum_{j=k}^{t} d_j \; .$$

It can be checked that by using the mixing procedure for any subset $I \subseteq \{t : t+1 \in G\} \cup \{n\}$ of these base inequalities, one gets, as inequalities of type (4), precisely the so-called (k, l, S, I) inequalities (see [18]) with the function $\bar{f}(x)$ defined as

$$\bar{f}(x) = s_{k-1} + \sum_{j \in F \cap (\cup_{t \in I}\{k,\dots,t\})} x_j \ .$$

Pochet and Wolsey ([18]) show that (k, l, S, I) inequalities generate all facets of $conv(X^{LCC})$ of the form

$$s_{k-1} + \sum_{j \in F} x_j + \sum_{j \in G} \alpha_j y_j \ \geq \ \pi_0$$

for all $2 \leq k \leq l \leq n$ and all partitions F, G of $\{k, \dots, l\}$. Moreover, it is known that adding only a subset of these inequalities (those with $F = \emptyset$, $G = \{k, \dots, l\}$) suffices to solve problem LCC by linear programming (without enumeration) when the objective function satisfies the Wagner-Whitin assumption (see Pochet and Wolsey [19]). The so-called (l, S) inequalities known to describe the convex hull of feasible solutions of the uncapacitated (i.e. $C = \infty$) single item lot sizing problem (see Barany, Van Roy and Wolsey [5]) can also be obtained by the mixing procedure.

3.2 Capacitated Facility Location.

The capacitated facility location problem consists of a set $M = \{1, \dots, m\}$ of potential depots with capacity C_j, $1 \leq j \leq m$, and a set $N = \{1, \dots, n\}$ of clients with demand d_k, $1 \leq k \leq n$. The objective is to decide which depots to open and to assign demands to open depots in order to minimize depot installation costs and distribution costs. Its feasible set X^{CFL} can be defined as

$$X^{CFL} = \Big\{ (v, y) \in \ R_+^{m \times n} \times B^m \ : \begin{array}{rcll} \sum_{j=1}^{m} v_{jk} & = & d_k & \text{for } 1 \leq k \leq n \\ \sum_{k=1}^{n} v_{jk} & \leq & C_j y_j & \text{for } 1 \leq j \leq m \\ v_{jk} & \leq & d_k y_j & \text{for } 1 \leq j \leq m \ , \ 1 \leq k \leq n \end{array} \Big\}$$

where the variable v_{jk} represents the amount of demand d_k delivered from depot j and the binary variable y_j indicates whether or not depot j is open
.

For any subset $K \subseteq N$ of clients, we can construct a flow cover relaxation X_K^{FC} of X^{CFL} as

$$X_K^{FC} = \Big\{(w,y) \in \quad R_+^m \times B^m \quad : \quad \sum_{j=1}^m w_j = \sum_{k \in K} d_k \qquad w_j \le C_j y_j \quad \text{for } 1 \le j \le m\Big\}$$

by defining the projected variable w_j as $w_j = \sum_{k \in K} v_{jk}$. When the capacities are equal (i.e. $C_j = C$ for all j), Padberg, Van Roy and Wolsey [17] prove that the convex hull of this relaxation X_K^{FC} is defined by the MIR inequalities (3) corresponding to the set of base inequalities

$$\sum_{j \in M \setminus S} w_j + C \sum_{j \in S} y_j \quad \ge \quad \sum_{k \in K} d_k \quad ,$$

for $S \subseteq M =$ with $|S| \ge \lceil \sum_{k \in K} d_k / C \rceil$.

Therefore, for $i = 1, \dots, r$, we can choose sets $S^i \subseteq M$, $K^i \subseteq N$ and mix the corresponding base inequalities. The (k, l, S, I) based inequalities defined in Aardal, Pochet and Wolsey [1] can be obtained by mixing these base inequalities, when $\{S^i, K^i\}$ form a "nested" family (i.e. $K^1 \subset K^2 \subset \dots \subset K^r$ and $S^1 \subset S^2 \subset \dots \subset S^r$). The corresponding mixed inequalities are typically strong (defining facets or high dimensional faces).

We next give an example of a facet defining mixed inequality, for the case of different capacities C_j, which does not correspond to such a nested family of base inequalities. This example inequality could also be derived as a lifted flow cover inequality by constructing an appropriate relaxation of X^{CFL} (see [17]).

Example 3.1 *Consider the following instance of X^{CFL} with $m = 5$ depots of different capacity $C = (4, 10, 10, 10, 50)$ and $n = 3$ clients whose demands are $d = (6, 9, 7)$.*
We define a first base inequality by $S = M \setminus \{5\}$ and $K = N$

$$(v_{51} + v_{52} + v_{53}) + 4y_1 + 10y_2 + 10y_3 + 10y_4 \ge 22 \quad .$$

¿From this base inequality, we define two other (relaxed) base inequalities by replacing the term $4y_1$ by its upper bound 4 in the first inequality, and by

$10y_1$ *in the second inequality:*

$$\begin{aligned} (v_{51}+v_{52}+v_{53})+10y_2+10y_3+10y_4 &\geq 18 \text{ , } \textit{and} \\ (v_{51}+v_{52}+v_{53})+10y_1+10y_2+10y_3+10y_4 &\geq 22 \text{ .} \end{aligned}$$

It can be checked that the mixed inequality (4) defined by

$$(v_{51}+v_{52}+v_{53}) \geq 2(3-y_1-y_2-y_3-y_4)+6(2-y_2-y_3-y_4)$$

defines a facet of this instance of $conv(X^{CFL})$.

3.3 Capacitated Network Design.

We next study the capacity expansion problem (CEP) that arises in telecommunication network design. The problem data consists of an undirected graph $G=(V,E)$ with existing capacities $k_e \geq 0$ for $e \in E$, and point-to-point traffic demands $\{t^{ij}\}$ to be routed through the network. Additional capacity has to be installed on edges in integer multiples of several modularities (corresponding to different technologies) in order to support the flow on the edges. The objective is to minimize the total routing and capacity installation cost. For simplicity, we consider the case when there is only one batch size (technology).

If we define $K=\{i \in V : \sum_{j \in V} t^{ij} > 0\}$, the set of feasible solutions X^{CEP} is defined by

$$\begin{aligned} X^{CEP} = \Big\{ (f,y) \in R_+^{2|K||E|} \times Z_+^{|E|} &: \\ \sum_{\{i,j\} \in E} f_{ji}^k - \sum_{\{i,j\} \in E} f_{ij}^k &= t^{ki} && \text{for } i \in V,\ k \in K \setminus \{i\} \\ \sum_{k \in K} f_{ij}^k &\leq Cy_e + k_e && \text{for } e=\{i,j\} \in E \\ \sum_{k \in K} f_{ji}^k &\leq Cy_e + k_e && \text{for } e=\{i,j\} \in E \Big\} \end{aligned}$$

where f_{ij}^k corresponds to the flow of commodity k (i.e. traffic with origin at node k) on the directed arc from i to j and y_e represents the number of batches of capacity C installed on edge $e=\{i,j\}$. The total flow on arc (i,j) cannot exceed the capacity Cy_e installed on edge e plus the existing capacity k_e. The first set of equalities in the above formulation simply represents the flow balance constraints.

In [6] Bienstock and Günlük present several classes of facet defining inequalities for $conv(X^{CEP})$. Most of their inequalities are based on the so-called cut-set and flow-cut-set inequalities. For any $S \subset V$, let $\delta(S) \in E$ be the set of edges with one end in S and the other in $V \setminus S$. It is easy to see that the cut-set inequality

$$\sum_{e \in \delta(S)} y_e \ \geq \ \left\lceil \frac{T(S)}{C} \right\rceil$$

where $T(S) = \max\{\sum_{i \in S} \sum_{j \in V \setminus S} t^{ij}, \sum_{i \in V \setminus S} \sum_{j \in S} t^{ij}\} - \sum_{e \in \delta(S)} k_e$, is valid for X^{CEP}.

