Optimal Income Taxation: An Ordinal Approach*

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Abstract

In a model where agents have unequal production skills and different preferences, we build social welfare functions which rely only on ordinal non-comparable information on individual preferences. Social welfare functions are required to satisfy properties of compensation for inequalities in skills, and responsibility for preferences. Then, assuming skills and preferences are unobservable, we use these social welfare functions to design optimal income tax schemes. We obtain ethical foundations for, among others, a maximized minimal income, a zero marginal tax rate for low incomes, and increasing marginal tax rates.

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1 Introduction

It is a common belief among public economists that social welfare functions, which are necessary for the design of optimal second best policies, must involve interpersonal utility comparisons. This is the main reason why the social planner's redistributive goals are almost always expressed in utilitarian terms.[1]

Utilities, however, are unobserved objects. As a consequence, all results obtained in the literature by using utilitarian-like social welfare functions either do not depend on specific utility functions, and, in this case, only deal with efficiency (or at best Lorenz dominance), or they depend on a particular specification of utility functions, and, in this case, are of little practical relevance.

This paper has the simple goal of showing that an ordinal approach to the theory of second best decentralization is possible. Hereafter, we propose social welfare functions which do not require any utility comparisons nor any cardinal measurement of utility. But the purpose of this paper is less to discuss such social welfare functions than to illustrate how useful they can be in the design of optimal income tax schemes. A thorough axiomatic analysis of ordinal social welfare functions is made in a companion paper (Fleurbaey and Maniquet 1998). This work is an application of suggestions made in Bossert, Fleurbaey and Van de gaer (1996) and Fleurbaey and Maniquet (1996b). The former paper made first proposals of social orderings adapted to a model similar to the one studied below, but was not focused on the ordinal approach, and the latter developed in much detail the general motivation for ordinal social orderings, but without any application.

The paper is organized as follows. In section 2, we sum up the main reasons why the ordinal approach in second best analysis is both possible and desirable. In section 3, we introduce a variant of Mirrlees' model, in which agents differ in two dimensions, namely, their preferences and their production skills. In this context, the laissez-faire allocation may be unsatisfactory as we could wish to compensate low-skill agents for their low productivity. At the same time, we may wish to compensate only for low productivity and

[1]The Nash social welfare function (Kaneko and Nakamura 1979, Roberts 1980) satisfies an axiom called *Cardinal non comparability* and is sometimes thought not to involve any interpersonal comparisons. Although we think this is debatable, at least is it clear that it needs cardinal information about individual utilities. In contrast we will stick here to a purely ordinal non-comparable approach.

leave the agents responsible for their choice of labor time. Compensation for inequalities in skills and responsibility for preferences are the two ethical values we analyse in section 4.

In sections 5 and 6, we study how social orderings based on properties of compensation and responsibility can be used to design optimal income tax schemes. A tax scheme gives the amount of tax paid as a function of personal pre-tax income. We assume that individual skills and preferences are unobservable, whereas their distribution in the population is known, and we compute tax rates maximizing these ordinal social orderings. All proofs are gathered in the Appendix.

2 Possibility and desirability of ordinalism

Our opinion is that the ordinal approach is not only possible but is desirable as well, and should be preferred to the traditional way of defining the planner's objective in utilitarian terms. First, as we already mentioned above, it is well known that observable behavior in market economies yields only information about individual preferences. Therefore non-ordinal data seem essentially impossible to collect and only ordinal social goals have any hope of application.

The second main argument in favor of ordinalism comes from political philosophy. Rawls (1971, 1982), followed by Dworkin (1981) and many authors, has argued that individuals should be held responsible for their ends, so that social goals should not take care of satisfaction levels. Along these lines, ordinalism seems a minimal requisite for any acceptable social goal. Actually, one would like to go further and to make social goals as independent of contingent individual preferences as possible, but there are limits to this move, in particular because preferences ultimately determine the value of economic goods. It is a well known problem of Rawls's theory that what he calls "primary goods" must in his theory have value independently of individual preferences whereas weighting them requires positing some particular preferences (Arrow 1973). Whether a good is a good or a bad cannot even be determined without knowing preferences. As a consequence, we think that ordinalism is a safe minimal condition that does not prevent any reference to individual preferences (for efficiency purposes, in particular) while conveying a substantial idea of responsibility for one's achievements at the same time.[2]

[2]Fleurbaey and Maniquet (1996b) and Gaspart (1996) study how much independence

Let us come back to the issue of possibility. The belief that ordinal non comparable social welfare functions are impossible is probably due to a misunderstanding of Arrow's theorem and of the historical development of social choice theory. It is widely thought that in his celebrated impossibility theorem, Arrow proved the non-existence of a satisfactory way of aggregating individual preferences into social preferences. It is also thought that reasonable social orderings of alternatives have been later legitimized by the theory of social choice only at the price of introducing interpersonal comparisons of utilities.

It is actually easy to find a host of interesting ordinal social orderings. In Fleurbaey and Maniquet (1996b, 1998), we compare in detail our possibility results with Arrow's theorem. The basic idea is that two departures from Arrow's framework must be done in order to make this possible. The first one is to introduce more information about the economic alternatives so that the agents' preferences have the usual structure of economic preferences, and the second one is to weaken the Independence of Irrelevant Alternatives axiom, which is the main factor of Arrow's impossibility, so as to allow the aggregation procedure to use information about individual indifference curves, and not only about individual pairwise preferences.[3] In brief, instead of introducing information about utilities, as done in the theory of social choice with utilities, we introduce information about indifference curves, and that is enough to provide an economically meaningful escape from the impossibility theorem.

3 The model

We take a variant of Mirrlees's (1971) model in order to illustrate our thesis. There is a set N of agents. Each agent $i \in N$ is characterized by a *skill* parameter $s_i \in [0, \overline{s}]$, and a *preference* relation $R_i \in \mathcal{R}$ defined over bundles $z = (l, c) \in X = [0, 1] \times \mathbb{R}_+$ where l denotes the labor time, and c denotes the consumption. The set $\mathcal{R}$ is the set of continuous and convex orderings

of individual preferences can be rendered compatible with Pareto conditions.

[3]Notice that Arrow's axiom of Independence of Irrelevant Alternatives can be justified on the basis of the idea developed above, namely, that the allocation of resources should be as independent from individual preferences as possible. The main lesson from Arrow's theorem is simply that Pareto and Anonymity conditions require dependence of the social decisions on more than individual pairwise preferences.

which are strictly increasing in consumption and strictly decreasing in labor. Let us note that good l can be viewed as labor time, or as a combination of labor time with effort (for instance, it may be that labor time can only take a couple of values, although effort can vary continuously and influence income accordingly). Skill parameters have the following interpretation. One unit of input l produces s units of consumption good for an agent working at skill level s. We also assume that there is a natural upper bound $\overline{s}$ on skills. Agent i can choose to work at any skill level $\hat{s}_i \leq s_i$, where s_i is her actual skill level. Actually, it is always in the planner's interest to induce agents to choose $\hat{s}_i = s_i$, so that we will forget this distinction in the sequel. We will call $y_i = s_i l_i$ agent i's *pre-tax income.*

For $R \in \mathcal{R}$, and $A \subset X$, let $m(R, A)$ denote the set of bundles (if any) which maximize preferences R over the set A, that is

$$m(R, A) = \{z \in A | \forall z' \in A,\ z\, R\, z'\}.$$

An *allocation* is a function $z_N \in X^N$ associating a bundle $z_i = (l_i, c_i)$ to each agent $i \in N$. An allocation $z_N \in X^N$ is *feasible* for the skill profile $s_N \in [0, \overline{s}]^N$ if

$$\bar{B}(z_N) = \int_N (c_i \Leftrightarrow s_i l_i) \leq 0$$

where the integration is made with respect to the appropriate measure.

A *social ordering* is an ordering over allocations (strict social preference and social indifference are denoted and respectively).

An *income tax scheme* is a continuous function $\tau : \mathbb{R}_+ \to \mathbb{R}$ which defines the amount of tax to be paid $\tau(y_i)$ by agent i as a function of her pre-tax income y_i, for $i \in N$.

