Preference Aggregation versus Truth-tracking: Asymptotic Properties of a Related Story

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Abstract: This paper is concerned with the asymptotic behavior of some global quantities relating to weighted decision rules when the number of small voters tends to infinity. First, voting is assumed to be motivated by interests, so that the collective decision is 'preference aggregation'. Here the quantity whose asymptotic behavior is analyzed is the 'complaisance' of the decision-making body which was introduced by Coleman in 1971 as the 'power of a collectivity to act'. Second, decision-making is assumed to be 'truth-tracking', so that there is a right answer but voters only have a partial information and imperfect competence for detecting the truth. The quantity considered here is the collective competence of the decision-making body: the probability of its arriving at the correct decision. This is the problem considered by Condorcet's Jury Theorem. The paper provides a generalization of this celebrated theorem by reinterpreting complaisance in terms of errors in a statistical sense.

Keywords: Majority games; Weighted voting games; Complaisance; Condorcet's Jury Theorem.
JEL Classification: C 71; D71

1. INTRODUCTION

Decision rules can be characterized in terms of the way in which voting power of individuals is distributed – as represented for example by the Shapley–Shubik index (Shapley 1953) or Banzhaf measure (1965) – or by some global values. This paper is concerned with the latter, specifically one that was introduced by Coleman in his 1971 as the 'power of a collectivity to act' and the 'collective competence' as introduced by Condorcet in 1785.

Coleman defined the power of a collectivity to act as the a priori probability that a committee representing this collectivity will be able to pass a random bill that comes before it. The measure is simply the cardinality of winning coalitions divided by all possible coalitions. Formally, a *simple voting game* (SVG) is a collection $\mathcal{W}$ of subsets of an assembly N representing the *winning* coalitions; hence any $S \subset N$ with $S \notin \mathcal{W}$ is a *losing* coalition. The *power of a collectivity to act* A is defined by

$$A[\mathcal{W}] := \frac{|\mathcal{W}|}{2^n}, \tag{1}$$

where $n = |N|$. If we read $|\mathcal{W}|$ as the number of outcomes that lead to action, then A is defined as the *relative* number of voting outcomes leading to action. It reflects the ease with which the individual members' interests in a collective action can be translated into actual collective action. This ease is at a *minimum* if the collectivity operated under a decision rule in which each member has a veto – unanimity – since only the grand coalition (the total assembly) can initiate action, i.e. $A = 1/2^n$. If the committee operates under simple majority rule and has an odd number of members, then exactly half of the coalitions can initiate action (for an even number of members it is slightly less than one half). The power of the collectivity is at a *maximum* under what Rae (1969) has called a 'rule of individual initiative': where action can be initiated by a single individual, for example when s/he gives a fire alarm. In this case A is obtained by $A = 1 - (1/2^n)$. Unless n is very small A will be close to one.[1]

Following Felsenthal & Machover (1998, p.62) we can think of Coleman's A as measuring the propensity of a committee to approve a random proposal, i.e. the *complaisance*[2] of the rule $\mathcal{W}$.

The interest in Coleman's A is that it allows us to say something about the ability of a collectivity that uses voting to make its decisions not only to act, but as Coleman himself said, '... to act in accord with the aims or interests of some members, but often against the aims or interests of others. Thus for a collectivity of a given size, the greater the power of the collectivity to act, the more power it has to act against the interests of some of the members' (1971, p. 277). Aspremont et al. (1987) present axiomatically a general index of aggregate power in organizations which contains Coleman's A as a special case.

As an example for the use of A consider the evaluation of the decision rules for the Council of Ministers (CM) of the European Union (EU). Table 1 and 4 (see Appendix C for the latter) are taken from Felsenthal & Machover (2001)[3]. Table 1 gives the decision rules of the CM from 1958 to 1995. The greatest number of issues in EU parlance, except those concerned with the constitution of the EU itself, is decided by a rule known as *qualified majority voting* (QMV). From from 1958 to 1995, the QMV has been a purely weighted decision rule. In a weighted voting game each board member is assigned to a non-negative number as weight and a proposed act is adopted if the combined weight of those affirming it achieves a fixed absolute quota. Table 4 is taken from the Treaty of Nice (2001) and represents the decision

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[1] This generally reflects the situation in which a public good, or a public bad, can be supplied by only a few members of a collectivity.

[2] Alternatively Felsenthal & Machover introduce a *resistance coefficient* R which is a simple linear transformation of A, however, R allows for easier comparisons of decision rules. We shall focus on A simply because it is technically easier to handle than R.

[3] For further work on evaluating decision rules for the CM consider e.g. Baldwin et. al. (2000) and Leech (2002).

Table 1. QMV weights and quota, first five periods

Country	1958	1973	1981	1986	1995
Germany	4	10	10	10	10
Italy	4	10	10	10	10
France	4	10	10	10	10
Neth'lnds	2	5	5	5	5
Belgium	2	5	5	5	5
Lux'mbrg	1	2	2	2	2
UK		10	10	10	10
Denmark		3	3	3	3
Ireland		3	3	3	3
Greece			5	5	5
Spain				8	8
Portugal				5	5
Sweden					4
Austria					4
Finland					3
Total	17	58	63	76	87
Quota	12	41	45	54	62
Quota %	70.59	70.69	71.43	71.05	71.26
min#	3	5	5	7	8
A %	21.88	14.65	13.67	9.81	7.78

Note: '*Total*' gives the total weight sum. '*Quota*' is the absolute quota; '*Quota* %' indicates the relative quota; 'min#' gives the least number of members whose total weight equals or exceeds the (absolute) quota; 'A %' is the Coleman index (1) in percentage terms.

rule designed for the QMV in the EU's Council of Ministers following its prospective enlargement to 27 member states[4].

Table 1 and 4 show a dramatic decrease of A from 21.88 % in 1958 to 7.78 % in 1995 and to 1.66 % in 2004. With each enlargement the quotient was kept in the range of $71 \pm 0.5\,\%$. These numbers suggest that if the CM keeps enlarging, while keeping the quota more or less constant, the EU will tend to paralysis. In this respect, Felsenthal & Machover (2001, p.456) have noted[5]:

> 'A very important fact, which is apparently not widely realized, about weighted decision rules is that if the quota is pegged at a constant

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[4] The decision rule is not stated in the treaty in this simple form, as a weighted voting game; but it can be reduced to the form shown in Table 4. For details see Felsenthal & Machover (2001, Section 3).

[5] Here, we are paraphrasing Felsenthal & Machover who quote this phenomenon in terms of resistance.

> percentage of the sum of weights, and if the percentage is greater than 50%, then as the number of voters increases the complaisance tends to fall ...'

Felsenthal & Machover do not provide a proof of this claim, which has major practical importance, particularly for ex ante evaluation of committee design. If it is generally true, then it seems to suggest that as a committee expands we may have to adjust the quota in order to avoid creating an undue bias in favor of the status quo. However, this would disregard the fact that the exercises to decide upon usually differ in an essential way:

(1) *Preference aggregation:* A decision has to be made aggregating the *views* or *interests* of its members.
(2) *Truth-tracking:* There is a *true ordering* of the alternatives, i.e. from 'best to worst', and the task is to make the best ('truest') collective choice.

(for the terminology, see for example List & Goodin 2001).

For preference aggregation consider e.g. a decision of a political body on how to distribute subsidies or a decision by shareholders whether to replace the board. Such a process is typically concerned with choosing among alternative proposals for action on the relative merits of which the members hold different views.