It is also possible to construct base inequalities of the form (1) by partitioning $\delta(S)$ into $E_1 \cup E_2$ and selecting a subset $Q \subseteq (K \cap S)$. If we let A_1 denote the set of directed edges obtained by orienting the edges in E_1 away from S, we can write the following valid inequality:

$$\sum_{(i,j) \in A_1} \sum_{k \in Q} f_{ij}^k + C \sum_{e \in E_2} y_e \ \geq \ \sum_{k \in Q} \sum_{j \in V \setminus S} t^{kj} - \sum_{e \in E_2} k_e.$$

The flow-cut-set inequalities are precisely the MIR inequalities (3) constructed from this base inequality.

It is also possible to mix these base inequalities to obtain new valid inequalities. We next give an example of a fractional solution feasible to the formulation consisting of the linear programming relaxation of X^{CEP} augmented by all cut-set and flow-cut-set inequalities. This fractional point violates a mixed inequality of the form (4).

Example 3.2 *Consider the following example with two nodes (representing S and $V \setminus S$), and two commodities a and b with origin in S. The batch capacity is $C = 10$ and the total demand for a and b in $V \setminus S$ is 17 and 8, respectively. The directed cut from S to $V \setminus S$ contains three arcs denoted by $e = 1, 2$ and 3. The existing capacities are $k = (0, 10, 0)$. The fractional point given in Table 1 satisfies all cut-set and flow-cut-set inequalities.*

edges e	y_e	f_e^a	f_e^b
1	3/4	1/4	19/4
2	0	6/4	4/4
3	7/4	61/4	9/4

Table 1: A fractional solution

Consider the following two base inequalities with $E_2 = \{3\},\ Q = \{a\}$ *and* $E_2 = \{1,3\},\ Q = \{a,b\}$ *:*

$$\begin{array}{rcl} f_1^a + f_2^a + & 10y_3 & \geq 17 \ , \\ f_2^a + f_2^b + & 10y_1 + 10y_3 & \geq 25 \end{array}$$

and note that their corresponding MIR inequalities (3) are satisfied at equality :

$$\begin{array}{rcccccl} f_1^a + f_2^a & = & 7/4 & \geq & 7(2 - y_3) & = & 7/4 \ , \\ f_2^a + f_2^b & = & 10/4 & \geq & 5(3 - y_1 - y_3) & = & 5/2. \end{array}$$

If we mix these two inequalities, the resulting inequality

$$f_1^a + f_2^a + f_2^b \ = \ 11/4 \ \geq \ 5(3 - y_1 - y_3) + 2(2 - y_3) \ = \ 5/2 + 2/4 \ = \ 12/4$$

is violated by the fractional point.

3.4 Multiple Knapsack.

As a final example we illustrate how the mixed inequalities generate facets of pure integer problems, and introduce the extensions of the mixed inequalities that will be presented in Section 5.

Example 3.3 *Consider the following feasible set* X *which consists in the intersection of two integer knapsack constraints.*

$$X = \{x \in Z_+^5 : x_1 + 3x_2 + 10x_4 \geq 25 \ , \ \text{and} \ x_1 + 2x_3 + 10x_5 \geq 37\}.$$

If we define two base inequalities by $f^1(x) = x_1 + 3x_2$, $g^1(x) = x_4$, $\pi^1 = 25$, $f^2(x) = x_1 + 2x_3$, $g^2(x) = x_5$, $\pi^2 = 37$, *and* $B = 10$, *then the corresponding mixed inequalities, respectively of type (4) and (5)*

$$x_1 + 3x_2 + 2x_3 \ \geq \ 5(3 - x_4) + 2(4 - x_5) \qquad (6)$$

$$x_1 + 3x_2 + 2x_3 \ \geq \ 5(3 - x_4) + 2(4 - x_5) + 3(2 - x_4) \qquad (7)$$

generate facets of $conv(X)$.

If we modify the set X *into* X' *defined by*

$$X' = \{x \in Z_+^5 : x_1 + 3x_2 + 10x_4 \geq 25 \ , \ \text{and} \ x_1 + 2x_3 + 8x_5 \geq 31\}.$$

then the above mixed inequality (6) is still valid and generates a facet of $conv(X')$. *We will see in the next section that this inequality can be obtained by taking two different values for* B, *namely* $B^1 = 10$ *and* $B^2 = 8$.

4 The Strength of the Mixing Procedure

We analyse now the strength of the valid inequalities (4) and (5) generated by the mixing procedure. Our main result, Theorem 4.5 established in Subsection 4.2, shows that the mixing procedure suffices to obtain a complete linear description of the convex hull of the mixed-integer region Q defined by the following set of base inequalities:

$$Q = \Big\{ y \in R_+ \times Z^m \;:\; y_0 + By_i \geq \pi^i \quad \text{for } i \in \{1, \ldots, m\} \Big\}$$

In subsection 4.1, we start by establishing an intermediate result, namely Corollary 4.4, needed to prove our main result. We end this section with the study of the separation problem associated with the mixed MIR inequalities (4) and (5).

4.1 The Coefficient Polyhedron $\mathcal{P}^C$.

It is possible to generalize (4) and (5) and write valid inequalities of the form $\bar{f}(x) + \alpha \geq \sum_{i \in \mathcal{I}} \delta^i(\tau^i - g^i(x))$ when α and δ satisfy certain properties. Without loss of generality, we again write $\mathcal{I} = \{1, \ldots, m\}$ with $\gamma^i \geq \gamma^{i-1}$ for all $m \geq i \geq 2$.

Lemma 4.1 *The following inequality is valid for S*

$$\bar{f}(x) + \alpha \geq \sum_{i \in \mathcal{I}} \delta^i(\tau^i - g^i(x)) \tag{8}$$

provided:
(i) $(\delta, \alpha) \in \mathcal{P}^C$, where

$$\mathcal{P}^C = \Big\{ (\delta, \alpha) \in R_+^{|\mathcal{I}|+1} \;:\; \begin{array}{rcl} \sum_{i \in \mathcal{I}} \delta^i & \leq & B \\ \sum_{j \leq i} \delta^j & \leq & \alpha + \gamma^i, \text{ for all } i \in \mathcal{I} \end{array} \Big\}$$

(ii) $\bar{f}(x) \geq f^i(x)$ for all $x \in S$ and for all $i \in \mathcal{I}$ with $\delta^i > 0$.

Proof. Similar to the proof of Lemma 2.1. ∎

Note that the coefficients (δ, α) in the inequalities (4) or (5) belong to the coefficient polytope $\mathcal{P}^C$, and inequality (8) with $(\delta, \alpha) \in \mathcal{P}^C$ appears thus to be a generalization of (4) and (5). We will show that, although inequality (8) seems to be more general, any inequality of the form (8) with $(\delta, \alpha) \in \mathcal{P}^C$ is equivalent to or is dominated by an inequality (4) or (5).

Lemma 4.1 defines sufficient conditions on the coefficients (δ, α) for inequality (8) to be valid for S. The following remark states the properties of the mixed-integer region S under which these conditions are also necessary.

Remark 4.2 *Let $(\delta, \alpha) \in R_+^{|\mathcal{I}|+1}$ with $\sum_{i\in\mathcal{I}} \delta^i \leq B$. Furthermore, let $\bar{f}$ be a function satisfying $\bar{f}(x) \geq f^i(x)$ for all $x \in S$ and for all $i \in I = \{i \in \mathcal{I} : \delta^i > 0\}$.*

If for all $t \in I$ there exists a point $\tilde{x}_t \in S$ with:
(i) $\tau^t - g^t(\tilde{x}_t) = 1$ for $i = 1, ..., t$ and $\tau^t - g^t(\tilde{x}_t) = 0$ for $i = t+1, ..., m$, and,
(ii) $\bar{f}(\tilde{x}_t) = \gamma^t$,

then, (8) is valid for S if and only if $(\delta, \alpha) \in \mathcal{P}^C$.