We will restrict our attention in this paper to continuous tax schemes τ such that the function $y \Leftrightarrow \tau(y)$ is non-negative and weakly increasing in y. The latter assumption is not restrictive since, by strict monotonicity of preferences in consumption and in leisure, any allocation obtained with a tax scheme that does not satisfy it can also be obtained with another tax scheme that satisfies it.[4] Let $\mathcal{T}$ denote the set of tax schemes satisfying these conditions.

We will restrict our attention to allocations obtained with tax schemes, namely to allocations $z_N \in X^N$ for which there exists $\tau \in \mathcal{T}$ such that for

[4] The continuity assumption is essential in order to guarantee that the individual budget sets are compact.

all $i \in N$,

$$z_i \in m(R_i, \{(l, c) \in X | c = s_i l \Leftrightarrow \tau(s_i l)\}).$$

Let $Z(\tau)$ denote the set of allocations obtained with τ, and

$$Z = \bigcup_{\tau \in \mathcal{T}} Z(\tau)$$

denote the set of what we will call "incentive compatible allocations". Notice that Z contains feasible allocations (for instance, the laissez-faire allocations). Although this restriction to incentive compatible allocations does not affect the essentials of our results, it helps making the presentation simple.[5]

The following assumptions are not imposed everywhere in the paper, but their presentation is gathered here for convenience. The first one says that the set of characteristics of the population is rich enough so that for any preferences represented in the population there is at least one agent with these preferences and a zero skill.

Assumption A1: For all $i \in N$, there exists $j \in N$ such that $R_j = R_i$ and $s_j = 0$.

The second assumption is also a richness assumption. It says that in any group of all agents having the same skill, and for any tax scheme, we can find an agent choosing in the neighborhood of any labor time, provided consumption is a strictly increasing function of labor at this labor time.

Assumption A2: For all $\tau \in \mathcal{T}$, $z_N \in Z(\tau)$, $s \in [0, \overline{s}]$, $y \in [0, s]$, $\varepsilon \in \mathbb{R}_{++}$, if $y = 0$ or $y \Leftrightarrow \tau(y)$ is strictly increasing over $[y \Leftrightarrow \varepsilon, y + \varepsilon] \cap [0, s]$, then there exists $j \in N$ such that $s_j = s$ and $y_j \in [y \Leftrightarrow \varepsilon, y + \varepsilon]$.

The third assumption is (much) weaker than the preceding one. It says that for any tax scheme, any income in an interval where consumption is a strictly increasing function of pre-tax income, it is possible to find an agent having chosen a labor time equal to 1 and having her pre-tax income in this interval.

[5]In particular, this will allow us to talk about a population with a continuum of characteristics without having to introduce measure theory in the definition of the axioms in the next section.

Assumption A3: For all $\tau \in \mathcal{T}$, $z_N \in Z(\tau)$, $y \in [0, \overline{s}], \varepsilon \in \mathbb{R}_{++}$, if $y \Leftrightarrow \tau(y)$ is strictly increasing over $[y \Leftrightarrow \varepsilon, y + \varepsilon] \cap [0, \overline{s}]$, then there exists $j \in N$ such that $l_j = 1$ and $s_j = y_j \in [y \Leftrightarrow \varepsilon, y + \varepsilon]$.

The fourth assumption is that no level of pre-tax income is negligible, in the sense that, if an agent has this level, the subpopulation of agents having their pre-tax income in a neighborhood of this level has positive measure.

Assumption A4: For all $\tau \in \mathcal{T}$, $z_N \in Z(\tau)$, $i \in N$, $\varepsilon \in \mathbb{R}_{++}$,

$$\int_{j:|y_j - y_i| < \varepsilon} y_j > 0$$

where the integration is made with respect to the appropriate measure.

The last two assumptions exclude the possibility for a tax scheme to be tangent at different points to the same indifference curve of a positive mass of agents. Assumption A5 excludes that possibility for any tax scheme, which essentially requires non-atomicity of the population, whereas Assumption A6 excludes it only when the tax scheme is convex (which corresponds to increasing marginal income tax rates).

Assumption A5: For all $\tau \in \mathcal{T}$, $\bar{B}(Z(\tau))$ is a singleton.

Assumption A6: For all $\tau \in \mathcal{T}$ such that τ is convex, $\bar{B}(Z(\tau))$ is a singleton.

Since the agents' preferences are assumed to be continuous, these assumptions entail that for any (resp., any convex) $\tau \in \mathcal{T}$, the function $\varepsilon \in \mathbb{R}_+ \mapsto \bar{B}(Z(\tau \Leftrightarrow \varepsilon))$ is continuous.[6]

Assumptions A1, A3, A4 and A6 are not very demanding (let us note that any population where agents have strictly convex preferences satisfy A6). Assumption 5 is more demanding but if preferences satisfy the single-crossing property, it is satisfied if the distribution of preferences is non-atomic.

Assumptions A2, on the contrary, is heroic. Let us give an example of a population satisfying this condition. Let $\mathcal{R}$ contain the family of all quasi-linear preferences $R_{l^*\varepsilon}$ composed of piecewise linear indifference curves having a marginal rate of substitution equal to ε at points $(l, c) \in X$ such

[6]In the general case, $\varepsilon \in \mathbb{R}_+ \mapsto \bar{B}(Z(\tau+\varepsilon))$ is a correspondence. By a direct application of the theorem of the maximum, it must be upper hemi-continuous (see e.g. Border 1985).

that $l < l^*$, and equal to $\frac{1}{\varepsilon}$ at points $(l, c) \in X$ such that $l > l^*$, for $l \in [0, 1]$, and $\varepsilon \in]0, 1]$. If the population is such that for all $s \in [0, \overline{s}]$ it is possible to find agents with any preferences in the family just described, and the distribution of s has full support in $[0, \overline{s}]$, then assumption A2 is satisfied (and all the other assumptions as well if s is non atomic).

In view of the complexity of this example, two remarks are in order. Firstly, it is important to note that only assumptions A1, A3 and A6 are needed for the two main results (that is, Proposition 1 and the upper bound property in Proposition 3) stated below. Secondly, the assumptions are of course sufficient but not necessary for our results. If the conditions stated in the assumptions are satisfied by those tax schemes τ which are, a priori, likely to be considered optimal (given, for example, that the planner is only interested in a small reform of the current system, or that she is committed to marginal income tax rates in a "reasonable" interval, etc), then the results still hold true.

4 Ethical principles for redistribution

The (efficient) no-redistribution equilibrium of this economy is such that every agent i chooses her preferred bundle in her budget set $\{(l, c) \in X | c = s_i l\}$. One can argue that this laissez-faire allocation is inequitable, in particular if differences in skills come from inherited features which cannot be attributed to the agents' responsibility, or if s_i is interpreted as the wage rate associated with the job assigned to agent i whereas all agents could be more productive but are constrained by the availability of higher wage rate jobs.

In such a context, one would like to neutralize the consequences of differential skills. In this respect, we will proceed in two steps. In the first step, we will study how to build orderings . These orderings are intended to capture the idea that consequences of differential skills should be neutralized. In the second step, we will study how to use such orderings to design income tax schemes. More precisely, assuming the planner does not observe individual skills and preferences, but only knows their distribution, we will study which income tax scheme she can design so that the resulting allocation maximizes the chosen ordering under the feasibility constraint and under the incentive constraint that agents freely choose their labor time.

We begin by defining three ethical principles bearing on social orderings. The first one is the standard Pareto efficiency condition.

Pareto *For all* $z_N, z'_N \in Z$*, if for all* $i \in N$*,* $z'_i \, R_i \, z_i$*, then* $z'_N \; z_N$*. If, in addition, for some* $j \in N$*,* $z'_j \, P_j \, z_j$*, then* $z'_N \, \mathrm{P} \, z_N$*.*

A well-known consequence of this requirement is that $z'_N \, \mathrm{I} \, z_N$ whenever $z'_i \, I_i \, z_i$ for all $i \in N$. It implies that for every tax scheme $\tau \in \mathcal{T}$, and corresponding allocations $z_N, z'_N \in Z(\tau)$, if R satisfies *Pareto*, then $z_N \, \mathrm{I} \, z'_N$. This fact allows the planner to extend an ordering on Z into an ordering on $\mathcal{T}$ in a straightforward manner.