However, truth-tracking asks for a decision from an epistemic point of view. Here, it makes sense to rate a decision in terms of 'optimality': the 'true' ordering of alternatives is, for example, an ordering from best to worst in terms of some ideal standard or criterion, such as the public interest or justice or efficiency, and the group is concerned to make the 'best' or 'correct' choice. Probably one of the most famous examples of a truth-tracking decision process occurred in 2003 when the US tried to legitimize the war against Iraq by an affirming vote of the United Nation Security Council. The affirmation of the Council hinged essentially upon the question whether it was true that Iraq had weapons of mass destruction. Further examples matching the second category are issues to vote upon like whether a relaxed dismissal protection leads to less unemployment or whether an investment strategy is profitable: there is a right answer, but voters only have partial information and imperfect competence for detecting the truth.

The probability of a single voter's choice being correct is taken to quantify the *competence* of a voter. Taking competence into account, the question arises if the pessimistic prognosis for an expanding committee with decreasing complaisance still holds on epistemic grounds, especially with regards to Condorcet's Jury Theorem: This theorem shows that on a dichotomous choice, individuals who all have the same competence above 0.5, can make collective decisions under simple majority rule with a probability of being right (collective competence) that approaches 1 as either the size of the group or the individual competence goes up. Briefly, if the aim is 'tracking the truth' then the Jury Theorem says that the *more* voters the *better* – contrary to the pessimistic prognosis of decreasing complaisance of a committee with an increasing member set.

This paper is concerned with the asymptotic behavior of complaisance and collective competence in weighted voting games when there are many small voters. Literature refers to this setup as 'oceanic games', however, research on asymptotic properties is almost exclusively concerned with *individual* voting power measures. To refer to the two classical indices, the work of Shapiro & Shapley (1978) gives an analysis of what happens to the Shapley-Shubik index in weighted majority games when the number of small voters tends to infinity. Dubey & Shapley (1979) investigate the corresponding properties of the Banzhaf index.[6]

The formal setup of this paper assumes that the voters are of two kinds: a fixed (possibly empty) set of 'major' voters with fixed weights (the *atomic* part), and a growing population of 'minor' voters whose total weight is also fixed but the individual weight of each minor voter becomes negligible (the *non-atomic* part). The paper provides a proof of the claim of Felsenthal & Machover as quoted above. Furthermore, it shows that although complaisance and collective competence differ conceptually the latter can be derived by a reinterpretation of complaisance in terms

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[6] In terms of affinity the present paper is closer to the paper of Dubey and Shapley (1979) since the Banzhaf and the Coleman measure are based on the same behavioral assumptions.

of errors in a statistical sense. This interpretation provides a formulation of collective competence by means of complaisance and hence allows the direct application of the asymptotic results of complaisance to the asymptotic behavior of collective competence.

Outline of the paper: We shall find that the (weighted) majority quota (50 percent quota) has an outstanding role. For weighted non-atomic games the paper shows that the 50 percent (weighted) majority rule is optimal with respect to both preference aggregating and truth tracking. The latter result is a generalization of the celebrated Condorcet Jury Theorem to weighted oceanic games.[7] In presence of big players the main result is that the asymptotic value of complaisance as well as collective competence is obtained by reducing the assembly of the game to the small set of major voters. Likewise the 50 percent (weighted) majority rule has an outstanding role. For preference aggregation this majority quota guarantees a suitable level of complaisance since this quota precludes the extreme values of zero (paralyzation) and one (anarchy). If the committee is interpreted as a knowledge aggregation machine heading for the truth of a matter, then this shift increases the probability that jury competence reaches infallibility. Moreover, the paper estimates the rate of convergence which turns out to be very high under reasonable smoothness conditions. Hence the limit values of both complaisance and collective competence serve as a convenient rule of thumb for large committees that apply weighted voting.

Section 2 sets up the probabilistic machinery that will be used throughout the paper. Section 3 formally defines the general setup of the games. Section 4 discusses the passage of the Coleman index A to the limit when the number of small voters grows to infinity. As an example Section 5 discusses the application of the results of the foregoing sections to the CM of the EU. Section 6 introduces the classical formulation of Condorcet's jury theorem. Section 7 proves a generalization of the latter which allows to discuss the previous results from an epistemic point of view.

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[7] Condorcet's statement refers to the simple majority rule - a simple and special case of a weighted voting game (an act is adopted if more voters vote for it than against it).

Section 8 provides statements estimating the rate of convergence. Finally, Section 9 concludes.

2. PRELIMINARIES

Let N be a nonempty finite set to which we shall refer as *assembly.* The elements of N are called *voters* and we shall often identify them with the integers $1, 2, ..., n$, where $n = |N|$. A play of the voting game consists in a *division*,[8] in which each voter chooses one of two options (usually, 'yes' and 'no'). Any subset of $S \subseteq N$ is called a *coalition.*

DEFINITION 2.1. *A weighted voting game – briefly, WVG –*

$$[c; w_1, w_2, ..., w_n] \tag{2}$$

is given by an assignment of a non-negative real weight w_k to each voter $k \in N$, and a relative Quota $c \in (0, 1)$ such that a coalition $S \subset N$ is a winning coalition iff

$$\sum_{k \in S} w_k \geq c \sum_{k \in N} w_k. \tag{3}$$

The loose inequality $\geq$ in (3) may be replaced by the strict inequality $>$. In this case we shall use the notation

$$< c; w_1, w_2, \ldots, w_n > . \tag{4}$$

We shall represent the choice of a voter $k \in N$ by a random variable X_k such that

$$X_k^{(\nu)} = \begin{cases} w_k & \text{if } k \text{ votes 'yes'}, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Our main tool, borrowed from probability theory, is a derived from a general version of the central limit theorem. We shall use the symbol Φ to denote the standard normal distribution. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of independent random

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[8] Here we follow Felsenthal and Machover (1997, p. 335) in borrowing the term from English parliamentary usage to denote the *collective* act of a voting body, whereby each individual member casts a vote.

variables, at least one of which has a non-degenerate distribution. Let the distribution of X_k be denoted by F_k, its expectation by $E[X_k] = \mu_k$ and assume its variance $Var[X_k] = \sigma_k^2$ to be finite. Further put

$$(s_n)^2 := Var\left[\sum_{k\leq n} X_k\right] = \sum_{k\leq n} \sigma_k^2,$$

$$S_n := \frac{1}{s_n}\sum_{k\leq n} X_k - \mu_k,$$

and

$$Q^{(n)} := \sum_{k\leq n} w_k^2.$$

Theorem 2.1. *For each k, let the independent random variable X_k be given by*

$$X_k = C_k w_k,$$

where the C_k are real-valued random variables with the same non-degenerate distribution on a compact set $[a,b]$ for all $k \in \mathbb{N}$. In order that

$$\lim_{n\to\infty} \sup_x |\text{Prob}\{S_n < x\} - \Phi(x)| = 0 \tag{6}$$

it is necessary and sufficient that the following condition be satisfied:

$$\lim_{n\to\infty} \frac{w_n}{\sqrt{Q^{(n)}}} = 0. \tag{7}$$

For a proof see Appendix A.

Remark 2.1. *For discussing the asymptotic properties of the global measures we shall use the normal distribution as an approximation tool. Theorem 2.1 validates this method* (6) *iff the weights of the voters are not too skewed* (7) *(iff the relative weights converge uniformly to zero).*

3. General Setup

Consider a partition of the set of voters N into two camps: we will denote the set of *major voters* in N by L which is given by $\{1, ..., l\}$, where l is a natural number. Note that $l = 0$ takes care of the case where L is empty by the general convention that $\{1, \ldots, 0\}$ is empty. The set of *minor voters* in N is denoted by $M^{(\nu)} = \{l+1, ..., l+m^{(\nu)}\}$.

We shall consider weighted voting situations as follows: there is a fixed quota c and a fixed set of major voters L, where each major voter is endowed with a fixed voting weight. These weights sum up to w_L, the combined voting weight of L. There is also a fixed total combined voting weight α of the minor voters $M^{(\nu)}$ such that the total weight sum is a fixed constant

$$W := w_L + \alpha. \tag{8}$$

However, the population number of $M^{(\nu)}$ grows to infinity whereas the individual weight of any minor voter tends to zero. Hence $M^{(\nu)}$ represents the non-atomic part of the game.