In the above remark, note that if points $\tilde{x}_t$ exist for all $t \in I$, then the mixing procedure to obtain inequality (4) can also be viewed as a sequential lifting procedure, starting from a simple MIR inequality, and where functions $(\tau^i - g^i(x))$ are lifted sequentially instead of variables. Similarly, if there exists $y \in S$ with $\tau^1 - g^1(y) = 2$, $\tau^t - g^t(y) = 1$ for $t = 2, \cdots, m$, and $\bar{f}(y) = \gamma^1 + B$, then inequality (5) can be obtained from (4) by lifting the function $(\tau^1 - g^1(x) - 1)$.

We show now that inequalities (4) and (5) contain all important inequalities of the form (8) with $(\delta, \alpha) \in \mathcal{P}^C$. Let $p = (\delta, \alpha)$ be an arbitrary, non-extreme point of the coefficient polyhedron $\mathcal{P}^C$ and consider the associated valid inequality (8). Since p can be expressed as a convex combination of other points in $\mathcal{P}^C$, that is, $p = \epsilon p_1 + (1-\epsilon)p_2$ for some $p_1, p_2 \in \mathcal{P}^C$ and some $\epsilon \in]0, 1[$, the inequality generated by p can also be obtained by simply combining the inequalities associated with p_1 and p_2. We are therefore mainly interested in valid inequalities generated by the extreme points and extreme directions of $\mathcal{P}^C$, and next we present a complete characterization of the extreme points.

Lemma 4.3 *Let $p = (\delta, \alpha)$ be an arbitrary, non-zero extreme point of $\mathcal{P}^C$, and define $I = \{i \in \mathcal{I} : \delta^i > 0\} = \{i_1, i_2, \ldots, i_n\}$ with $i_1 < i_2 < \ldots < i_n$. The extreme point $p = (\delta, \alpha)$ is characterized by:*

$$\alpha \in \{0, B - \gamma^{i_n}\}$$

$$
\begin{aligned}
\delta^{i_1} &= \gamma^{i_1} + \alpha \\
\delta^{i_j} &= \gamma^{i_j} - \gamma^{i_{j-1}} \quad \textit{for } j = 2, \ldots, n \\
\delta^{i} &= 0 \quad \textit{for } i \in \mathcal{I} \setminus I
\end{aligned}
$$

Proof. Note that the only extreme point of $\mathcal{P}^C$ with $\delta = 0$ is the origin, and thus $|I| > 0$. First we show that if p is an extreme point of $\mathcal{P}^C$, then $\sum_{j \le i} \delta^j = \alpha + \gamma^i$ for all $i \in \{i_1, i_2, \ldots, i_{n-1}\}$. Assume $\sum_{j \le i_k} \delta^j < \alpha + \gamma^{i_k}$ for some $1 \le k < n$. In this case, p can be expressed as a convex combination of two other points in $\mathcal{P}^C$. The first point p^1 is obtained from p by simultaneously increasing γ^{i_k} and decreasing $\gamma^{i_{k+1}}$ by a small $\epsilon > 0$. Similarly, p^2 is obtained from p by simultaneously decreasing γ^{i_k} and increasing $\gamma^{i_{k+1}}$ by the same ϵ. Clearly $p^1, p^2 \in \mathcal{P}^C$, and $p = 1/2p^1 + 1/2p^2$.

Next, consider i_n and assume $\sum_{j \le i_n} \delta^j < \alpha + \gamma^{i_n}$. If $\sum_{i \in \mathcal{I}} \delta^i < B$, then two points $p^1, p^2 \in \mathcal{P}^C$ can be obtained by simply increasing and decreasing δ^i_n by a small $\epsilon > 0$ and $p = 1/2\ (p^1 + p^2)$. On the other hand, if $\sum_{i \in \mathcal{I}} \delta^i = B$, then we have $\sum_{i \in \mathcal{I}} \delta^i = B < \alpha + \gamma^{i_n}$ and thus $\alpha > 0$. In this case, we obtain $p^1 \in \mathcal{P}^C$ by simultaneously increasing δ^i_1 and α, and decreasing δ^i_n by a small $\epsilon > 0$. The second point $p^2 \in \mathcal{P}^C$ is obtained by decreasing δ^i_1 and α, and increasing δ^i_n by the same $\epsilon > 0$, and $p = 1/2p^1 + 1/2p^2$. (If $|I| = 1$, we construct p^1 and p^2 by perturbing α only.)

We therefore established that $\sum_{j \le i} \delta^j = \alpha + \gamma^i$ for all $i \in I$ which, in turn, implies that $\alpha = \sum_{i \in \mathcal{I}} \delta^i - \gamma^{i_n}$ and thus $\alpha \le B - \gamma^{i_n}$. Finally, we need to show that α is either zero or $B - \gamma^{i_n}$. Assume $0 < \alpha < B - \gamma^{i_n}$, and therefore $\sum_{i \in \mathcal{I}} \delta^i < B$. We construct the following two points: p^1 is obtained from p by increasing α and γ^{i_1} by $\epsilon > 0$, and p^2 is obtained by decreasing α and γ^{i_1} by the same ϵ. Clearly $p^1, p^2 \in \mathcal{P}^C$, and $p = 1/2\ (p^1 + p^2)$. ∎

Although inequalities of the form (8) appear to be more general than the valid inequalities generated by the mixing procedure, any inequality generated by an extreme point of $\mathcal{P}^C$ coincides with one of (4) or (5). More precisely, for an inequality (8) associated with an extreme point of $\mathcal{P}^C$, we construct I by choosing all $i \in \mathcal{I}$ with $\delta^i > 0$. If α is zero, then the inequality coincides with (4), and if α is positive, it coincides with (5). Since the only extreme direction of $\mathcal{P}^C$ is $(\delta, \alpha) = (0, 1)$, we have the following corollary:

Corollary 4.4 *Inequality (8) generated by $(\delta, \alpha) \in \mathcal{P}^C$ is equivalent to or dominated by a positive combination of inequalities (4) and (5).*

4.2 A Related Mixed-integer Set.

Given a function $\bar{f}$, let $I = \{i \in \mathcal{I} \ : \ \bar{f}(x) \geq f^i(x) \text{ for all } x \in S\}$. The inequalities produced by the mixing procedure are actually valid for the following continuous relaxation of the mixed-integer set S:

$$S^{R(I)} = \Big\{x \in R^{m_1+m_2} \ : \quad \begin{array}{rcll} \bar{f}(x) & \geq & 0 & \\ g^i(x) & \in & Z & \text{for } i \in I \\ \bar{f}(x) + Bg^i(x) & \geq & \pi^i & \text{for } i \in I \end{array} \Big\}.$$

Furthermore, since we are not considering the interdependencies among functions g^i, $i \in I$, or $\bar{f}$, we are essentially producing valid inequalities for the following mixed-integer set:

$$Q^I = \Big\{y \in R_+ \times Z^{|I|} \ : \ y_0 + By_i \geq \pi^i \quad \text{for } i \in I \Big\}.$$

Note that if an inequality $\beta y \geq \theta$ is valid for Q^I, then

$$\beta_0 \bar{f}(x) + \sum_{i \in I} \beta_i g^i(x) \geq \theta$$

is valid for $S^{R(I)}$ and thus also valid for S.