The second one is called *Compensation.* It is easily justified as follows: if two agents differ only in their skills, then no difference in welfare is ethically founded, and the skill differences should be perfectly counter-balanced.[7] We apply this idea to social orderings as follows. Let us suppose that we compare two allocations. The first one is socially preferred if, for any agent who prefers the second one, there is another agent with identical preferences who is worse-off and prefers the first allocation.

Compensation *For every* $z_N, z'_N \in Z$ *such that for some* $k \in N$*,* $z_k \, P_k \, z'_k$*, one has* $z'_N \, \mathrm{P} \, z_N$ *if for all* $i \in N$

$$z_i \, P_i \, z'_i \Rightarrow \exists j \in N, \; R_i = R_j, \;\; z'_j \, P_j \, z_j \;\; z'_i \, R_j \, z'_j \, .$$

By contrast, if one follows the ethical line described in the previous section, the various preferences in the population should not be treated differently, that is, the agents should not receive any differential amount of resources on the pure basis that they have "favorable" or "unfavorable" preferences. Let us note that a weak consequence of this principle is that no redistribution is ethically grounded in economies where all agents have the same skill.

We capture this ethical principle by requiring that agents with identical skills should be free to choose their preferred bundle in the same budget set. Notice that by construction any allocation in Z is such that agents with the same skill s choose their bundles from the same budget set

$$\{(l, c) \in X |, c = sl \Leftrightarrow \tau(sl)\}.$$

[7]If skill levels could at least partly be attributed to agents' responsibility, then other ethical principles would have to be proposed. We will adhere to the compensation principle defined above throughout this paper, however, but mainly for the purpose of illustration.

But this "second best" budget set is not the one we would like to consider for the definition of a social ordering. It is more appropriate to build a social ordering on a "first best" reference if we want to make it consistent with good social choices in a first best context as well. The definition of the "first best" budget set that is associated with a given bundle is simple in this framework: This is the budget set that would give the same satisfaction to the agent, and would involve a lump sum transfer only, without any taxation.[8] We call it the agent's *implicit budget*. Formally, for $s \in [0, \bar{s}]$, $R \in \mathcal{R}$, $z \in X$, the implicit budget $IB(s, R, z) \subset X$ is defined by

$$z \, I \, m(R, IB(s, R, z))$$

$$\exists e^* \in I\!R_+, \; IB(s, R, z) = \{(l, c) \in X \mid c \Leftrightarrow sl \leq e^*\}.$$

Let us note that

$$IB(s, R, z) \supseteq IB(s, R, z') \Leftrightarrow zRz'.$$

We apply the idea of equalizing implicit budget sets to social orderings as follows. Let us suppose that we compare two allocations. The first one is socially preferred if, for any agent who prefers the second one, there is another agent with identical skill who has a lower implicit budget and prefers the first allocation.

Responsibility *For every $z_N, z'_N \in Z$ such that for some $k \in N$, $z_k \, P_k \, z'_k$, one has $z'_N \, \mathrm{P} \, z_N$ if for all $i \in N$*

$$z_i \, P_i \, z'_i \Rightarrow \exists j \in N, \; s_i = s_j, \; z'_j \, P_j \, z_j \; IB(s_i, R_i, z'_i) \supseteq IB(s_j, R_j, z'_j) \, .$$

Notice that the agents in the above condition have implicit budget sets such that

$$IB(s_i, R_i, z_i) \supset IB(s_i, R_i, z'_i) \supseteq IB(s_j, R_j, z'_j) \supset IB(s_j, R_j, z_j)$$

which illustrates the equalization idea.

If the population satisfies *Assumption A1*, there exists a social ordering satisfying *Pareto*, *Compensation* and *Responsibility*, namely, the ordering which focuses on the minimum consumption:

$$z'_N z_N \Leftrightarrow \min\{c'_i | i \in N\} \geq \min\{c_i | i \in N\}.$$

[8] In a model where first best prices could vary as a function of the profile of characteristics of the population, things would be more difficult. Here the first best price of agent i's time is always s_i (taking the consumption good as the numeraire).

Such a compatibility between the *Compensation* and *Responsibility* requirements, however, comes as a surprise because incompatibility between similar kinds of axioms embodying the same pair of ethical principles is commonplace in the first-best analysis of allocation rules (an allocation rule selects a set of best allocations, and does not allow one to compare any pair of non selected allocations).[9] Actually, it is shown in Fleurbaey and Maniquet (1998) that when all feasible allocations are considered, these axioms of *Compensation* and *Responsibility* are incompatible.[10]

This means that this ordering cannot be extended meaningfully to the set of all allocations. We think it more meaningful to study orderings which are projections on Z of complete orderings on the set of all allocations (or at least, all feasible allocations). As a consequence, we will separately focus in the next sections on orderings satisfying *Pareto* and *Compensation*, and on orderings satisfying *Pareto* and *Responsibility.*

5 Compensation

In this section, we study the income tax schemes a planner can design if she is interested in maximizing a social ordering satisfying *Pareto* and *Compensation.* First we derive an analytical result. We consider it as the main result of this section. It says that the income tax scheme the planner must obtain is the one which maximizes the minimum income, $\Leftrightarrow\tau(0)$. Maximizing the minimum income is a social objective which has been advocated by many authors (e.g. Van Parijs 1995).

Proposition 1 *Under Assumption A1, if a planner wishes to maximize a social ordering satisfying* Pareto *and* Compensation*, then she chooses $\tau \in \mathcal{T}$ so as to maximize the minimal after tax income, that is, for all $\tau' \in \mathcal{T}$, $\Leftrightarrow\tau(0) \geq \Leftrightarrow\tau'(0)$.*

This result rests on a straightforward intuition. Those who lose when the minimum income is increased can only be those who have a strictly positive

[9]See e.g. Fleurbaey and Maniquet (1996a).

[10]Fleurbaey and Maniquet (1998) restrict their attention to economies with a finite number of (types of) agents, but one can show, with appropriate formulations of the axioms for a framework with a continuum of agents, that the same incompability occurs.

level of skill, and they are always better off than those who have the same preferences but lower skills. Raising the minimum income tends to equalize, according to the maximin criterion, the welfare levels (in terms of indifference curves) of agents with the same preferences, whatever their skills.

Our second result is that maximizing the minimum income $\Leftrightarrow \tau(0)$ is obtained with a flat tax rate of 50% in economies where preferences are quasi-linear and spread uniformly over the population, on a wide enough range. We mention this result only because the literature on taxation has so far focused on models in which agents differ only along one dimension. It is well known that multidimensional screening is a difficult topic, but this result is rather simple and striking because it holds for a large class of economies.

Proposition 2 *Assume that the economy has the following characteristics:*

1. *preferences R_i are represented by functions $u_i(l, c) = c \Leftrightarrow \alpha_i v(l)$, where v is continuously differentiable over $[0, 1)$, and $v' > 0$, $v'' > 0$;*

2. *for every s, the preference parameter α_i is uniformly distributed on some interval $[0, \bar{\alpha}_s]$, where $\bar{\alpha}_s \geq s/v'(0)$.*

Then the tax with a constant marginal rate of 50% maximizes the minimum after tax income.

It is remarkable that this result requires no assumption on the distribution of s, and does not even require that the distribution of preferences be independent from s.[11]

The intuition behind that result is the following one. Maximizing the minimal after tax income amounts to maximizing the total taxes paid by all agents and redistributing it uniformly. Let us fix s and let us try to maximize the total taxes levied on agents with skill equal to s. Now, given the quasi-linearity of the preferences, and given that α is distributed uniformly, the total taxes are computed as the integral of a function proportional to $\tau'(1 \Leftrightarrow \tau')$. The return of the tax is therefore maximized with at a constant marginal rate equal to 50 percent.

[11] With the weaker assumption that for all s, $2\bar{\alpha}_s \geq s/v'(0)$, one proves by the same kind of proof that the same tax yields a *local* maximum.