Let $\{\Gamma^{(\nu)}\}_{\nu \in \mathbb{N}}$ be a sequence of WVGs, as follows

$$\Gamma^{(\nu)} = [c; w_1, ..., w_l, \alpha_1^{(\nu)}, ..., \alpha_{m^{(\nu)}}^{(\nu)}]. \tag{9}$$

Put $Q^{(\nu)} := \sum_{k \leq m^{(\nu)}} \left[\alpha_k^{(\nu)}\right]^2$. Let $\{\Gamma^{(\nu)}\}_{\nu \in \mathbb{N}}$ evolve such that

$$\sum_{k \leq m^{(\nu)}} \alpha_k^{(\nu)} = \alpha, \quad \text{for each } \nu, \tag{10}$$

for a fixed $\alpha > 0$, and

$$\lim_{\nu \to \infty} \alpha_{max}^{(\nu)} / \sqrt{Q^{(\nu)}} = 0, \tag{11}$$

where $\alpha_{\max}^{(\nu)} := \max_{k \leq m^{(\nu)}} \alpha_k^{(\nu)}$.

Remark 3.1. *Note that (11) ensures*

$$\alpha_{\max}^{(\nu)} \to 0, \quad \textit{as } \nu \to \infty, \tag{12}$$

which implies $m^{(\nu)} \to \infty$.

However, it can be shown that $Q^{(\nu)}$ tends to zero so that condition (11) is stricter than (12).[9]

4. COMPLAISANCE IN WVGs

In the following we shall generalize Coleman's assumption that each coalition is equally likely.

ASSUMPTION 4.1. *We assume that the minor voters act independently and make an affirmative vote with probability $p \in (0,1)$.*

We shall use the notation $w(B) := \sum_{k \in B} w_k$ for the sum of the weights of the major 'yes' voters $B \subseteq L$. We shall regard $\alpha^{(\nu)}(S) = \sum_{k \in S} \alpha_k^{(\nu)}$ as the random weight sum of the affirming minor voters S, where S is a random subset of $M^{(\nu)}$. Let the random variable Y denote a random coalition among the major voters.

From Definition 2.1 of a WVG and complaisance (1) we get

$$A[\Gamma^{(\nu)}] = \sum_{B \subseteq L} \text{Prob}\,\{Y = B\}\,\text{Prob}\left\{\alpha^{(\nu)}(S) \geq c - w(B)\right\}. \tag{13}$$

Due to Assumption 4.1 we should expect that in the limit the continuous 'ocean' of randomly voting minor voters would be divided such that the affirming voters represent $p\alpha$ of the total minor weight sum α. In other words we should expect a p share of the minor voters stating 'yes'. This suggests to focus on the following games

$$\begin{aligned} \Gamma_0 &= [c - p\alpha; w_1, ..., w_l], \\ \Gamma_0' &= < c - p\alpha; w_1, ..., w_l > . \end{aligned} \tag{14}$$

Let $\mathcal{B}_l$ denote the unanimity game which consists of the grand coalition L only. Let $\mathcal{B}_l^*$ denote the dual[10] of $\mathcal{B}_l$, representing what Rae (1969) has called a 'rule of individual initiative': any coalition with size larger than or equal one is winning.

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[9] Let $\beta_\nu = \alpha_{max}^\nu$. Then $\sum_k (\alpha_k)^2 \leq \sum_k \alpha_k \beta_\nu \leq \beta_\nu \sum_k \alpha_k = \beta_\nu \alpha$. Now, since α is fixed, but $\beta_\nu \to 0$, it follows that $\sum_k (\alpha_k)^2$ also goes to 0.

[10] The *dual* of v is the game v^*, with the same grand coalition L as v, such that, for every $B \subseteq L$, $v^*(B) = v(B) - v(L - B)$.

Put

$$\mathcal{R} := \{c \,|\, 0 < c < w_L + \alpha\}\,, \tag{15}$$

$$\mathcal{J} := \{c \,|\, p\alpha < c < w_L + p\alpha\}\,. \tag{16}$$

Theorem 4.1. *In the sequence of games described by* (9)– (11), *we have*

$$\lim_{\nu\to\infty} A[\Gamma^{(\nu)}] = \frac{1}{2}A[\Gamma_0] + \frac{1}{2}A[\Gamma_0'] \qquad \text{if } c \in \mathcal{J}. \tag{17}$$

For other values of c *we have*

$$\lim_{\nu\to\infty} A[\Gamma^{(\nu)}] = \begin{cases} 1 & \text{if } c < p\alpha, \\ 1/2\,(1 + A[\mathcal{B}_l^*]) & \text{if } c = p\alpha, \\ 1/2\,A[\mathcal{B}_l] & \text{if } c = w_L + p\alpha, \\ 0 & \text{if } c > w_L + p\alpha. \end{cases} \tag{18}$$

For a proof see Appendix B.

Figure 4.1 illustrates the result for $p = 1/2$.

Figure 4.1: Complaisance in WVGs for $p = 1/2$

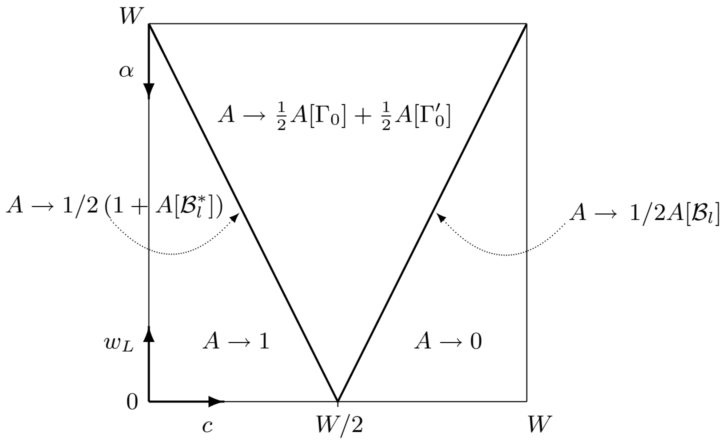

Figure 4.1 shows that the limit value of A is determined by the relation of the quota c to the total major weight w_L. At the interior region $\mathcal{J}$ the limit follows as

the arithmetical mean of complaisance of finite major games as defined in (14). Let the closure of $\mathcal{J}$ be denoted by $\bar{\mathcal{J}} := \{c \,|\, p\alpha \leq c \leq w_L + p\alpha\}$. In the domain $\mathcal{R} - \bar{\mathcal{J}}$ the influence of the major voters is 'destroyed': in the limit we have a combined voting weight of exactly $p\alpha$ affirming minor voters such that $c < p\alpha$ always ensures the passage of a proposal. The opposite holds for $w_L + p\alpha < c$ - even with all major voters affirming the quota c is too high for a motion to pass. With an increasing $p \geq 1/2$ the result as displayed in Figure 4.1 stays topologically the same, however, the intersection point $w_L = 0$ and $c = W/2$ shifts to the right. This is the *direct* effect of p. At the interior region $\mathcal{J}$ a changing p affects complaisance *indirectly* by manipulating the quota of the major games as defined in (14). Note that Theorem 4.1 does not require to specify any probabilities of affirmative voting of the major voters.

5. An Example: the EU Council of Ministers

This section applies the results on complaisance to the evolution of CM of the EU since 1958. The substantial characteristic of the process is that with each enlargement of the EU the maximal normalized voting weight decreases while the relative quota was kept more or less constant at 71% as indicated by Table 1 and Table 4. Hence the development of the CM can be described by games

$$\Gamma^{(\nu)} = [c; \alpha_1, ..., \alpha_\nu] \tag{19}$$

with an empty atomic part, i.e. we put the set of major voters $L = \emptyset$. We shall identify the five scenarios from $1958 - 1995$ and the QMV following its prospective enlargement to 27 as sequence elements $\Gamma^{(6)}, \Gamma^{(9)}, ..., \Gamma^{(15)}, \Gamma^{(27)}$, where the index denotes the size of the Council. Without loss of generality put $\alpha = 1$. The second row of Table 2 suggests that these games can be interpreted as elements of a sequence satisfying condition (11).