We next show that inequalities (4) and (5), or equivalently inequalities of the form (8), are sufficient to obtain a complete polyhedral description of $Q \equiv Q^{\mathcal{I}}$. To study Q, we use the same notation as before. For $i \in \mathcal{I} = \{1, \ldots, m\}$, we define $\tau^i = \lceil \pi^i / B \rceil$ and $\gamma^i = \pi^i - (\tau^i - 1)B$ with $\gamma^i \geq \gamma^{i-1}$. We use $\{i_1, i_2, \ldots, i_{|I|}\}$ to denote the members of any subset $I \subseteq \mathcal{I}$ and assume $i_1 < i_2 < \ldots < i_{|I|}$. We also define $\gamma^{i_0} = 0$.

Theorem 4.5

$$\begin{aligned} conv(Q) \ = \ \Big\{ & y \in R_+ \times R^{|\mathcal{I}|} : \\ & y_0 \geq \sum_{j=1}^{|I|} (\gamma^{i_j} - \gamma^{i_{j-1}})(\tau^{i_j} - y_{i_j}) \qquad \text{for all } I \subseteq \mathcal{I}, \\ & y_0 \geq \sum_{j=1}^{|I|} (\gamma^{i_j} - \gamma^{i_{j-1}})(\tau^{i_j} - y_{i_j}) + (B - \gamma^{i_{|I|}})(\tau^{i_1} - y_{i_1} - 1) \qquad \text{for all } I \subseteq \mathcal{I} \Big\} \end{aligned}$$

Proof. Let $\beta y \geq \theta$ be an arbitrary valid inequality for Q with $\beta \neq 0$. Using Corollary 4.4, it suffices to show that the inequality can be re-written as:

$$y_0 + \alpha \geq \sum_{i \in \mathcal{I}} \delta^i (\tau^i - y_i) \tag{9}$$

for some $(\delta, \alpha) \in \mathcal{P}^C$.

Consider a valid inequality $\beta y \geq \theta$ and note that for any $y \in Q$, and $i \in \{0, 1, \ldots, m\}$, one can obtain a new point $\bar{y} \in Q$ by increasing the i'th component of y by one. This implies that $\beta \geq 0$. Also note that $p^\mu = (\mu B, \tau_1 - \mu, \tau_2 - \mu, \ldots, \tau_m - \mu) \in Q$ for any non-negative integer μ and if $\beta_0 = 0$, then for some $\mu \in Z_+$, p^μ violates the inequality. Therefore, we established that any inequality $\beta y \geq \theta$ can be re-written as (9) and $\delta^i \geq 0$ for all $i \in \mathcal{I}$. Finally, $p^0 \in Q$ implies that $\alpha \geq 0$.

Now we need to show that (i) $\sum_{i \in \mathcal{I}} \delta^i \leq B$ and (ii) $\sum_{j \leq i} \delta^j \leq \alpha + \gamma^i$, for all $i \in \mathcal{I}$. First note that $p^\mu \in Q$ for all $\mu \in Z_+$ implies that $\mu B + \alpha \geq \sum_{i \in \mathcal{I}} \delta^i \mu$ for all $\mu \in Z_+$, and thus $\sum_{i \in \mathcal{I}} \delta^i$ cannot exceed B. Finally, for $i \in \mathcal{I}$, let $q^i \in R^{|\mathcal{I}|+1}$ be such that $q^i_0 = \gamma^i$ and $q^i_k = \tau^i - 1$ for all $1 \leq k \leq i$ and $q^i_k = \tau^i$ for all $i < k \leq m$. Since $q^i \in Q$ for $i \in \mathcal{I}$, the inequality has to satisfy (ii) for all $i \in \mathcal{I}$ and the proof is complete. ∎

If we define Q^+ to be the intersection of Q with the non-negative orthant:

$$Q^+ = \left\{ y \in R_+ \times Z_+^{|\mathcal{I}|} \; : \; y_0 + By_i \geq \pi^i \quad \text{for } i \in \mathcal{I} \right\},$$

then it is possible to show that the mixed inequalities (4) and (5) are still sufficient for a complete linear decription of the convex hull of Q^+. In other words:

$$conv(Q^+) = conv(Q) \cap R_+^{|\mathcal{I}|+1}.$$

This requires minor modifications in the proof of Lemma 4.5. In this case, inequality (5) becomes redundant unless the set $I \subseteq \mathcal{I}$ has $y^{i_1} < \tau^{i_1} - 1$ for some $y \in Q^+$ (see Example 2.2). We also note that an alternative proof of Theorem 4.5 can be adapted from the study of the capacitated lot-sizing polyhedron in [19].

4.3 Finding Violated Inequalities.

Given a point $\bar{x} \in X$, we next address the (separation) problem of finding a $(\hat{\delta}, \hat{\alpha}) \in \mathcal{P}^C$ that maximizes the right-hand-side of:

$$\bar{f}(\bar{x}) \geq \sum_{i \in \mathcal{I}} \hat{\delta}^i (\tau^i - g^i(\bar{x})) - \hat{\alpha}. \tag{10}$$

Let $h^i(x) = (\tau^i - g^i(x))$ and $(\hat{\delta}, \hat{\alpha})$ be such that it has the minimum number of non-zero components. Due to the structure of the extreme points of $\mathcal{P}^C$, it is easy to see that $\hat{\alpha}$ is non-zero if and only if $max_{j \in \mathcal{I}} \{h^j(\bar{x})\} > 1$.

Similarly, for all $i \in \mathcal{I}$, $\hat{\delta}^i$ is non-zero if and only if (i) $h^i(\bar{x}) > h^j(\bar{x})$ for all $j > i$, (ii) $h^i(\bar{x}) > max_{j\in\mathcal{I}}[h^j(\bar{x}) - 1]$, and (iii) $h^i(\bar{x}) > 0$.

Using the above observations and the correspondence between extreme points of $\mathcal{P}^C$ and subsets of $\mathcal{I}$ given in Lemma 4.3, we next present an algorithm that constructs a set $I \subseteq \mathcal{I}$ and fixes the value of $\hat{\alpha}$. The values $\hat{\delta}^i$ for $i \in I$ are then fixed according to Lemma 4.3.

1. Order the base inequalities in non-increasing order of γ^i and set $h^{max} = max_{j\in\mathcal{I}}\{h^j(\bar{x})\}$. If $h^{max} \leq 0$, then $(\hat{\delta}, \hat{\alpha}) = 0$, stop.
2. Set $best = \max\{0, h^{max} - 1\}$, $I = \emptyset$ and start from the top of the list,
3. Scan the list until $h^i(\bar{x}) > best$ or the list is exhausted.
4. If the list is not finished, set $I = I \cup \{i\}$ and $best = h^i(\bar{x})$, go to step 3.
   If the list is finished, goto step 5.
5. If $h^{max} > 1$, fix $\hat{\alpha}$ to $B - max_{j\in I}\{\gamma^j\}$.

   Fix it to zero otherwise. Return the set I and $\hat{\alpha}$

We also note that finding a most violated inequality of the form (8) (or (4)-(5)) is more complicated than simply maximizing the right-hand-side of (10). This is mainly because the choice of the set I may in turn affect the choice of $\bar{f}$ and thus the value of $\bar{f}(\bar{x})$. If $\bar{f}(\bar{x})$ does not depend on the set I, which is the case for the mixed integer set Q, then the above procedure gives an exact separation algorithm. Otherwise, it should be considered as a heuristic. We next give an example to emphasize this point.