6 Responsibility

In this section, we study the income tax schemes a planner can design if she is interested in maximizing a social ordering satisfying *Pareto* and *Responsibility*. First of all, let us observe that a laissez-faire allocation has the property that all implicit budgets are equal to actual budgets, and they all are equalized among agents with identical skill. This means that the laissez-faire policy is consistent with the *Responsibility* principle as defined above. But as noticed above, this principle would also condone the maximization of the minimum income, which, in some sense, lies at the other end of the spectrum of redistribution. This suggests that the *Responsibility* principle is less constraining in our setting than the *Compensation* principle.

But the laissez-faire policy is inequitable, under the assumption that skill differentials are not deserved, and therefore it is worth combining *Responsibility* with compensation requirements. In Fleurbaey and Maniquet (1998) an axiomatic study of orderings bearing on all allocations is made, and yields a characterization of a family of orderings, on the basis of *Responsibility* and a weakening of the *Compensation* principle.

Each ordering in this family is parameterized by reference preferences $\tilde{R} \in \mathcal{R}$. These are not preferences that the social planner would like the agents to adopt. Instead, they identify the subset of agents among which compensation will be as much carried out as possible (recall that it is impossible to satisfy *Responsibility* while compensating in all groups of agents having the same preferences). As we will see shortly, these reference preferences also serve to calibrating the degree of redistribution that the planner wishes to implement. Let an ordering in this family be called the $\tilde{R}$-Implicit Budget Leximin ordering. The comparison between two allocations, according to this ordering, works as follows. The bundle of each agent is evaluated by the reference preferences applied to the agent's implicit budget. Then, the leximin ordering is applied to these evaluations. The formal definition of this ordering requires the following terminology. For a given set E and any $R \in \mathcal{R}$, and two lists of bundles $(x_e)_{e\in E}, (x'_e)_{e\in E} \in X^E$, we write

$$(x_e)_{e\in E} \, (R) \, (x'_e)_{e\in E}$$

to denote that the leximin order of the elements of the first list is at least as good for preferences R as the leximin order of elements of the second list (that is, the least preferred bundle in the first list is strictly preferred to the least preferred one in the second list, or they are equivalent but the second least preferred bundle in the first list is strictly preferred to the second least preferred one in the second list, or ..., or they are all equivalent). In the case of a continuum of agents, the definition is similar but involves cumulative distribution functions of welfare levels (for some fixed representation of preferences R).

$\tilde{R}$-Implicit Budget Leximin ordering: For every $z_N, z'_N \in Z$,

$$[z_N \;\; z'_N] \Leftrightarrow$$

$$\left(m\left(\tilde{R}, IB(s_i, R_i, z_i)\right)\right)_{i\in N} \;(\tilde{R})\; \left(m\left(\tilde{R}, IB(s_i, R_i, z'_i)\right)\right)_{i\in N}$$

It is clear that a $\tilde{R}$-Implicit Budget Leximin ordering satisfies *Pareto* and *Responsibility*. It also satisfies *Compensation* restricted to the subpopulation of agents with preferences $\tilde{R}$. Moreover, it is proven in Fleurbaey and Maniquet (1998) that it is the only ordering satisfying these three properties together with some additional minor requirement. As a consequence, we will concentrate, in this section, on the income tax schemes a planner must design if she wishes to maximize $\tilde{R}$-Implicit Budget Leximin orderings. The properties of such optimal tax schemes are presented in the following proposition, which is the main result of this section. This proposition allows us to study the relationship between the choice of reference preferences $\tilde{R}$ and the shape of the corresponding optimal tax scheme. First we introduce some terminology and definition. The Reference Tax Scheme associated with some reference preferences is the lowest feasible tax scheme having the property that the reference agent is indifferent between the implicit budget sets of all agents choosing a labor time equal to 1.

Given $\tilde{R} \in \mathcal{R}$, the **Reference Tax Scheme** $\tilde{\tau}_{\tilde{R}}$: for all $z_N \in Z(\tilde{\tau}_{\tilde{R}})$, $i, j \in N$,

1. if $l_i = l_j = 1$, then

$$m\left(\tilde{R}, IB(s_i, R_i, z_i)\right) \; \tilde{I}\, m\left(\tilde{R}, IB(s_j, R_j, z_j)\right),$$

2. $\overline{B}(z_N) = 0$, and

3. for all $\tau \in \mathcal{T}$ satisfying properties 1 and 2, $\tau(y) \geq \tilde{\tau}_{\tilde{R}}(y)$, for all $y \in [0, \overline{s}]$.

Lemma *Under* Assumption A6*, for all* $\tilde{R} \in \mathcal{R}$*, the Reference Tax Scheme* $\tilde{\tau}_{\tilde{R}}$ *exists and is convex.*

As shown below, the reference tax scheme always plays the role of an upper bound on the optimal tax scheme, and moreover, is itself the optimal tax scheme in large classes of cases. The convexity property (which implies increasing marginal income tax rates) stated in the above lemma is, therefore, particularly interesting.

We also introduce this notation. For $R \in \mathcal{R}$, and $\overline{c} \in \mathbb{R}_+$, let $MRS_R(0, \bar{c})$ denote the supremum of numbers s such that

$$(0, \bar{c}) \in m\left(R, \{(l, c) \in X \mid c \leq \bar{c} + sl\}\right).$$

Note that $MRS_R(0, \bar{c})$ is equal to the marginal rate of substitution at $(0, \bar{c})$ when it is well defined.

Proposition 3 *If a social planner wishes to maximize an* $\tilde{R}$-Implicit Budget Leximin *ordering, then she chooses a tax scheme* $\tau \in \mathcal{T}$ *having the following properties:*

- *[lower bound] under* Assumptions A2, A4 *and* A5*, for all* $y \in [0, \overline{s}]$*,*

  $$\tau(y) \geq \tau(0),$$

- *[upper bound] under* Assumptions A3 *and* A6*, for all* $y \in [0, \overline{s}]$ *such that there is* $i \in N : s_i l_i = y$*,*

  $$\tau(y) \leq \tilde{\tau}_{\tilde{R}}(y)$$

- *[zero marginal tax rate for low income] under* Assumptions A2*,* A4 *and* A5*, for all* $y \in [0, MRS_{\tilde{R}}(0, \Leftrightarrow\tau(0))]$*,*

  $$\tau(y) = \tau(0).$$

The lower bound property forbids a policy similar to the Earned Income Tax Credit that would subsidize low incomes by means of a negative marginal tax rate.

The property of zero marginal tax rate for low income is striking because it justifies the widespread practice of tax exemption below an income threshold.

But notice that it also implies the less common practice of granting the whole amount of the minimum income to all agents below that threshold.

It is interesting to give examples of reference tax schemes as a function of the reference preferences, and of the optimal tax schemes which can be deduced from them. Let us first study the case where $\tilde{R}$ is defined by $(l,c)\,\tilde{R}\,(l',c') \Leftrightarrow c \Leftrightarrow Ml \geq c' \Leftrightarrow Ml'$, with a very large M, that is, the reference agent would almost always choose a labor time equal to zero. We easily compute $\tilde{\tau}_{\tilde{R}}(y) = 0$ for all $y \in [0, \overline{s}]$, so that the optimal tax scheme is simply $\tau(y) = 0$ for all $y \in [0, \overline{s}]$. This means that the laissez-faire allocations can actually be justified with respect to *Responsibility* when combined with *Compensation* restricted to "lazy" agents.

At the other extreme, let us study the case where $\tilde{R}$ is defined by (l,c) $\tilde{R}$ $(l',c') \Leftrightarrow c \Leftrightarrow \varepsilon l \geq c' \Leftrightarrow \varepsilon l'$ for a very small ε, that is, the reference agent would almost always choose a maximal labor time. We easily see that the reference agent's best point associated with the implicit budget of any agent is equal to or above $(1, \Leftrightarrow\tau(0))$, so that maximizing the social ordering is equivalent to maximizing $\Leftrightarrow\tau(0)$ (this tax scheme may actually be chosen for a large range of ε, depending on the distribution of characteristics). We come back to the result that maximizing the minimal income can also be justified from a responsibility point of view (see the comments at the end of section 4, combined with Proposition 1).