Following Coleman we set $p = 1/2$. The limit scenario for non-atomic weighted voting games (i.e. no major voters) is depicted by the horizontal axis $w_L = 0$ in

Table 2. Evolution of the CM

	$\Gamma^{(6)}$	$\Gamma^{(9)}$	$\Gamma^{(10)}$	$\Gamma^{(12)}$	$\Gamma^{(15)}$	$\Gamma^{(27)}$	$\dots$	
$\alpha_{\max}^{(\nu)}$	0.2353	0.1724	0.1587	0.1316	0.1149	0.0852	$\dots$	$\to 0$
$\alpha_{\max}^{(\nu)}/\sqrt{Q^{(\nu)}}$	0.5298	0.4603	0.4486	0.4131	0.3994	0.3627	$\dots$	$\to 0$

Figure 4.1. Theorem 4.1 provides for games with an empty set of major voters and $\alpha = 1$

$$\lim_{\nu\to\infty} A\left[\Gamma^{(\nu)}\right] = \begin{cases} 1 & \text{if } c < 1/2, \\ 1/2 & \text{if } c = 1/2, \\ 0 & \text{if } c > 1/2. \end{cases} \tag{20}$$

Figures 5.1 and 5.2 illustrate the scenario. Figure 5.1 gives $A\left[\Gamma^{(\nu)}\right]$ for the six scenarios $\Gamma^{(6)}, \Gamma^{(9)}, ..., \Gamma^{(27)}$ as a function of $c \in (0, 1)$. The points marked with ' $*$ ' are the corresponding realizations of c in CM as given by Table 1 and Table 4. The step function in the front indicates the limit values with increasing number of voters. As a reference scenario, Figure 5.2 provides the same picture for the symmetric weight distribution 'one person one vote' with $\alpha_k^{(\nu)} = \alpha/\nu$ for all $k = 1, ..., \nu$ which are qualitatively similar to the setting of the CM.

In all scenarios we observe that convergence tends to be relatively quick for $c \neq 0.5$. For a detailed convergence analysis see Section 8. The convergence rate is the higher, the closer c gets to the boundaries $c = 0$ and $c = 1$ as indicated by Theorem 8.2. This theorem provides an explaination for the rapid convergence of complaisance of the CM to zero by setting $\epsilon = c - 0.5 = 0.71 - 0.5 = 0.21$ and $m^{(\nu)} = \nu$.

Interestingly Figures 5.1 and 5.2 identify another indicator for the dramatic decrease of A in the process of enlarging EU: both display a high sensitivity in changes in c especially for low values of ν. It turns out that this sensitivity has a large impact. The first three columns of Table 1 indicate that the decrease of A had its origin partly in a slight increase in the quota from 70.59% in $\Gamma^{(6)}$ to 70.69% in $\Gamma^{(9)}$ and 71.43% in $\Gamma^{(10)}$. Row A_{71} in Table 3 gives complaisance of the voting systems if the quota c had been kept at constant 0.71% which represents the arithmetic mean of the quotas from 1958-1995. In this case the first scenario would have started already with a lower value $A\left[\Gamma^{(6)}\right]$ and hence the difference in comparison to the subsequent scenario would have been less significant. In fact, with a fixed quota $c = 0.71$ complaisance would have increased from 1973 to 1981.

Table 3. Complaisance of QMV with fixed 71%, first five periods

Country	1958	1973	1981	1986	1995
Quota %	70.59	70.69	71.43	71.05	71.26
A	0.2188	0.1465	0.1367	0.0981	0.0778
A_{71}	0.1562	0.1309	0.1367	0.0981	0.0778

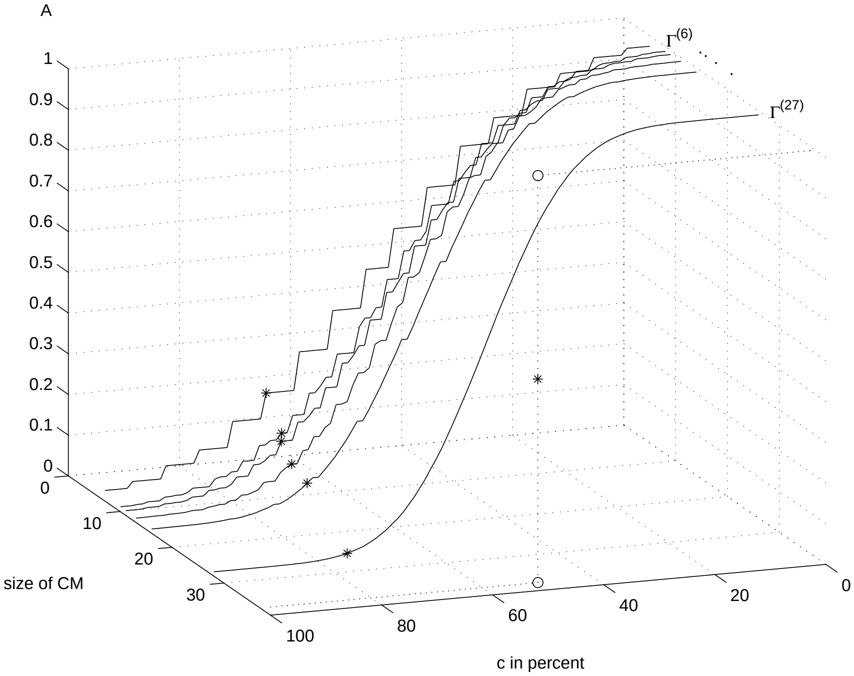

Figure 5.1: Complaisance in EU Council of Ministers

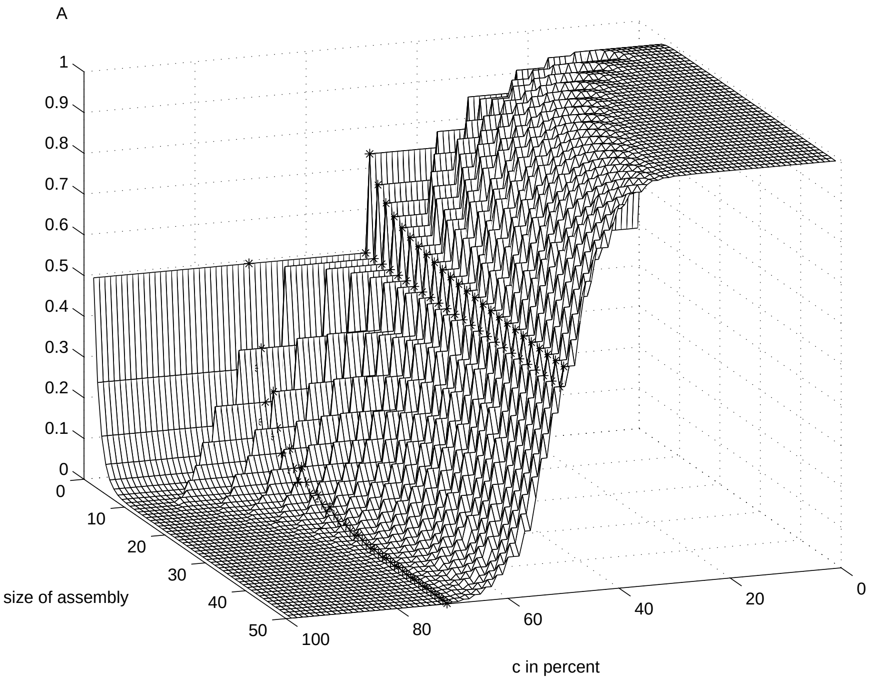

Figure 5.2: One Person One Vote

6. Introduction to Condorcet's Jury Theorem

The research in this section is rooted in a tradition which goes back to Condorcet (1785). Consider a group N confronting a dichotomous choice, while the members of the group are all assumed to possess more or less reliable perceptions of which alternatives 'ought' to be chosen. The fundamental premise is that there exists some procedure-independent fact of the matter as to what the best or right outcome is. We shall base the discussion on the following cover story: assume that in a jury trial the probability that the defendant is guilty of the offense charged is $\theta \in [0,1]$. Hence there is a truth independent of the jury, yet unknown to the jury members. We will assume that each member k possesses a more or less reliable perception about the truth. This degree of knowledge is modelled by $p_k \in (0,1)$, the *judgemental competence of voter k*. It is the probability that the voter will make the correct choice (the 'better') of the two available to him or her. If the proposal to be voted upon is whether the defendant shall be convicted then p_k is the probability that k votes 'yes' if the defendant is guilty, 'no' if he is innocent respectively.