Example 4.6 *Let* $P' = \{x \in R^2_+, y \in Z^2_+ : (1)\, x_1 + 10y_1 \geq 3,\ (2)\, x_1 + 10y_2 \geq 5,\ (3)\ x_1 + x_2 + 10y_3 \geq 9\}$ *and* $\bar{p} = (\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{y}_3) = (3, 9, 0, 0.4, 0.5)$. *If we apply the above procedure with* $g^i = y_i$, *we obtain* $I = \{1, 2, 3\}$, $\alpha = 0$, *and the resulting inequality of type (4) is:*

$$\bar{f}(p) \geq 3(1 - y_1) + 2(1 - y_2) + 4(1 - y_3). \qquad (c)$$

Since we are mixing all three inequalities, $\bar{f}$ *is (at least)* $x_1 + x_2$ *and thus* $\bar{p}$ *satisfies* (c). *However* $\bar{p}$ *does not satisfy* (ab') *of Example 2.2 and thus* $\bar{p} \notin conv(P')$.

5 Variations.

In this section we show how the mixing procedure can be applied when some of the base inequalities do not have the form (1) or do not satisfy (2).

First we consider the case when $f^i(x) \geq 0$ does not hold for a base inequality $i \in \mathcal{I}$. In this case the MIR inequality (3) is not valid. If $f^i(x)$ can be bounded from below, i.e. if we have $f^i(x) \geq LB^i$, then we rewrite the related inequality as follows:

$$\left(f^i(x) - LB^i\right) + Bg^i(x) \geq \pi^i - LB^i \tag{11}$$

so that $\hat{f}^i(x) = f^i(x) - LB^i \geq 0$. It is now possible to apply the mixing procedure, with $\hat{f}^i(x)$ and by computing $\hat{\tau}^i$ and $\hat{\gamma^i}$ using the new right-hand-side $\pi^i - LB^i$, to obtain valid inequalities of type (4) and (5).

Another possibility is to have base inequalities of the form,

$$f^i(x) + B^i g^i(x) \geq \pi^i$$

where B^is are not necessarily the same for all $i \in I$. In this case, we define $\tau^i = \lceil \pi^i / B^i \rceil$ and $\gamma^i = \pi^i - (\tau^i - 1)B^i$, and check if $\bar{B} = \min_{i \in I}\{B^i\} \geq \bar{\gamma} = \max_{i \in I}\{\gamma^i\}$ holds. If $\bar{B} \geq \bar{\gamma}$, then we can apply the mixing procedure and the resulting inequality (4) is valid for S (the proof is similar to that of Lemma 2.1.) On the other hand if $\bar{B} < \bar{\gamma}$, it is possible to relax the base inequalities with small B^i's by replacing B^i with $\bar{\gamma}$ and then apply the mixing procedure to obtain a valid inequality of type (4). Another possible approach is to "scale" the base inequalities so as to increase $\bar{B}$ or to decrease $\bar{\gamma}$.

Let $\alpha^i > 0$ be the scaling coefficients for $i \in I$, and consider the scaled base inequality of type (1)

$$\alpha^i f^i(x) + (\alpha^i B^i) g^i(x) \geq \alpha^i \pi^i \tag{12}$$

and the related MIR inequality,

$$\alpha^i f^i(x) \geq \hat{\gamma}^i(\hat{\tau}^i - g^i(x)) = \alpha^i \gamma^i (\tau^i - g^i(x))$$

where $\hat{\tau}^i = \lceil \alpha^i \pi^i / \alpha^i B^i \rceil = \lceil \pi^i / B^i \rceil = \tau^i$ and $\hat{\gamma}^i = \alpha^i \pi^i - (\hat{\tau}^i - 1)\alpha^i B^i = \alpha^i(\pi^i - (\tau^i - 1)B^i) = \alpha^i \gamma^i$. Using scaled inequalities (12) as the base set, we can apply the mixing procedure if $\min_{i \in I}\{\alpha^i B^i\} \geq \max_{i \in I}\{\alpha^i \gamma^i\}$ holds, and obtain new inequalities of type (4).

We note that this idea can also be helpful if one can use the same $\bar{f}$ after scaling some inequalities up, or if one can find a better $\bar{f}$ after scaling some inequalities down. This is illustrated in the next example.

Example 5.1 *Consider mixing the following inequalities where $x_1, x_2 \in R^1_+$ and $x_3, x_4 \in Z_1$:*

$$\begin{aligned} x_1 + 2x_2 + 7x_3 &\geq 15 \\ 3x_1 + 5x_2 + 10x_4 &\geq 14 \end{aligned}$$

and note that if we set $g^1(x) = x_3$ and $g^2(x) = x_4$, then $\tau_1 = 3$, $\tau_2 = 2$, $\gamma_1 = 1$, and $\gamma_2 = 4$. Since $\bar{B} \geq \bar{\gamma}$, we can apply the mixing procedure with $\bar{f}(x) = 3x_1 + 5x_2$ to obtain,

$$3x_1 + 5x_2 \geq (3 - x_3) + 3(2 - x_4). \tag{13}$$

It is also possible to apply the mixing procedure after scaling the first inequality by $5/2$ and obtain

$$3x_1 + 5x_2 \geq (5/2)(3 - x_3) + (3/2)(2 - x_4).$$

which is stronger than (13) when $3 - x_3 > 2 - x_4$.

6 Mixing Independent Constraints.

In the examples presented in Section 3, the functions f^i were always "dependent" with respect to $\bar{f}$ in the sense that, due to their common terms, we did not have $\bar{f}(x) \geq f^i(x) + f^j(x)$ for two different $i, j \in I$. However, when this occurs, inequalities assocated with i and j can be treated as "independent" base inequalities and the mixed inequality (4) can be tightened to give a stronger inequality. Before formalizing this idea, we first present an example.

Example 6.1 *Consider a 5 depots, 3 clients instance of the capacitated facility location problem (CFL) with demands $d = (9, 2, 15)$ and capacity $C = 10$. We take the three base inequalities*

$$\begin{aligned} v_{21} + v_{31} + v_{41} + v_{51} + 10y_1 &\geq 9 \quad (S = K = \{1\}) \\ v_{13} + v_{23} + v_{53} + 10y_3 + 10y_4 &\geq 15 \quad (S = \{3,4\}, K = \{3\}) \\ v_{51} + v_{52} + v_{53} + 10y_1 + 10y_2 + 10y_3 + 10y_4 &\geq 26 \quad (S = \{1,2,3,4\}, K = \{1,2,3\}) \end{aligned}$$

Combining these base inequalities using the mixing procedure of section 2, we obtain

$$\begin{aligned} \bar{f}(x) &= v_{13} + (v_{21} + v_{23}) + v_{31} + v_{41} + (v_{51} + v_{52} + v_{53}) \\ &\geq 5(2 - y_3 - y_4) + 1(3 - y_1 - y_2 - y_3 - y_4) + 3(1 - y_1). \end{aligned}$$

Now, as $\bar{f}(x) \geq f^1(x) + f^2(x) = v_{13} + (v_{21} + v_{23}) + v_{31} + v_{41} + (v_{51} + v_{53})$, *the first two base inequalities can be treated as independent, and the following inequality*

$$\bar{f}(x) \geq f^1(x) + f^2(x) \geq 5(2 - y_3 - y_4) + 9(1 - y_1)$$

is valid. Since the third base inequality is not independent of the other two (i.e. $\bar{f}(x) \ngeq f^1(x) + f^3(x)$ *and* $\bar{f}(x) \ngeq f^2(x) + f^3(x)$*) , when introducing this third inequality into the mixed inequality we use the basic mixing procedure. This gives the following inequality*

$$\begin{aligned} \bar{f}(x) &\geq 5(2 - y_3 - y_4) + (6 - 5)(3 - y_1 - y_2 - y_3 - y_4) + (9 - (6 - 5))(1 - y_1) \\ &= 5(2 - y_3 - y_4) + 1(3 - y_1 - y_2 - y_3 - y_4) + 8(1 - y_1). \end{aligned}$$

This inequality is not only valid but also facet inducing for this instance of $conv(X^{CFL})$*.*