There are of course a large class of intermediary cases. From a practical point of view, the most interesting cases are certainly those where $\tilde{R}$ is quasi linear with piecewise linear indifference curves. If preferences $\tilde{R}$ can be defined by parameters $[l_0, l_1, \ldots, l_T] \in [0,1]^{T+1}$ and $[s_0, s_1, \ldots, s_T] \in [0, \overline{s}]^{T+1}$ such that $l_0 = 0$, $s_0 = 0$, $l_T = 1$, and each indifference curve has slope s_t in the interval (l_{t-1}, l_t), for $t \in \{1, \ldots, T\}$, then the upper bound tax scheme is defined by: incomes in the interval (s_{t-1}, s_t) are taxed at the marginal rate of l_{t-1}, for $t \in \{1, \ldots, T\}$.

Our last result, stated below, identifies a rather large class of cases where $\tilde{\tau}_{\tilde{R}}$ is binding, and, therefore, in these cases, gives a general formula for the explicit shape of an optimal τ.

Proposition 4 *Assume that the economy has the following characteristics:*

1. *preferences R_i are represented by functions $u_i(l,c) = c \Leftrightarrow \alpha_i v(l)$, where v is continuously differentiable over $[0,1)$, and $v' > 0$, $v'' > 0$;*

2. *there is* $\bar{\alpha} > \overline{s}/v'(0)$ *such that for every* s*, the preference parameter* α_i *is uniformly distributed on the interval* $[0, \bar{\alpha}]$.

Let the reference preferences $\tilde{R}$ *be represented by* $\tilde{u}(l, c) = c \Leftrightarrow \tilde{\alpha} v(l)$. *If* $\tilde{\alpha} \geq \bar{s}/v'(1/2)$*, then the upper bound tax* $\tilde{\tau}_{\tilde{R}}$ *defined by:*

$$\tilde{\tau}'_{\tilde{R}}(y) = \begin{cases} 0 & y < \tilde{\alpha} v'(0) \\ (v')^{-1}(y/\tilde{\alpha}) & y > \tilde{\alpha} v'(0) \end{cases}$$

(and $\tilde{\tau}_{\tilde{R}}(0)$ *being determined by the budget constraint) is a local optimum for the* $\tilde{R}$*-Implicit Budget Leximin ordering.*

In conclusion, the family of $\tilde{R}$-Implicit Budget Leximin orderings authorizes a large class of optimal tax schemes, and is compatible with all degrees of inequality aversion with respect to incomes, by an appropriate choice of the reference preferences. But this does not mean that it covers the whole spectrum of second best allocations (allocations that are Pareto optimal in the subset of incentive compatible allocations), as shown by Proposition 3.

7 Conclusion

To conclude, we comment on the methodological point made in this paper, and on public economics lessons which can be drawn from our inquiry.

1. Our main objective in this paper was methodological. We wanted to prove that a second best policy involving redistribution can be chosen without any interpersonal comparison of well-being. In order to demonstrate this possibility, we have shown here how precise policy recommendations can be derived from ethical values captured by purely ordinal social orderings.

   The model adopted for this purpose is a variant of the basic income tax model due to Mirrlees, and the social objectives that have been relied upon have an essentially illustrative purpose, although we think that the underlying ethical values are sufficiently appealing to make the exercize meaningful. A more precise justification of these social objectives is made in Fleurbaey and Maniquet (1998).

2. Although our point is primarily methodological, we consider that the results have some interest of their own. Two insights seem to be worth emphasizing in this conclusion. First, the social objective of maximizing the minimum income is rather easy to justify in our approach, when agents with zero productivity are present in the population. Second, it is well known that it is difficult in the standard optimal tax framework to find increasing marginal tax rates, and even more rare to find tax exemption under an income threshold, whereas both features are displayed by some optimal tax schemes in our setting.

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Appendix

Proof of Proposition 1 Notice that for all $\tau \in \mathcal{T}$, $z_N \in Z(\tau)$, $i, j \in N$ such that $R_i = R_j = R$,

$$s_i > s_j \Rightarrow z_i \, R \, z_j.$$

Now, assume that τ and τ' in $\mathcal{T}$ are such that $\Leftrightarrow\tau'(0) > \Leftrightarrow\tau(0)$ and consider any $z_N \in Z(\tau)$ and $z'_N \in Z(\tau')$. One has $z'_i \, P_i \, z_i$ for all $i \in N$ such that $s_i = 0$.

First case. If for all $i \in N$, $z'_i \, R_i z_i$, then by *Pareto* $z'_N \mathrm{P} z_N$.

Second case. There is $i \in N$, $z_i \, P_i z'_i$. For every $R \in \mathcal{R}$, we define $S^R \subseteq [0, \overline{s}]$ by: $S^R = \{s \in [0, \overline{s}] \mid \exists i \in N, \; R_i = R, \; s_i = s, \; z_i \, P \, z'_i\}$. Necessarily $S^R \subset [0, \overline{s}]$. Choose any $R \in \mathcal{R}$ such that $S^R \neq \emptyset$, any $s \in S^R$, and $i \in N$ such that $R_i = R$ and $s_i = s$. By A1 there is $j \in N$ such that $R_j = R$ and $s_j = 0$, and by the above remark, $z'_i \, R \, z'_j$. But since $z'_j \, P \, z_j$, by *Compensation* one must have $z'_N \mathrm{P} z_N$. This implies that τ cannot be chosen if there is $\tau' \in \mathcal{T}$ such that $\Leftrightarrow\tau'(0) > \Leftrightarrow\tau(0)$. Q.E.D.

Proof of Proposition 2. Let $T(y) = \tau(y) \Leftrightarrow \tau(0)$. Under the feasibility constraint, maximizing $\Leftrightarrow\tau(0)$ is equivalent to maximizing the total amount of "net taxes" $T(y)$ paid by all agents. Let us fix s, and consider the reduced problem of maximizing $T(y)$ paid by agents having skill s. Let G_s^T denote the cumulative distribution function of l over $[0, 1]$ in some allocation obtained when the "net tax" function T is adopted. The total "net tax" TR is equal to:

$$TR = \int_0^1 T(sl) dG_s^T(l).$$

Assume that T is continuously differentiable. By applying an integration by part to this Stieltjes integral (which is allowed in particular if G_s^T, which is monotone and bounded on $[0,1]$, is also continuous on $(0,1)$), one obtains:

$$\begin{aligned} TR &= T(s)G_s^T(1) \Leftrightarrow T(0)G_s^T(0) \Leftrightarrow \int_0^1 sT'(sl)G_s^T(l)dl \\ &= T(s) \Leftrightarrow \int_0^1 sT'(sl)G_s^T(l)dl. \end{aligned}$$

An agent with parameter α will choose l such that

$$\alpha = \frac{s(1 \Leftrightarrow T'(sl))}{v'(l)}.$$

There may be several solutions l to this equation, including a continuum of solutions. By assumption, $T'(l) \leq 1$. And there is no loss of generality in considering only functions T such that for any α, the set of solutions l to the above equation is connected (actually a closed interval, since T' and v' are continuous). Indeed, if an agent with α is indifferent between optimal choices l_1 and l_2, no other agent will ever choose $l \in (l_1, l_2)$, and one can replace T with $\hat{T}$ so that an agent with exactly α is indifferent between all $l \in [l_1, l_2]$, without changing TR, since the distribution of α is non-atomic. Then the expression $\frac{s(1-\hat{T}'(sl))}{v'(l)}$ is constant over $[l_1, l_2]$. With these restrictions on T, the function

$$G(l) = 1 \Leftrightarrow \frac{s(1 \Leftrightarrow T'(sl))}{\bar{\alpha}_s v'(l)}.$$

is continuous on $[0,1)$, weakly increasing and such that $G(l) \leq 1$. If $G(l) < 0$, one has

$$\frac{s(1 \Leftrightarrow T'(sl))}{v'(l)} > \bar{\alpha}_s,$$

so that no agent will choose l. If $l \in [0,1)$ and $G(l) \geq 0$, then $G(l)$ is the proportion of agents who work l or less. One sees that

$$\forall l \in [0,1),\ G_s^T(l) = \max\{G(l), 0\}, \quad G_s^T(1) = 1.$$

This shows in particular that G_s^T is continuous on $(0,1)$, and justifies the integration by part made above. Let $\hat{l}$ be defined by

$$\forall l \in [0,\hat{l}),\ G_s^T(l) = 0 \quad \forall l \in [\hat{l},1),\ G_s^T(l) = G(l).$$