The *jury's competence* or *collective competence* is measured by the likelihood of the verdict being correct. For a given voting game Γ let $C[\Gamma]$ denote the probability that the decision rule in Γ leads to a correct choice. We shall refer to $C[\Gamma]$ as *jury competence* or *group judgemental accuracy.*[11] We further define $\mathcal{M}_n$ as the *simple majority game*. This is the game whose winning coalitions are just those subsets of the voter set N with cardinality larger than $n/2$, i.e. $\mathcal{M}_n = \{S \subseteq N |\ |S| > n/2\}$. Condorcet's Jury Theorem provides a statement for the jury competence of $\mathcal{M}_n$. Assume for simplicity n to be odd and put $m = (n + 1)/2$.

Theorem 6.1. *(Condorcet Jury Theorem) (Condorcet, 1785; see also Grofman et al., 1983) Assume that the voters' choices are independent of one another and are homogenous, i.e. the probability that voter k's choice is correct is given by* $p_k = p$ *for all* $k \in \{1, 2, , \ldots, n\}$. *Then*

$$C[\mathcal{M}_n] = \sum_{h=m}^{n} \binom{n}{h} p^h (1-p)^{n-h}.$$

Moreover, if $1 > p > 1/2$, *then* $C[\mathcal{M}_n]$ *is monotonically increasing in* n *and* $\lim_{n\to\infty} C[\mathcal{M}_n] = 1$; *if* $0 < p < 1/2$, *then* $C[\mathcal{M}_n]$ *is monotonically decreasing in* n *and* $\lim_{n\to\infty} C[\mathcal{M}_n] = 0$; *while if* $p = 1/2$ *then* $C[\mathcal{M}_n] = 1/2$ *for all* n *(and* $\lim_{n\to\infty} C[\mathcal{M}_n] = 1/2$).

The result says that if each voter makes the correct choice with a given probability larger than 1/2, the correct option of being the majority winner converges to certainty monotonically as the number of voters tends to infinity. This result constitutes an important pro-democratic argument and has been extended in many ways by statisticians, economists, political scientists, etc. For example, Shapley & Grofman (1984) show that the decision rule that maximizes jury competence is a

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[11] For the terminology, see for example Shapley & Grofman (1984).

weighted majority voting rule that assigns weights w_k equal to $\log[p_k/(1-p_k)]$ to any voter k. For our purpose to offset the results in context of Coleman's A of the previous sections, we shall prove a generalized statement for weighted majority games when there are many small voters as defined by the setting (9) – (11).

7. GENERALIZATION OF CONDORCET'S JURY THEOREM

ASSUMPTION 7.1. *(homogeneity) There are exactly two alternatives, only one of which is correct (or equivalently, one of which is 'better' than the other) with probability* $\theta \in [0,1]$. *We fix an arbitrary real* $p \in (0,1)$ *and assume that each minor voter* $k+l \in M^{(\nu)}$ *acts independently and makes the correct choice (i.e. the 'better' choice) with probability* p.

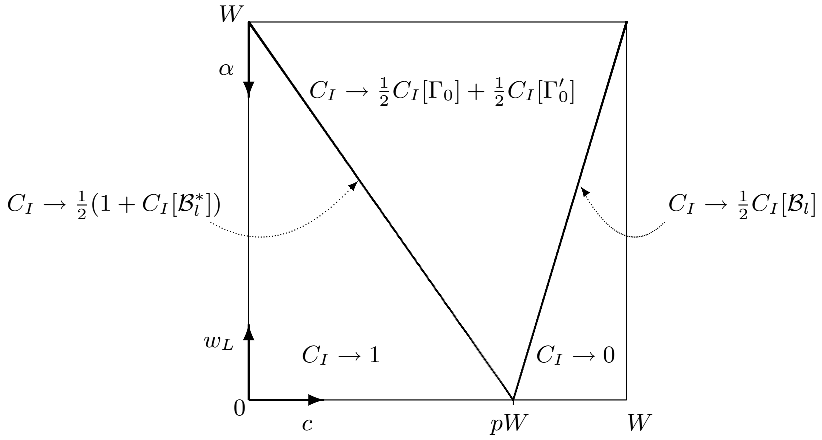
For our purpose it will prove useful to decompose C into probabilities of avoiding errors of Type I and II in a statistical sense. Let C_I denote the probability of avoiding a Type I error or equivalently $(1-C_I)$ the probability of a true hypothesis being rejected. In terms of the cover story it implies that if the hypothesis is that the defendant is guilty C_I is the probability that a guilty defendant will be convicted. Analogously, let C_{II} denote the probability of avoiding an error of Type II i.e. that an innocent defendant will be found guilty. Jury competence then follows as

$$C = \theta C_I + (1-\theta) C_{II}. \tag{21}$$

For the moment, put $\theta = 1$ (the defendant is guilty). Hence the correct choice is an affirmative vote. We can then interpret $C = C_I$ as complaisance of the jury since an affirmative outcome leads to conviction of the defendant. In this interpretation Theorem 4.1 is a statement about jury competence avoiding an error of Type I. Note that there is no need to specify the competence of the major voters. Their competence enters generally by means of the major collective competencies of the games Γ_0 and Γ_0', the games $\mathcal{B}_l$ and $\mathcal{B}_l^*$ respectively.

The results for $p > 0.5$ are illustrated in Figure 7.1.

Figure 7.1: Limit scenario for C if $\theta = 1$

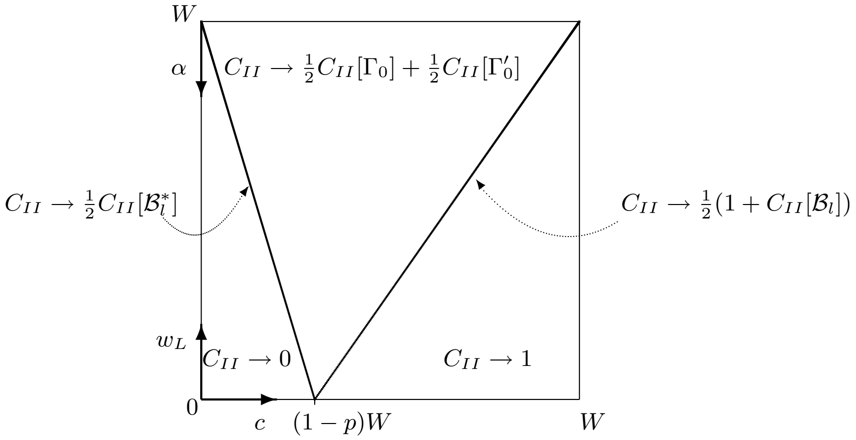


The scenario is topologically equivalent to Figure 4.1. However, $p > 0.5$ has led to a distortion effect of the inner area.