We next formalize this idea and describe how to generate stronger mixed inequalities when the base inequalities can be partitioned into several "independent" groups with respect to a given function $\bar{f}$. Assume that the base inequalities are partitioned into $n + 1$ groups:

$$f_j^i(x) + Bg_j^i(x) \geq \pi_j^i \qquad i \in I(j)$$

for $j = 0, 1, \cdots, n$ with $n \geq 2$. Define f_j^* such that $f_j^*(x) = \max_{i \in I(j)}\{f_j^i(x)\}$ for $j = 0, 1, \cdots, n$ and all $x \in S$, and choose a function $\bar{f}(x)$ satisfying:

$$\begin{aligned} &(i) \quad \bar{f}(x) \geq f_0^*(x) \text{ for all } x \in S, \text{ and,} \\ &(ii) \quad \bar{f}(x) \geq \sum_{j=1}^{n} f_j^*(x) \text{ for all } x \in S. \end{aligned}$$

Condition (ii) is imposed on $\bar{f}(x)$ to represent the "independence" of the base inequalities belonging to different groups $I(j)$, $j = 1, \cdots, n$ (see the first two base inequalities in example 6.1). Having $\bar{f}(x) \geq \sum_{j=1}^{n} f_j^*(x)$ instead of $\bar{f}(x) \geq \max_{j=1}^{n} f_j^*(x)$, for all $x \in S$, implies that the polytope of feasible coefficients for inequality (8) is bigger than the one defined in Lemma 4.1, and thus new valid inequalities can be defined.

For all $0 \leq j \leq n$ and $i \in I(j)$ we define τ_j^i and γ_j^i as in Section 2, and without loss of generality assume that the base inequalities are ordered in such a way that $\gamma_j^1 \leq \gamma_j^2 \leq \ldots \leq \gamma_j^{|I(j)|}$ for all $j = 0, 1, \cdots, n$.

Let U be an ordering of all the base constraints so that $U(i,j)$ denotes the position of constraint i of $I(j)$. Furthermore assume that for all $i^1, i^2 \in I(j)$, if $i^1 < i^2$, then $U(i^1,j) < U(i^2,j)$. We define the coefficient polytope $\mathrm{P}^\delta(U)$ relative to a given partition of the base inequalities and a global ordering U as follows:

$$\begin{aligned}
\mathrm{P}^\delta(U) = \Big\{ \ \delta \in \mathcal{R}^{\sum_{j=0}^{n} |I(j)|} \ &: \\
\sum_{U(i,j)\,\leq\, U(k,j)} \delta^i_j + \sum_{U(i,0)\,\leq\, U(k,j)} \delta^i_0 \ &\leq \ \gamma^k_j \qquad k \in I(j) \quad j = 1, \cdots, n. \qquad (14) \\
\sum_{j=0}^{n} \sum_{U(i,j)\,\leq\, U(k,0)} \delta^i_j \ &\leq \ \gamma^k_0 \qquad k \in I(0) \qquad (15) \\
\delta^i_j \ &\geq \ 0 \quad \Big\}
\end{aligned}$$

Lemma 6.2 *Given an ordering U of the base inequalities, and any $\delta \in P^\delta(U)$. The inequality*

$$\bar{f}(x) \geq \sum_{j=0}^{n} \sum_{i \in I(j)} \delta^i_j(\tau^i_j - g^i_j(x)) \qquad (16)$$

is valid for S.

Proof. See Appendix 1.

For a fixed ordering U there is a one-to-one correspondence between extreme points of $\mathrm{P}^\delta(U)$ and subsets of the base inequalities that is analogous to the correspondence between extreme points of $\mathcal{P}^C$ with $\alpha = 0$ and subsets of $\mathcal{I}$ given by Lemma 4.3 . We also note that the global ordering U basically corresponds to the lifting sequence of these subsets of base inequalities (or functions $(\tau^i_j - g^i_j(x))$). Different subsets of base inequalities or different orderings U may result in different inequalities (16).

We give now an example of a facet defining valid inequality (16) to illustrate the notation used.

Example 6.3 *Consider the instance of a single commodity capacitated network flow problem defined by*

$$X^{CF} = \Big\{ (f,y) \in \; R_+^{11} \times B^{11} \; : \; \begin{array}{rcl} f_1 + f_2 & = & 9 + f_3 \\ f_3 + f_4 & = & 3 + f_5 \\ f_6 + f_7 & = & 3 + f_8 \\ f_8 + f_9 & = & 4 + f_{10} \\ f_5 + f_{10} + f_{11} & = & 7 \\ f_i & \leq & 10 y_i \qquad \text{for} \;\; 1 \leq i \leq 11 \end{array} \Big\}$$

where the f_i variables represent the flow variables and the y_i variables represent the setup for the flow variables. This problem corresponds to an instance of a network flow problem defined on the in-tree shown in Figure 2.

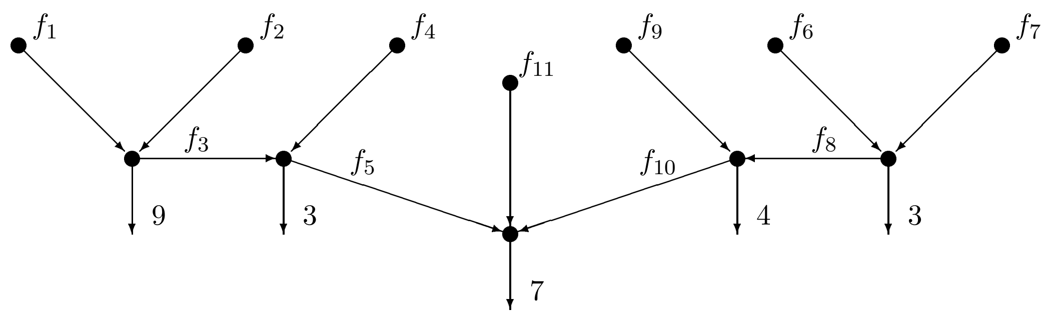

Figure 2: The in-tree of Example 6.3

Each equation in the definition of X^{CF} corresponds to the flow conservation constraint at one node of the tree. Variables f_3, f_5, f_8 and f_{10} represent the flow between the nodes of the in-tree. The other flow variables f_i, $i \neq 3, 5, 8, 10$, represent inflows at the nodes of the in-tree.

We follow now the construction of inequality (16) by giving first three sets of base inequalities, then the ordering of the base inequalities and the associated polyhedron $P^{\delta}(U)$.

The base inequalities constructed using the general aggregation principle presented in section 3 are defined as (with $B = 10$) :

$I(1):$	$f_1^1(x) = f_1$	$g_1^1(x) = y_2 + y_4$	$\pi_1^1 = 12$	$\tau_1^1 = 2$	$\gamma_1^1 = 2$
$I(1):$	$f_1^2(x) = f_1$	$g_1^2(x) = y_2$	$\pi_1^2 = 9$	$\tau_1^2 = 1$	$\gamma_1^2 = 9$
$I(2):$	$f_2^1(x) = f_6$	$g_2^1(x) = y_7$	$\pi_2^1 = 3$	$\tau_2^1 = 1$	$\gamma_2^1 = 3$
$I(2):$	$f_2^2(x) = f_6$	$g_2^2(x) = y_7 + y_9$	$\pi_2^2 = 7$	$\tau_2^2 = 1$	$\gamma_2^2 = 7$
$I(0):$	$f_0^1(x) = f_1 + f_6$	$g_0^1(x) = y_2 + y_4 + y_7 + y_9 + y_{11}$	$\pi_0^1 = 26$	$\tau_0^1 = 3$	$\gamma_0^1 = 6$

The groups of base inequalities $I(1)$ and $I(2)$ are "independent" because they are associated to the two independent branches of the in-tree. These base inequalities give the following standard MIR inequalities (3) to be combined

$$
\begin{array}{llcl}
f_1 & & \geq & 2(2-y_2-y_4) \\
f_1 & & \geq & 9(1-y_2) \\
 & f_6 & \geq & 3(1-y_7) \\
 & f_6 & \geq & 7(1-y_7-y_9) \\
f_1 + & f_6 & \geq & 6(3-y_2-y_4-y_7-y_9-y_{11})
\end{array}
$$

with $f_1^*(x) = f_1,\ f_2^*(x) = f_6,\ f_0^*(x) = f_1 + f_6 = \bar{f}(x)$.