One computes

$$
\begin{aligned}
TR &= T(s) \Leftrightarrow \int_{\hat{l}}^{1} sT'(sl)G(l)dl \\
&= T(s) \Leftrightarrow \int_{\hat{l}}^{1} sT'(sl)dl + \int_{\hat{l}}^{1} sT'(sl)\frac{s(1 \Leftrightarrow T'(sl))}{\bar{\alpha}_s v'(l)}dl \\
&= T(s\hat{l}) + \frac{s^2}{\bar{\alpha}_s}\int_{\hat{l}}^{1} T'(sl)\frac{(1 \Leftrightarrow T'(sl))}{v'(l)}dl.
\end{aligned}
$$

Notice that since $\bar{\alpha}_s \geq s/v'(0)$, the only way to have $G(0) < 0$ is to have $T'(0) < 0$. But then either there are some agents who work l with $T(sl) < 0$, in which case $T(s\hat{l}) < 0$, or there is no change in TR by changing T to $\hat{T}$ such that $\hat{T} \geq 0$, and the corresponding $\hat{G}(0) \geq 0$. As a consequence, maximizing TR is obtained by setting $T'(y) = 1/2$, so that $T(y) = y/2$.

Since this solution is independent of s, it is the general solution, for any distribution of the parameter s. Q.E.D.

Proof of the Lemma: Let us fix $\tilde{R} \in \mathcal{R}$. Let $g : [0, \overline{s}] \times \mathbb{R}_+ \Leftrightarrow\!\rightarrow \mathbb{R}_+$ be defined by

$$g(s, \overline{c}) = y \Leftrightarrow (0, \bar{c})\, \tilde{I}\, m \left(\tilde{R}, \{(l, c) \in X \mid c \leq y \Leftrightarrow s + sl\}\right)$$

that is, g associates skill s and consumption level $\overline{c}$ with the full income[12] corresponding to the budget set leaving a reference agent with skill s indifferent with consuming the bundle $(0, \overline{c})$. See Fig. 1. It is easy to check that g is continuous and strictly increasing in both arguments.

Take two skills s and s', and consider the two budget lines of equations

$$c = g(s, \overline{c}) \Leftrightarrow s + sl$$

and

$$c = g(s', \overline{c}) \Leftrightarrow s' + s'l$$

respectively. The intersection of these two lines has a first coordinate equal to:

$$l = 1 \Leftrightarrow \frac{g(s, \overline{c}) \Leftrightarrow g(s', \overline{c})}{s \Leftrightarrow s'}$$

[12]Let us recall that the full income corresponding to a given budget set is the income an agent facing this budget set would earn, should she choose the maximal labor time.

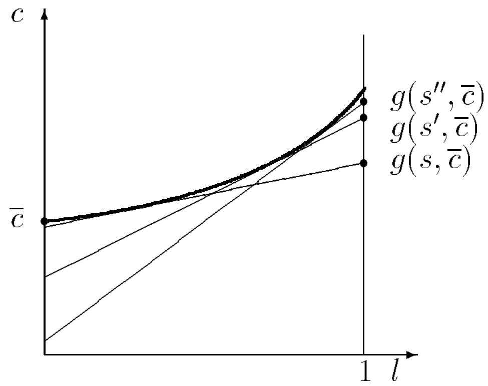

Figure 1:

By a straightforward revealed preference argument (see Fig. 1), $s < s' < s''$ implies that intersections of budget lines are located such that

$$1 \Leftrightarrow \frac{g(s,\overline{c}) \Leftrightarrow g(s',\overline{c})}{s \Leftrightarrow s'} \leq 1 \Leftrightarrow \frac{g(s,\overline{c}) \Leftrightarrow g(s'',\overline{c})}{s \Leftrightarrow s''}.$$

Define $\lambda \in (0,1)$ by

$$s' = \lambda s + (1 \Leftrightarrow \lambda) s''.$$

One then has:

$$1 \Leftrightarrow \frac{g(s,\overline{c}) \Leftrightarrow g(\lambda s + (1 \Leftrightarrow \lambda) s'',\overline{c})}{s \Leftrightarrow \lambda s \Leftrightarrow (1 \Leftrightarrow \lambda) s''} \leq 1 \Leftrightarrow \frac{g(s,\overline{c}) \Leftrightarrow g(s'',\overline{c})}{s \Leftrightarrow s''}$$

or equivalently,

$$\lambda g(s,\overline{c}) + (1 \Leftrightarrow \lambda) g(s'',\overline{c}) \leq g(\lambda s + (1 \Leftrightarrow \lambda) s'',\overline{c})$$

which proves that g is concave in s.

Notice that

$$1 \Leftrightarrow \frac{g(s,\overline{c}) \Leftrightarrow g(s',\overline{c})}{s \Leftrightarrow s'} \geq 0$$

also reads:

$$[s \Leftrightarrow g(s,\overline{c}) \Leftrightarrow (s' \Leftrightarrow g(s',\overline{c}))]\,[s \Leftrightarrow s'] \geq 0$$

implying that the expression $s \Leftrightarrow g(s,\overline{c})$ is weakly increasing in s.

Let us consider the family $\tau_{\overline{c}}$ defined by $\overline{c} \in [0, \overline{s}]$ and $\tau_{\overline{c}}(y) = y \Leftrightarrow g(y, \overline{c})$. By concavity of g in its first argument, all functions $\tau_{\overline{c}}$ are convex. Moreover, they are weakly increasing.

By A6, $\overline{B}(Z(\tau_{\overline{c}}))$ are all singletons, so that by the continuity properties of g, and therefore of $\tau_{\overline{c}}$, $\overline{B}(Z(\tau_{\overline{c}}))$ is continuous with respect to $\overline{c}$. Now, by construction, for all $s \in [0, \overline{s}]$, $g(s, 0) \leq s$, so that for all $y \in [0, \overline{s}]$, $\tau_0(y) \geq 0$ and $\overline{B}(Z(\tau_0)) \geq 0$. On the other hand, for all $y \in [0, \overline{s}[$, $\tau_{\overline{s}}(y) < 0$ so that $\overline{B}(Z(\tau_0)) < 0$. By an intermediary value argument, at least one tax scheme exists with a balanced budget, and if several such tax schemes exist, one of them is associated with the highest $\overline{c}$, say $\overline{c}_m$. We easily check that $\tau_{\overline{c}_m}$ is the *Reference Tax Scheme* associated with $\tilde{R}$. This proves its existence. Q.E.D.

Proof of Proposition 3: We first prove the *lower bound* property. Let $\tau \in \mathcal{T}$ be chosen by the planner, with $\tau(y^*) < \tau(0)$ for some $y^* \in \mathbb{R}_+$. Let $z_N \in Z(\tau)$ be given. Let $\hat{\tau}$ be defined by:

$$\hat{\tau}(y) = \max\{\tau(y), \tau(0)\}.$$

The function $\hat{\tau}$ is continuous, belongs to $\mathcal{T}$ and satisfies: for all $y \in \mathbb{R}_+$, $\hat{\tau}(y) \geq \hat{\tau}(0)$. For all $\hat{z}_N \in Z(\hat{\tau})$, $i \in N$, $\hat{\tau}(\hat{y}_i) \geq \tau(y_i)$, and $\tau(y_i) < \tau(0)$ implies $\hat{\tau}(\hat{y}_i) > \tau(y_i)$.