For $\theta = 0$ we have that $C = C_{II}$ is the likelihood that an innocent defendant is found 'not guilty' in which case voting 'no' is the correct choice. This implies that the voters vote 'yes' with probability $(1-p)$ leading to a distortion effect of the inner area opposite to the shift of Figure 7.1, as well as another adjustment of the quotas of the games played among the major voters by replacing p by $(1-p)$ in (14).

Figure 7.2 illustrates this scenario – a precise statement is given in Theorem 7.1.

Figure 7.2: Limit scenario for C if $\theta = 0$



In the following let the additional argument p stress the dependance of the minor voters' competence, the probability of the minor voters' affirmative vote respectively. Group competence follows as

$$C[\Gamma^{(\nu)}, p] = \theta A[\Gamma^{(\nu)}, p] + (1-\theta)(1 - A[\Gamma^{(\nu)}, 1-p]). \tag{22}$$

In summary, we get

THEOREM 7.1. **(Generalized Condorcet Jury Theorem)** *In a jury trial, let the probability that the defendant is guilty of the offense charged be given by* $\theta \in [0,1]$. *Assume that the minor voters' choices are independent of one another and are homogenous, i.e. the probability that voter k's choice is correct is given by* $p_k = p \in (0,1)$ *for all* $k \leq m^{(\nu)}$. *In the sequence of games described by* (9) - (11) *the jury competence follows as*

$$\lim_{\nu \to \infty} C[\Gamma^{(\nu)}] = \theta C_I + (1-\theta) C_{II}, \tag{23}$$

where C_I *is given by*

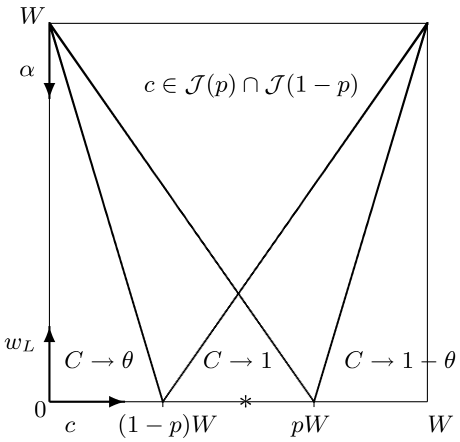
$$C_I = \frac{1}{2} A[\Gamma_0] + \frac{1}{2} A[\Gamma'_0], \qquad \textit{if } c \in \mathcal{J}(p).$$

For other values of c we have

$$C_I = \begin{cases} 1 & \textit{if } c < p\alpha, \\ 1/2\,(1 + A[\mathcal{B}^*_l]) & \textit{if } c = p\alpha, \\ 1/2\, A[\mathcal{B}_l] & \textit{if } c = w_L + p\alpha, \\ 0 & \textit{if } c > w_L + p\alpha. \end{cases} \tag{24}$$

C_{II} *is given by replacing* p *by* $1-p$.

Figure 7.3: Generalized Jury Theorem



For values of the quota c outside the closure of $\mathcal{J}(p) \cup \mathcal{J}(1-p)$ jury competence is independent from the competence level of either the major or minors voters as illustrated in Figure 7.3. The relative quotas in the area around $c = 50\%$ prove to be the best 'truth-trackers' since jury competence reaches infallibility, i.e. $C = 1$. This area enlarges with an increasing competence p of the minor voters due to a shift of the overlapping triangles in opposite direction. However, for any fixed p and c an increasing total weight w_L of the major voters leads to areas in which the jury competence depends exclusively on the competence of the major voters. Hence the limit value C can be computed from WVGs with an assembly consisting of the small set of major voters. Minor voters only have an indirect effect of manipulating the quota. This effect represents a shift of the quota c in 'truth direction'. If the defendant is guilty then the threshold (quota) $c - p\alpha$ necessary for conviction decreases with increasing competence p of the minor voters (analogously, if the defendant is not guilty the likelihood of acquittal increases with increasing competence of the minor voter).

The result of Theorem 6.1 – the classical version of Condorcet – is indicated by the point marked with '$*$' on the horizontal axis $w_L = 0$. However, the statement of Condorcet refers to the special symmetric case where any (minor) voter has the same voting weight. Theorem 7.1 states that at $(c, w_L) = (0.5, 0)$ the collective competence of *any* non-atomic game under the condition (9) – (11) tends to one.

The convergence behaves qualitatively similar to the convergence of complaisance A as discussed in Section 8. Thus the limit C serving as an approximation for finite real-world scenarios is justified in (c, w_L) areas with high convergence rates.

8. CONVERGENCE CHARACTERISTICS

According to (13) complaisance A can be derived as finite sums of (weighted) terms in the shape of

$$\mathrm{Prob}\left\{\alpha^{(\nu)}(S) \geq z\right\}, \qquad z \in \mathbb{R}.$$

These determine the convergence behavior which in turn hinges on the relation of z to the mean value $\mu = p\alpha$ of the random variable $\alpha^{(\nu)}(S)$. We shall see that the rate of convergence is high if $z \neq \mu$ and if the distribution of the minor votes is reasonably smooth. This is due to the fact that under the smoothness condition the 'tails' of the sum of random variables display high convergence rates. In the following analysis of A we shall focus on weight distributions of the major voters and the quota c such that $z \neq \mu$ is always matched. Similar results for collective competence C follow easily by the fact that it can be derived by A by means of an easy linear combination (see (22)).

For a 'tail' estimation we shall exclude the set P given by

$$P := \{c \,|\, c - w(B) = p\alpha \text{ for some } B \subseteq L\}\,. \tag{25}$$

Note that P is a null set under the uniform distribution.

LEMMA 8.1. *Let $\{Z_k\}_{k=1}^{\infty}$ be a sequence of independent real-valued random variables such that $|Z_k| \leq 1$ for all k. Let further $\{c_k\}_{k=1}^{\infty}$ be a sequence of real constants such that*

$$s^2 = \sum_{k=0}^{\infty} {c_k}^2 < \infty. \tag{26}$$

Then

$$\tilde{Z} = \sum_{k=0}^{\infty} c_k (Z_k - \mu_k), \tag{27}$$

with $\mu_k = E\left[X_k\right]$, *satisfies*

$$\text{Prob}\left\{\tilde{Z} > \delta s\right\} \leq \exp\left[-\frac{\delta^2}{2}\right], \quad \text{Prob}\left\{\tilde{Z} < -\delta s\right\} \leq \exp\left[-\frac{\delta^2}{2}\right] \tag{28}$$

for each number $\delta > 0$.

For a proof see Kemperman (1964).

Theorem 8.1. *In the sequence of games defined by* (9) – (11) *we have for* $c \notin P$

$$\left| A\left[\Gamma^{(\nu)}\right] - \lim_{i \to \infty} A\left[\Gamma^{(i)}\right] \right| = \mathcal{O}\left(\exp\left[- \left(\frac{\lambda}{\sqrt{m^{(\nu)}}\alpha_{\max}^{(\nu)}} \right)^2 \right] \right), \tag{29}$$

where λ *is a positive constant.*

Proof. For a fixed ν let

$$Z_k = \begin{cases} \alpha_k^{(\nu)}/\alpha_{\max}^{(\nu)} & \text{with probability } p, \\ 0 & \text{otherwise,} \end{cases} \tag{30}$$

for each $k = 1, ..., m^{(\nu)}$. For $k > m^{(\nu)}$ set $Z_k \equiv 0$. For the constants c_k put

$$c_k = \begin{cases} \alpha_{\max}^{(\nu)} & \text{for } k \leq m^{(\nu)}, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