In order to combine these MIR inequalities we define the ordering $U(i,j)$ of the different terms by

$$U(1,1) \ < \ U(1,2) \ < \ U(1,0) \ < \ U(2,2) \ < \ U(2,1).$$

This ordering defines the polyhedron $P^\delta(U)$ as

$$
P^\delta(U) = \left\{ \delta \in R_+^5 \quad : \begin{array}{rcl}
\delta_1^1 & \leq & \gamma_1^1 = 2 \\
\delta_2^1 & \leq & \gamma_2^1 = 3 \\
\delta_1^1 + \delta_2^1 + \delta_0^1 & \leq & \gamma_0^1 = 6 \\
\delta_2^1 + \delta_2^2 + \delta_0^1 & \leq & \gamma_2^2 = 7 \\
\delta_1^1 + \delta_1^2 + \delta_0^1 & \leq & \gamma_1^2 = 9
\end{array} \right\}
$$

If we take the following extreme point of $P^\delta(U)$

$$\delta_1^1 = 2\ ,\ \delta_2^1 = 3\ ,\ \delta_0^1 = 1\ ,\ \delta_2^2 = 3\ ,\ \delta_1^2 = 6 \quad ,$$

we obtain the valid inequality (16)

$$
\begin{aligned}
f_1 + f_6 \quad \geq \quad & 2(2-y_2-y_4) + 3(1-y_7) + 1(3-y_2-y_4-y_7-y_9-y_{11}) + \\
& 3(1-y_7-y_9) + 6(1-y_2)
\end{aligned}
$$

which can be shown to generate a facet of $conv(X^{CF})$.

7 Conclusion.

In this paper we present a general procedure for generating valid inequalities for a mixed-integer set S based on mixing or combining a set of mixed-integer rounding valid inequalities for S. The performance of the procedure strongly depends on identifying a set of "good" starting valid inequalities (called the base inequalities) from which the standard MIR inequalitites are generated, and this requires some understanding of the structure of the mixed-integer set. Since we are more familiar with mixed-integer linear programming problems, in Section 3 we consider some well known MIP problems and show that using the mixing procedure we can derive new facet defining valid inequalities. The mixing procedure also provides a unifying approach to proving validity for some known classes of inequalities (such as the (k, l, S, I) inequalities).

In Sections 5 and 6 we have investigated how to generalize and strengthen the inequalities generated by this mixing procedure. In particular, in section 5 we have studied the generalization of the mixing procedure to cases where the set of base inequalities do not satisfy our initial assumptions. This leads to a class of valid inequalities obtained by combining scaled versions of the mixed-integer rounding inequalities. In section 6, we have presented a first approach for strengthening the mixed inequalities when the base inequalities contain some more structure (i.e. when the continuous part of the base inequalities satisfy additional requirements). In this direction, we are currently investigating the extension of these ideas to generate valid inequalities for capacitated network flow problems on directed in-trees (see example 6.3) or out-trees.

Finally, apart from the work on (k, l, S, I) inequalities for production planning problems, the valid inequalities generated by the mixing procedure have not yet been used as cutting planes in the solution of optimization problems by branch and cut approaches. For applications where a separation procedure for mixed-integer rounding inequalities (as the flow-cut-set inequalities for the capacity expansion problem mentioned in section 3) is available, one way to generate our mixed inequalities would be to try to combine the mixed-integer rounding inequalities that are satisfied at equality at the current fractional point, and that have already been generated during the cutting plane phase (see example 3.2). This merits further investigation.

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References

[1] K. Aardal, Y. Pochet and L.A. Wolsey, "Capacitated Facility Location : Valid Inequalities and Facets", *Mathematics of Operations Research* 20 (1995), 562-582.

[2] E. Balas, "Disjunctive Programming", *Annals of Discrete Mathematics* 5 (1979), 3-51.

[3] E. Balas, S. Ceria and G. Cornuéjols, "A Lift-and-Project Cutting Plane Algorithm for Mixed 0-1 Programs", *Mathematical Programming* 58 (1993), 295-324.

[4] E. Balas, S. Ceria, G. Cornuéjols and Natraj, "Gomory Cuts Revisited", *Operations Research Letters* 19 (1996), 1-9.

[5] I. Barany, T.J. Van Roy and L.A. Wolsey, "Uncapacitated Lot Sizing : the Convex Hull of Solutions", *Mathematical Programming Study*, 22 (1984), 32-43.

[6] D. Bienstock, and O. Günlük, "Capacitated Network Design - Polyhedral structure and Computation," to appear in *ORSA Journal on Computing* .

[7] D. Bienstock, S. Chopra, O. Günlük, and C-Y. Tsai, "Minimum Cost Capacity Installation for Multicommodity Network Flows" to appear in *Mathematical Programming.*

[8] V. Chvátal "Edmonds Polytopes and a Hierarchy of Combinatorial Problems", *Discrete Mathematics* 4 (1973), 305-337.

[9] W. Cook, R. Kannan and A. Schrijver, "Chvátal Closures for Mixed Integer Programming Problems", *Mathematical Programming* 47 (1990), 155-174.

[10] H. Crowder, E.L. Johnson and M.W. Padberg, "Solving Large Scale Zero-One Linear Programming Problems", *Operations Research* 31 (1983), 803-834.

[11] R.E. Gomory, "An Algorithm for the Mixed Integer Problem", RM-2597, The RAND Corporation, (1960).

[12] Z. Gu, G.L. Nemhauser and M.W.P. Savelsbergh, "Sequence Independent Lifting", LEC-9508, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, (1995).

[13] H. Marchand and L.A. Wolsey, "The 0-1 Knapsack Problem with a Single Continuous Variable", CORE Discussion Paper 9720, Université Catholique de Louvain, Louvain-la-Neuve, (1997).

[14] T. Magnanti, P. Mirchandani and R. Vachani, "Modeling and Solving the Two Facility Capacitated Network Loading Problem", *Operations Research* 43 (1995), 142-157.

[15] G.L. Nemhauser and L.A. Wolsey, "A Recursive Procedure for Generating all Cuts for 0-1 Mixed Integer Programs", *Mathematical Programming* 46 (1990), 379-390.

[16] G.L. Nemhauser and L.A. Wolsey, *Integer and Combinatorial Optimization*, Wiley, New York, (1988).

[17] M.W. Padberg, T.J. Van Roy and L.A. Wolsey, "Valid Linear Inequalities for Fixed Charge Problems" *Operations Research* 33 (1985), 842-861.

[18] Y. Pochet and L.A. Wolsey, "Lot Sizing with Constant Batches : Formulation and Valid Inequalities", *Mathematics of Operations Research* 18 (1993), 767-785.

[19] Y. Pochet and L.A. Wolsey, "Polyhedra for Lot Sizing with Wagner-Whitin Costs", *Mathematical Programming* 67 (1994), 297-323.

[20] Y. Pochet and L.A. Wolsey, "Integer Knapsack and Flow Covers with Divisible Coefficients: Polyhedra, Optimization and Separation", *Discrete Applied Mathematics* 59 (1995), 57-74.

[21] L.A. Wolsey, "Valid Inequalities and Superadditivity for 0-1 Integer Programs", *Mathematics of Operations Research* 2 (1977), 66-77.