Necessarily there exist $y_1 < y_2$ such that the function $y \Leftrightarrow \tau(y)$ is strictly increasing over (y_1, y_2) and for all $y \in (y_1, y_2)$, $\tau(y) < \tau(0)$. By A2 and A5,

$$\int_{y_i \in (y_1, y_2)} \tau(y_i) > 0$$

and moreover

$$\int_{y_i \in (y_1, y_2)} \hat{\tau}(\hat{y}_i) \Leftrightarrow \tau(y_i) > 0$$

This entails that since by the feasibility constraint, $\bar{B}(z_N) \leq 0$, one must have $\bar{B}(\hat{z}_N) < 0$ for all $\hat{z}_N \in Z(\hat{\tau})$. As a consequence, by A5 there exists $\varepsilon > 0$ such that the tax defined by $\check{\tau}(x) = \hat{\tau}(x) \Leftrightarrow \varepsilon$ belongs to $\mathcal{T}$ and satisfies: for all $\check{z}_N \in Z(\check{\tau})$, $\bar{B}(\check{z}_N) \leq 0$.

We now prove that for all $s \in [0, \overline{s}]$, the lowest implicit budget among agents $i \in N$ having $s_i = s$ is higher at any $\check{z}_N \in Z(\check{\tau})$ than at z_N. Fix s and let $\underline{l}_{s\tau} \in [0, 1]$ be such that $\tau(sl) \leq \tau(s\underline{l}_{s\tau})$ for all $l \in [0, 1]$. If there is $i \in N$ with $s_i = s$ and $l_i = \underline{l}_{s\tau}$, then this agent i's implicit budget is the lowest, and is equal to:

$$\underline{IB}_{s\tau} = \{(l, c) \in X | c \leq sl \Leftrightarrow \tau(s\underline{l}_{s\tau})\}.$$

If the function $y \Leftrightarrow \tau(y)$ is strictly increasing over some interval $(s\underline{l}_{s\tau} \Leftrightarrow \mu, s\underline{l}_{s\tau} + \mu)$, then by A2 $\underline{IB}_{s\tau}$ is the infimum of implicit budgets among agents with skill s. If the function $y \Leftrightarrow \tau(y)$ is not strictly increasing over any interval $(s\underline{l}_{s\tau} \Leftrightarrow \mu, s\underline{l}_{s\tau} + \mu)$, then by a slight perturbation it can be made strictly increasing over some such interval. And by continuity of preferences and of tax functions, in this perturbation the agents' implicit budgets can be made arbitrarily close to their implicit budgets under τ. Therefore it is true, in all cases, that $\underline{IB}_{s\tau}$ is the infimum of implicit budgets among agents with skill s. Similarly, under $\check{\tau}$, the infimum of implicit budgets is equal to

$$\underline{IB}_{s\check{\tau}} = \{(l,c) \in X | c \leq sl \Leftrightarrow \check{\tau}(s\underline{l}_{s\check{\tau}})\}.$$

If $\tau(s\underline{l}_{s\check{\tau}}) \geq \tau(0)$, then $\check{\tau}(s\underline{l}_{s\check{\tau}}) = \tau(s\underline{l}_{s\check{\tau}}) \Leftrightarrow \varepsilon \leq \tau(s\underline{l}_{s\tau}) \Leftrightarrow \varepsilon < \tau(s\underline{l}_{s\tau})$. If, on the contrary, $\tau(s\underline{l}_{s\check{\tau}}) < \tau(0)$, then $\check{\tau}(s\underline{l}_{s\check{\tau}}) = \tau(0) \Leftrightarrow \varepsilon \leq \tau(s\underline{l}_{s\tau}) \Leftrightarrow \varepsilon < \tau(s\underline{l}_{s\tau})$. Therefore, in both cases, $\check{\tau}(s\underline{l}_{s\check{\tau}}) < \tau(s\underline{l}_{s\tau})$. This proves that $\underline{IB}_{s\tau} \subset \underline{IB}_{s\check{\tau}}$.

Let us now prove the *upper bound* property. Let $z_N \in Z(\tau)$. Let $\tilde{\tau}_{\tilde{R}}$ be the Reference Tax Scheme associated to $\tilde{R}$. Let us assume that there exist $y^* \in [0, \overline{s}]$, and $i \in N$ such that $s_i l_i = y^*$ and $\tau(y^*) > \tilde{\tau}_{\tilde{R}}(y^*)$. By strict monotonicity of i's preferences, $y \Leftrightarrow \tau(y)$ is strictly increasing over some interval $[y^* \Leftrightarrow \mu, y^*]$. Choose μ small enough so that for all $y \in [y^* \Leftrightarrow \mu, y^*]$, $\tau(y) > \tilde{\tau}_{\tilde{R}}(y)$. By A3, there exists $j \in N$ such that $s_j \in [y^* \Leftrightarrow \mu, y^*]$ and $l_j = 1$.

In the proof of the Lemma, it was shown that $\tilde{\tau}_{\tilde{R}}$ is weakly increasing. Therefore, for all $s \in [0, \overline{s}]$, $\underline{l}_{s\tilde{\tau}} = 1$, where $\underline{l}_{s\tilde{\tau}}$ is defined as above. As a consequence, by A3, the infimum of implicit budgets for agents with skill s, under $\tilde{\tau}_{\tilde{R}}$, is equal to

$$\underline{IB}_{s\tilde{\tau}_{\tilde{R}}} = \{(l,c) \in X | c \leq \Leftrightarrow \tilde{\tau}(s) + sl\}$$

Now, it is clear that

$$IB(s_j, R_j, z_j) \subseteq \{(l,c) \in X | c \leq \Leftrightarrow \tau(s_j) + s_j l\}.$$

Since $\tau(s_j) > \tilde{\tau}(s_j)$, we have $IB(s_j, R_j, z_j) \subset \underline{IB}_{s_j\tilde{\tau}_{\tilde{R}}}$. But by construction of $\tilde{\tau}_{\tilde{R}}$, $\underline{IB}_{s_j\tilde{\tau}_{\tilde{R}}}$ (like any $\underline{IB}_{s\tilde{\tau}_{\tilde{R}}}$) has the lowest value for the reference preferences, that is, for all $i \in N$, $\tilde{z}_N \in Z(\tilde{\tau}_{\tilde{R}})$,

$$m(\tilde{R}, IB(s_i, R_i, \tilde{z}_i))\tilde{R}m(\tilde{R}, \underline{IB}_{s_j\tilde{\tau}_{\tilde{R}}})$$

As a consequence, necessarily $\tilde{z}_N \text{ P } z_N$, the desired contradiction.

Finally, let us prove the property of *zero marginal tax rate for low income.* Let $z_N \in Z(\tau)$. First we claim that for all $i \in N$,

$$m(\tilde{R}, IB(s_i, R_i, z_i))\, \tilde{R}\, (0, \Leftrightarrow\tau(0)).$$

Let us suppose that the claim is false. Then there exist $c^* < \Leftrightarrow\tau(0)$ and $j \in N$, such that

$$(0, c^*)\, \tilde{I}\, m\left(\tilde{R}, IB(s_j, R_j, z_j)\right).$$

Let $\tau^* \in \mathcal{T}$ be defined by:

$$\tau^*(y) = \{\Leftrightarrow c^*, \tau(y)\}.$$

The function τ^* is continuous, and there exists $x^* > 0$ such that for all x, $0 \leq x \leq x^*$, one has $\tau^*(x) > \tau(x)$. Let $z_N^* \in Z(\tau^*)$. For all $i \in N$, $\tau^*(y_i^*) \geq \tau(y_i)$. Either the function $y \Leftrightarrow \tau(y)$ is constant over $[0, x^*]$, or it is strictly increasing over some interval contained in $[0, x^*]$. In both cases, by A2 there exists an agent $k \in N$ such that $\tau^*(y_k^*) > \tau(y_k)$. Therefore, by A4 and A5, $\bar{B}(z_N^*) < 0$. As a consequence, there exists $\varepsilon \in \mathbb{R}_{++}$ and $\hat{\tau} \in \mathcal{T}$ defined by $\hat{\tau}(x) = \tau^*(x) \Leftrightarrow \varepsilon$, such that for all $\hat{z}_N \in Z(\hat{\tau})$, $\bar{B}(\hat{z}_N) \leq 0$.