From (26) follows

$$s = \sqrt{m^{(\nu)}}\alpha_{\max}^{(\nu)} \ .$$

This setting allows us to identify the $\tilde{Z}$ in (27) with $\alpha^{(\nu)}(S) - \mu$, where $\mu = p\alpha$, and we get from (28)

$$\text{Prob}\left\{|\alpha^{(\nu)}(T) - \mu| > \delta s\right\} \leq \exp\left[-\frac{\delta^2}{2}\right],$$

for any positive number $\delta > 0$. Putting $\varepsilon := \delta s$ yields the reformulation

$$\text{Prob}\left\{|\alpha^{(\nu)}(T) - \mu| > \varepsilon\right\} \leq \exp\left[-\left(\frac{\varepsilon}{2\sqrt{m^{(\nu)}}\alpha_{\max}^{(\nu)}}\right)^2\right]. \tag{32}$$

Finally, from definition (13) follows that $A\left[\Gamma^{(\nu)}\right]$ is a weighted sum of finitely many terms where each term can be estimated by (32). For $c \notin P$ we have $\varepsilon > 0$ for each of those terms which proves (29). □

From Theorem 8.1 follows that the rate of converges hinges on $\sqrt{m^{(\nu)}}\alpha_{\max}^{(\nu)}$. Note that this term also reflects the ratio of $\alpha_{\max}^{(\nu)}$ to the mean minor weight $\alpha/m^{(\nu)}$. If the weight distribution among the minor voters is sufficiently smooth so that $\sqrt{m^{(\nu)}}\alpha_{\max}^{(\nu)}$ tends to zero sufficiently fast we can expect high rates of convergence as indicated by (29). For the symmetric case $\alpha_{\max}^{(\nu)} = \alpha/m^{(\nu)}$ this implies that $A[\Gamma^{\nu}]$ converges exponentially in $m^{(\nu)}$.

We shall see that if the number of voters increases in non-atomic games, while at the same time the quota is pegged at a constant percentage, the scenario is qualitatively comparable to the 'one person one vote' situation (the symmetric case). However, there is a distortion effect due to unequal weight distribution. The following theorem provides a statement for $l = 0$ and $p = 1/2$, considering the ratio $\alpha_{\max}^{(\nu)}/\alpha_{\min}^{(\nu)}$.

THEOREM 8.2. *Let $\alpha_1, ..., \alpha_{m^{(\nu)}}$ be positive numbers totalling α, and S a random subset of $M^{(\nu)}$. If every subset is equally probable then for any $\varepsilon > 0$,*

$$\text{Prob}\left\{\alpha(S) > \alpha/2 + \varepsilon\right\} \leq \exp\left[-\frac{8m^{(\nu)}\varepsilon^2\theta}{\alpha^2(1+\theta)^2}\right],$$

where θ denotes $\alpha_{\max}^{(\nu)}/\alpha_{\min}^{(\nu)}$.

For a proof see Hoeffding (1963).

9. Concluding Remarks

We agree with the claim of Felsenthal & Machover that for non-atomic games if the quota is pegged at a constant percentage which is greater than 50% then as the number of voters increases complaisance tends to fall. The non-atomic setup generally reflects the scenario when the distribution of the voting weights of a large committee is not too skewed (in other words, if the ratio of the largest weight to the smallest is not very high). Examples are the US Presidential Electoral College or the CM of the EU.

It is tempting to criticize the impact of this result on complaisance because it is based on the assumption of each 'yes' and 'no' choice being equally likely. This assumption does not match the observation of many real-world voting scenarios, as for example the CM: when it comes to voting the affirmative votes usually represent a majority. However, this argument disregards two essential aspects of measuring the power of a collectivity to act.

First, complaisance as introduced by Coleman (1971) is in the spirit of *a priori* analysis. Contrary to *actual* (a posteriori) analysis, it models the voting system as an 'abstract shell', without taking into consideration voters' preferences, the range of issues over which a decision is taken or the degree of affinity between the voters. This abstraction seems to be necessary to evaluate the decision rule itself. Roth (1988, p. 9) puts it this way:

'Analyzing voting rules that are modeled as simple games abstracts away from the particular personalities and political interests present in particular voting environments, but this abstraction is what makes the analysis focus on the rules themselves rather than on other aspects of the political environment. This kind of analysis seems to be just what is needed to analyze the voting rules in a new constitution, for example, long before the specific issues to be voted on arise or the specific factions and personalities that will be involved can be identified.'

Second, a vote held is usually the result of a foregoing bargaining process. Before the formal vote is taken there is usually a whole series of shadow or straw divisions

– which comes to a halt when a majority can be expected.[12] In that sense, complaisance can be thought of as measuring the barrier that members of a committee have to overtake via negotiations and bargaining in order to approve a given proposal. A decreasing A increases this barrier which is usually reflected by a long pre-vote period – a clear indicator of paralysis.

However, the generalization of Condorcet's Jury Theorem has shown that if the intention is to arrive at a collective decision in terms of a 'correct' judgement (truth-tracking), the prognosis of the assembly's reliability depends on the competence level of its members. The variety of the issues to vote upon suggests a symmetric a priori setting of θ to $1/2$ (note that values for θ other than $1/2$ make sense if the jury is e.g. exclusively concerned with penal jurisdiction; in this case θ could represent a measure of the crime rate). For non-atomic games Theorem 7.1 implies that if the committee acts as a jury with more or less homogenous competence of its members $p > 0.5$, then the ex ante evaluation of an expanding committee is positive: in the worst case the judgemental competence tends to 50% if $p \leq c$. For $p > c$, however, the committee tends to *infallibility* with an increasing member set. Therefore, the pessimistic prognosis for an expanding committee with decreasing complaisance does not hold on epistemic grounds.

The classification of problems into two main categories has also an impact on the assessments of large decision making bodies with an atomic part. These are characterized by a small set of voters with a large voting weight and a large 'pool' of small voters – a typical scenario in, for example, shareholding (the main stockholder of a company might hold 10 percent of total shares while the other stockholders' holdings are very small). Likewise the 50% quota has an outstanding role. For preference aggregation this majority quota implies a suitable level of complaisance, as measured by A since it excludes the areas where A tends to he extremes 0 and 1. If the committee is interpreted as a knowledge aggregation machine heading for the truth of a matter this shift increases the probability that the jury is infallible.

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[12] Also, many committees as e.g. the CM seem to publish only positive outcomes, i.e. when acts have been adopted.

Appendix A. Proof of Theorem 2.1

Theorem A.1. **(Lindeberg-Feller)** *In order that*

$$\lim_{n\to\infty} \max_{k\leq n} \frac{\sigma_k}{s_n} = 0 \tag{33}$$

and

$$\lim_{n\to\infty} \sup_x |\mathrm{Prob}\,\{S_n < x\} - \Phi(x)| = 0 \tag{34}$$

it is necessary and sufficient that the following condition (the Lindeberg condition) be satisfied:

$$\lim_{n\to\infty} L_n(\varepsilon) = 0 \tag{35}$$

with

$$\begin{aligned} L_n(\varepsilon) &:= s_n^{-1} \sum_{k\leq n} E\left[(X_k-\mu_k)^2; |X_k-\mu_k| \geq \varepsilon s_n\right] \\ &= s_n^{-1} \sum_{k\leq n} \int_{\{|x-\mu_k|\geq\varepsilon\sqrt{s_n}\}} (x-\mu_k)^2 dF_k(x) \end{aligned} \tag{36}$$

for every fixed $\varepsilon > 0$.