[22] E. Zemel, "Lifting the Facets of Zero-One Polytopes", *Mathematical Programming* 15 (1978), 268-277.

Appendix 1: Proof of Lemma 6.2

As in the proof of Lemma 2.1, for any fixed $\bar{x} \in S$, let $\beta_j = \max_{i \in I(j)}\{h_j^i(\bar{x})\}$ and $v_j = \max\{i \in I(j) : \beta_j = h_j^i(\bar{x})\}$ for $j = 0, \ldots, n$. Without loss of generality, we can assume that $\beta_j > 0$ for $j = 0, \cdots n$, because otherwise the validity of (16) at $\bar{x}$ is implied by the validity at $\bar{x}$ of the inequality (16) obtained by removing all sets $I(j)$ with $\beta_j \leq 0$ from the set of base inequalities and using the coefficients δ_k^i for $i \in I(k)$ and all k with $\beta_k > 0$. Note that if $\beta_0 \leq 0$, the validity of (16) follows from Lemma 4.1 and the fact that $\bar{f}(\bar{x}) \geq \sum_{j=1}^n f_j^*(\bar{x})$.

If we let $\Delta_j = \sum_{i \in I(j)} \delta_j^i$ and $\epsilon_j = \sum_{\{i \in I(j): i \leq v_j\}} \delta_j^i$ for $j = 0, \ldots, n$, then

$$\sum_{j=0}^n \sum_{i \in I(j)} \delta_j^i h_j^i(\bar{x}) \leq \sum_{j=1}^n [(\beta_j - 1)\Delta_j + \epsilon_j] + (\beta_0 - 1)\Delta_0 + \epsilon_0$$

and using $\Delta_0 + \Delta_j \leq max\{\gamma_j^{|I(j)|}, \gamma_0^{|I(0)|}\} \leq B$ for $j = 1, \cdots, n$,

$$\begin{aligned}\sum_{j=0}^n \sum_{i \in I(j)} \delta_j^i h_j^i(\bar{x}) &\leq \sum_{j=1}^n (\beta_j - 1)(B - \Delta_0) + (\beta_0 - 1)\Delta_0 + \epsilon_0 + \sum_{j=1}^n \epsilon_j \\ &= \sum_{j=1}^n (\beta_j - 1)B + \Delta_0 \left[\beta_0 - \sum_{j=1}^n \beta_j + n - 1\right] + \epsilon_0 + \sum_{j=1}^n \epsilon_j.\end{aligned}$$

We next define $V = \{\ j\ :\ U(v_j, j) > U(v_0, 0)\ \}$,
$Y = \{\ j\ :\ U(v_j, j) > U(|I(0)|, 0)\ \}$ and note that $Y \subseteq V$.

If $\left[\beta_0 - \sum_{j=1}^n \beta_j + n - 1\right] \geq |Y| + 1$, then

$$\begin{aligned}\sum_{j=0}^n \sum_{i \in I(j)} \delta_j^i h_j^i(\bar{x}) &\leq B\left(\sum_{j=1}^n \beta_j - n\right) + \Delta_0\left[\beta_0 - \sum_{j=1}^n \beta_j + n - 1 - (|Y| + 1)\right] + \epsilon_0 \\ &\quad + \sum_{j \in Y}(\Delta_0 + \epsilon_j) + \left(\Delta_0 + \sum_{j \notin Y} \epsilon_j\right)\end{aligned}$$

and, using $B \geq \gamma_0^{|I(0)|} \geq \Delta_0 + \sum_{j \notin Y} \epsilon_j$, $B \geq \gamma_j^{v_j} \geq \Delta_0 + \epsilon_j$ for all $j \in Y$, and $\epsilon_0 \leq \gamma_0^{v_0}$,

$$\begin{aligned}\sum_{j=0}^n \sum_{i \in I(j)} \delta_j^i h_j^i(\bar{x}) &\leq B\left(\sum_{j=1}^n \beta_j - n\right) + B\left[\beta_0 - \sum_{j=1}^n \beta_j + n - 1 - (|Y| + 1)\right] + \epsilon_0 \\ &\quad + |Y|B + B \leq B(\beta_0 - 1) + \gamma_0^{v_0} \leq f_0^*(\bar{x}) \leq \bar{f}(\bar{x}).\end{aligned}$$

On the other hand, if $\left[\beta_0 - \sum_{j=1}^n \beta_j + n - 1\right] \leq |Y|-1$, then, as $\epsilon_0 \leq \Delta_0$,

$$\begin{aligned}
\sum_{j=0}^{n}\sum_{i\in I(j)} \delta_j^i h_j^i(\bar{x}) &\leq \sum_{j=1}^{n}(\beta_j - 1)B + \Delta_0\left[|Y|-1\right] + \epsilon_0 + \sum_{j=1}^{n}\epsilon_j \\
&\leq \sum_{j=1}^{n}(\beta_j - 1)B + \sum_{j\in Y}(\Delta_0 + \epsilon_j) + \sum_{j\notin Y}\epsilon_j \\
&\leq \sum_{j=1}^{n}(\beta_j - 1)B + \sum_{j\in Y}\gamma_j^{v_j} + \sum_{j\notin Y}\gamma_j^{v_j} \leq \sum_{j=1}^{n} f_j^*(\bar{x}) \leq \bar{f}(\bar{x}).
\end{aligned}$$

Therefore, the only remaining case is when $\left[\beta_0 - \sum_{j=1}^n \beta_j + n - 1\right] = |Y|$. In this case, if $V = Y$, then using $\epsilon_0 + \sum_{j\notin V}\epsilon_j \leq \gamma_0^{v_0}$,

$$\begin{aligned}
\sum_{j=0}^{n}\sum_{i\in I(j)} \delta_j^i h_j^i(\bar{x}) &\leq B\left(\sum_{j=1}^{n}\beta_j - n\right) + \Delta_0\left[\beta_0 - \sum_{j=1}^{n}\beta_j + n - 1 - |V|\right] + \\
&\quad \sum_{j\in V}(\Delta_0 + \epsilon_j) + \left(\epsilon_0 + \sum_{j\notin V}\epsilon_j\right) \\
&\leq B\left(\sum_{j=1}^{n}\beta_j - n\right) + B\left[\beta_0 - \sum_{j=1}^{n}\beta_j + n - 1 - |V|\right] + |V|B + \gamma_0^{v_0} \\
&\leq B(\beta_0 - 1) + \gamma_0^{v_0} \leq f_0^*(\bar{x}) \leq \bar{f}(\bar{x}).
\end{aligned}$$

Finally, if $|V \setminus Y| \geq 1$, then we note that $\epsilon_0 + \epsilon_j \leq \gamma_j^{v_j}$ for all $j \in V$ and $\Delta_0 + \epsilon_j \leq \gamma_j^{v_j}$ for all $j \in Y$, and write,

$$\begin{aligned}
\sum_{j=0}^{n}\sum_{i\in I(j)} \delta_j^i h_j^i(\bar{x}) &\leq \sum_{j=1}^{n}(\beta_j - 1)B + \Delta_0\,|Y| + \sum_{j\in Y}\epsilon_j + \left(\epsilon_0 + \sum_{j\in V\setminus Y}\epsilon_j\right) + \sum_{j\notin V}\epsilon_j \\
&\leq \sum_{j=1}^{n}(\beta_j - 1)B + \sum_{j\in Y}(\Delta_0 + \epsilon_j) + \sum_{j\in V\setminus Y}\gamma_j^{v_j} + \sum_{j\notin V}\gamma_j^{v_j} \\
&\leq \sum_{j=1}^{n}(\beta_j - 1)B + \sum_{j=1}^{n}\gamma_j^{v_j} \leq \sum_{j=1}^{n} f_j^*(\bar{x}) \leq \bar{f}(\bar{x}).
\end{aligned}$$

This completes the proof. ∎