As above, by A2, for any tax function τ_0, the infimum implicit budget among agents with skill s is equal to $\underline{IB}_{s\tau_0}$. Let $\underline{IB}_{\tau_0}$ denote (a selection of) the $\underline{IB}_{s\tau_0}$ such that for all $s' \in [0, \overline{s}]$,

$$m\left(\tilde{R}, \underline{IB}_{s'\tau_0}\right) \tilde{R} m\left(\tilde{R}, \underline{IB}_{s\tau_0}\right)$$

For all $s \in [0, \overline{s}]$, by construction, $\underline{IB}_{s\tau^*} \subset \underline{IB}_{s\hat{\tau}}$ and $\underline{IB}_{s\tau^*} \subseteq \underline{IB}_{s\tau}$. Now, either $\underline{IB}_{s\tau^*} = \underline{IB}_{s\tau}$ or

$$\underline{IB}_{s\tau^*} = \{(l, c) \in X | c \leq c^* + sl\}$$

and in the latter case

$$m\left(\tilde{R}, \underline{IB}_{s\tau^*}\right)\, \tilde{R}\, (0, c^*)$$

Since

$$(0, c^*)\, \tilde{I}\, m\left(\tilde{R}, IB(s_j, R_j, z_j)\right),$$

it turns out that

$$m\left(\tilde{R}, \underline{IB}_{\tau^*}\right) \tilde{I} m\left(\tilde{R}, \underline{IB}_{\tau}\right).$$

And as

$$m\left(\tilde{R}, \underline{IB}_{\hat{\tau}}\right) \tilde{P} m\left(\tilde{R}, \underline{IB}_{\tau^*}\right),$$

one must have $\hat{z}_N \mathrm{P} z_N$, the desired contradiction. This proves the claim.

Let $y \leq MRS_{\tilde{R}}(0, \Leftrightarrow\tau(0))$. If $\tau(y) > \tau(0)$, then

$$\underline{IB}_{y\tau} \subseteq \{(l,c) \in X | c \leq \Leftrightarrow\tau(y) + yl\} \subset \{(l,c) \in X | c \leq \Leftrightarrow\tau(0) + yl\}.$$

But since

$$(0, \Leftrightarrow\tau(0)) \in m\left(\tilde{R}, \{(l,c) \in X | c \leq \Leftrightarrow\tau(0) + yl\}\right),$$

this implies that

$$(0, \Leftrightarrow\tau(0))\tilde{P} m\left(\tilde{R}, \underline{IB}_{y\tau}\right),$$

in contradiction to the above claim. By the lower bound property, $\tau(y) \geq \tau(0)$. Therefore $\tau(y) = \tau(0)$. Q.E.D.

Proof of Proposition 4: One has to show that there is no other continuous tax function τ satisfying the budget constraint and such that $\tau \leq \tilde{\tau}$, $\tau \neq \tilde{\tau}$. It is enough to show the result for a Dirac distribution of s, concentrated on any value $0 < s \leq \bar{s}$. Indeed, the reasoning below gives us the value of optimal marginal income tax rates. Since this result is independent of s, and since preferences are quasi linear, the optimal tax scheme associated with a population with any distribution of skills is obtained by using the same marginal rates and computing the tax amounts so as to balance the budget.

Let $\tilde{T}(y) = \tilde{\tau}(y) \Leftrightarrow \tilde{\tau}(0)$, $T(y) = \tau(y) \Leftrightarrow \tau(0)$, and $\delta(y) = T(y) \Leftrightarrow \tilde{T}(y)$. Since function g introduced in the proof of the Lemma is increasing in its first argument, for all $y, y' \in [0, \overline{s}]$ such that $y < y'$, $\tilde{\tau}(y') \Leftrightarrow \tilde{\tau}(y) \leq y' \Leftrightarrow y$. Therefore, by the assumption that $\overline{\alpha} > \frac{\overline{s}}{v'(0)}$, for all $\tilde{z}_N \in Z(\tilde{\tau})$, for all $s \in [0, \overline{s}]$, $\inf\{\tilde{l}_i | i \in N, s_i = s\} = 0$. Since τ is in a neighborhood of $\tilde{\tau}$, we also have for all $\tilde{z}_N \in Z(\tau)$, for all $s \in [0, \overline{s}]$, $\inf\{l_i | i \in N, s_i = s\} = 0$. Therefore, by computations similar to those in the proof of Proposition 2, we get

$$\begin{aligned}
\tilde{\tau}(0) &= \Leftrightarrow \frac{s^2}{\bar{\alpha}} \int_0^1 \tilde{T}'(sl) \frac{(1 \Leftrightarrow \tilde{T}'(sl))}{v'(l)} dl, \\
\tau(0) &= \Leftrightarrow \frac{s^2}{\bar{\alpha}} \int_0^1 T'(sl) \frac{(1 \Leftrightarrow T'(sl))}{v'(l)} dl.
\end{aligned}$$

One has

$$\begin{aligned}
\forall y,\ \tau(y) &\leq \tilde{\tau}(y) \Longleftrightarrow \forall y,\ \tau(0) + T(y) \leq \tilde{\tau}(0) + \tilde{T}(y) \\
&\Longleftrightarrow \forall y,\ \delta(y) \leq \tilde{\tau}(0) \Leftrightarrow \tau(0) \\
&\Longleftrightarrow \forall y,\ \delta(y) \leq \frac{s^2}{\bar{\alpha}} \int_0^1 \frac{1}{v'(l)} \Big[T'(sl)(1 \Leftrightarrow T'(sl)) \\
&\quad \Leftrightarrow \tilde{T}'(sl)(1 \Leftrightarrow \tilde{T}'(sl)) \Big] dl \\
&\Longleftrightarrow \forall l,\ \delta(sl) \leq \frac{s^2}{\bar{\alpha}} \int_0^1 \frac{1}{v'(l)} \delta'(sl)(1 \Leftrightarrow \tilde{T}'(sl) \Leftrightarrow T'(sl)) dl \\
&\Longleftrightarrow \forall l,\ \delta(sl) \leq \frac{s^2}{2\bar{\alpha}} \int_0^1 \frac{1}{v'(l)} \delta'(sl)(1 \Leftrightarrow 2\tilde{T}'(sl)) dl \\
&\quad + \frac{s^2}{2\bar{\alpha}} \int_0^1 \frac{1}{v'(l)} \delta'(sl)(1 \Leftrightarrow 2T'(sl)) dl \\
&\Longleftrightarrow \forall l,\ \delta(sl) \leq \frac{1}{2} \int_0^1 \delta(sl) dG_1(l) + \frac{1}{2} \int_0^1 \delta(sl) dG_2(l),
\end{aligned}$$

the last line being obtained by integration by part, with

$$G_1(l) = 1 \Leftrightarrow \frac{s(1 \Leftrightarrow 2\tilde{T}'(sl))}{\bar{\alpha} v'(l)}, \quad G_2(l) = 1 \Leftrightarrow \frac{s(1 \Leftrightarrow 2T'(sl))}{\bar{\alpha} v'(l)}$$

and $G_1(1) = G_2(1) = 1$. If $\tilde{T}'(sl) \leq 1/2$, the function G_1 is strictly increasing over $[0, 1)$, and if T is close to $\tilde{T}$, then G_2 is also increasing. Moreover, since $\bar{\alpha} v'(l) \geq \bar{\alpha} v'(0) \geq \bar{s} \geq s$, one then has $0 \leq G_1 \leq 1$ and $0 \leq G_2 \leq 1$. Therefore, if $\tilde{T}'(sl) \leq 1/2$ and if T is close to $\tilde{T}$, the two integrals above are means of the function δ computed with respect to two distributions whose cumulative distribution functions are G_1 and G_2, respectively. The inequality

$$\forall l,\ \delta(sl) \leq \frac{1}{2} \int_0^1 \delta(sl) dG_1(l) + \frac{1}{2} \int_0^1 \delta(sl) dG_2(l)$$

can then only hold if δ is constant. But since $\delta(0) = 0$, one must have $\delta \equiv 0$, that is, $T \equiv \tilde{T}$.

It remains to find conditions guaranteeing $\tilde{T}'(sl) \leq 1/2$. This is equivalent to $(v')^{-1}(sl/\tilde{\alpha}) \leq 1/2$. Since v' is increasing, this will hold for all l and all s if $(v')^{-1}(\bar{s}/\tilde{\alpha}) \leq 1/2$, namely, if $\bar{s}/\tilde{\alpha} \leq v'(1/2)$. Q.E.D.