For a proof see e.g. Petrov (1975), p.100-101. We put

$$Q^{(n)} := \sum_{k\leq n} w_k^2.$$

Lemma A.1. *For each k, let the independent random variable X_k be given by*

$$X_k = C_k w_k,$$

where the C_k are real-valued random variables with the same non-degenerate distribution on a compact set $[a,b]$ for all $k \in \mathbb{N}$. Then $\{X_k\}_{k=1}^{\infty}$ satisfies the Lindeberg condition (35) *iff*

$$\lim_{n\to\infty} \frac{w_n}{\sqrt{Q^{(n)}}} = 0. \tag{37}$$

Proof. For each k follows

$$E\,[X_k] = \lambda_1 w_k \tag{38}$$

$$Var\,[X_k] = \lambda_2^2 w_k^2, \tag{39}$$

where λ_1 and λ_2 are reals independent of k, with $\lambda_2 > 0$ (since each C_k has a non-degenerate distribution). Hence

$$s_n = \lambda_2 \sqrt{Q^{(n)}}. \tag{40}$$

Now suppose the Lindeberg condition (35) is satisfied. Then by Theorem A.1 we have (33), from which (37) follows at once in view of (39) and (40).

Conversely, suppose that (37) holds. We now show that

$$\lim_{n\to\infty} \max_{k\le n} \frac{w_k}{\sqrt{Q^{(n)}}} = 0. \tag{41}$$

For any $\varepsilon > 0$ fix n' so large that $w_k/\sqrt{Q^{(k)}} < \varepsilon$ for all $k > n'$. Thus, for all $n > n'$ we have

$$\frac{w_k}{\sqrt{Q^{(n)}}} \le \frac{w_k}{\sqrt{Q^{(k)}}} < \varepsilon \qquad \text{for } k = n' + 1, ..., n.$$

Thus (41) holds. Now observe that for every k, the integral in (36) follows as

$$\int_{|x-\lambda_1 w_k| > \varepsilon \lambda_2 \sqrt{Q^{(n)}}} (x - \lambda_1 w_k)^2 dF_k(x). \tag{42}$$

But from $|x - \lambda_1 w_k| = |y - \lambda_1| w_k$ for all $y \in [\alpha, \beta]$ and (41) it follows that, for any given $\varepsilon > 0$, if n is sufficiently large, then

$$|y - \lambda_1| w_k < \varepsilon \lambda_2 \sqrt{Q^{(n)}}$$

for all $y \in [\alpha, \beta]$ and all $k \le n$. That implies the integral (42) vanishes for all $k \le n$. Hence (35) holds. □

Appendix B. Proof of Theorem 4.1

For the proof of Theorem 4.1 we shall use the following Lemma.

Lemma B.1. *Let* $0 \le z \le \alpha$ *and choose a subset* $S \subseteq M^{(\nu)}$ *at random. For the sequence of games (9) - (11) we have*

$$\lim_{n\to\infty} \text{Prob}\{\alpha^{(\nu)}(S) \ge z\} = \begin{cases} 1 & \text{if } z < p\alpha, \\ 1/2 & \text{if } z = p\alpha, \\ 0 & \text{if } z > p\alpha. \end{cases} \tag{43}$$

Proof. We represent the vote of each minor voter $k+l \in M^{(\nu)}$ as the random variable

$$X_k^{(\nu)} = \begin{cases} \alpha_k^{(\nu)} & \text{if } k \text{ votes 'yes'}, \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

Put

$$\mu := \sum_{k \leq m^{(\nu)}} E\left[X_k^{(\nu)}\right] = p\alpha. \tag{45}$$

and

$$\left[s^{(\nu)}\right]^2 := \sum_{k \leq m^{(\nu)}} Var\left[X_k^{(\nu)}\right] = p(1-p)Q^{(\nu)}. \tag{46}$$

Then, by Theorem 2.1,

$$\lim_{\nu\to\infty} \text{Prob}\{\alpha^{(\nu)}(S) < z\} = \lim_{\nu\to\infty} \Phi(\frac{z-\mu}{s^{(\nu)}})$$

$$= \lim_{\nu\to\infty} \Phi(\frac{(z/\alpha - p)\alpha}{s^{(\nu)}}). \tag{47}$$

With increasing ν the standard deviation $s^{(\nu)}$ from (46) tends to zero. This implies that in (47) the sign of the term $(z/\alpha - p)$ determines whether the argument of Φ converges to infinity or is constantly zero. This provides (43). □

Proof of Theorem 4.1. For the terms in (13) follows with Lemma B.1

$$\text{Prob}\left\{\alpha^{(\nu)}(S) \geq c - w(B)\right\} \to \begin{cases} 0 & w(B) < c - p\alpha, \\ 1/2 \quad \text{if} & w(B) = c - p\alpha, \\ 1 & w(B) > c - p\alpha. \end{cases} \tag{48}$$

For $c \in \mathcal{J}$ the games Γ_0 and Γ_0' are well defined, and for any $B \subseteq L$ for which the limit of $\text{Prob}\left\{\alpha^{(\nu)}(S) \geq c - w(B)\right\}$ is 1 we have that B is a winning coalition in both Γ_0 and Γ_0'. If the limit is 1/2 the coalition B is winning in Γ_0 but not Γ_0'. This yields for (13)

$$\begin{aligned} & \lim_{\nu\to\infty} A[\Gamma^{(\nu)}] \\ = & \sum_{w(B)>c-p\alpha} \text{Prob}\{Y = B\} + \frac{1}{2} \sum_{w(B)=c-p\alpha} \text{Prob}\{Y = B\} \\ = & \frac{1}{2} \sum_{w(B)>c-p\alpha} \text{Prob}\{Y = B\} + \frac{1}{2} \sum_{w(B)\geq c-p\alpha} \text{Prob}\{Y = B\} \\ = & \frac{1}{2}A[\Gamma_0] + \frac{1}{2}A[\Gamma_0'], \end{aligned}$$

and hence (17).

To see (18) note that from $c < p\alpha$ follows that the third condition in (48) is fulfilled for any $B \subseteq L$ and hence $\text{Prob}\left\{\alpha^{(\nu)}(S) \geq c - w(B)\right\} \to 1$ for all $B \subseteq L$.

The equality $c = p\alpha$ implies $\text{Prob}\left\{\alpha^{(\nu)}(S) \geq c - w(B)\right\} \to 1/2$ for $B = \emptyset$ and 1 otherwise which yields

$$\begin{aligned} & \lim_{\nu\to\infty} A[\Gamma^{(\nu)}] \\ = \; & \sum_{B\neq\emptyset} \text{Prob}\{Y = B\} + \frac{1}{2}\text{Prob}\{Y = \emptyset\} \\ = \; & A[\mathcal{B}_l^*] + 1/2\,(1 - A[\mathcal{B}_l^*]). \end{aligned}$$

If $c = w_L + p\alpha$ then $\text{Prob}\left\{\alpha^{(\nu)}(S) \geq c - w(B)\right\} \to 1/2$ for $B = L$ and 0 otherwise.

Finally, from $c > w_L + p\alpha$ follows that $\text{Prob}\left\{\alpha^{(\nu)}(S) \geq c - w(B)\right\} \to 0$ for any $B \subseteq L$.

$\square$

Appendix C. Qualified Majority Voting in the CM

Table 4. QMV under $\mathcal{N}_{27}$

Country	(1) w	(2) $\bar{w}\,(\%)$
Germany	118	8.5199
UK	117	8.4477
France	117	8.4477
Italy	117	8.4477
Spain	108	7.7978
Poland	108	7.7978
Romania	56	4.0433
Netherlands	52	3.7545
Greece	48	3.4657
Czech Rep	48	3.4657
Belgium	48	3.4657
Hungary	48	3.4657
Portugal	48	3.4657
Sweden	40	2.8881
Bulgaria	40	2.8881
Austria	40	2.8881
Slovakia	28	2.0217
Denmark	28	2.0217
Finland	28	2.0217
Ireland	28	2.0217
Lithuania	28	2.0217
Latvia	16	1.1552
Slovenia	16	1.1552
Estonia	16	1.1552
Cyprus	16	1.1552
Luxembourg	16	1.1552
Malta	12	0.8664
Total	1 385	100.0001

Quota: 1 034 = 74.66% of 1 385.
Complaisance: A=0.0166.

Note For explanations see the Introduction.

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