CORE DISCUSSION PAPER
2004/07

INVARIANTS IN THE RIEMANNIAN GEOMETRY OF CONVEX SETS

Roland HILDEBRAND[1]

March 2004

Abstract

In this contribution we study some aspects of the Riemannian geometry induced on a convex set by a barrier function of the set. Using Noether's theorem, we link the symmetries of the set to invariants of the geodesic flow. This allows to lower the dimension of the differential system defining the geodesics and gives insights in the structure of the geodesic flow, specifically on the configuration of geodesic submanifolds. We use the developed apparatus to completely integrate the geodesic equations for the convex hulls of the sphere, the paraboloid, the hyperboloid and the standard symmetric cones and to obtain explicit formulae for the geodesics on these sets.

Keywords: Riemannian geometry, barrier function, convexity, symmetry
JEL Classification: C69

[1]Laboratoire de Modelisation et Calcul, IMAG, UJF, 51 rue des Mathématiques, BP 53, 38041 Grenoble Cedex 9, France; tel. 0033-(0)476-514555; e-mail `roland.hildebrand@imag.fr`.

This text presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

1 Introduction

A standard problem in Mathematical Programming is that of minimizing a linear function on a convex set X,

$$\min_{x \in X} \langle c, x \rangle. \tag{1}$$

To avoid dealing with a constrained optimization problem, one usually transforms this problem to a nearly unconstrained one. To this end, the information on the boundary of the convex set X is condensed to a scalar function F on X, which tends to infinity as the argument tends to the boundary of X. This function is added as a penalty term to the linear objective function. Thus we obtain the nearly unconstrained problem

$$\min_{x} \tau \langle c, x \rangle + F(x), \tag{2}$$

where $\tau > 0$ is a scalar parameter expressing the weight of the penalty function.

Modern interior-point methods use a special class of penalty functions F, namely self-concordant barrier functions. A detailed treatment of these methods can be found in [6]. A function F on X is said to be a barrier function if it is strictly convex, of class C^3, tends to infinity as the argument tends to ∂X, and satisfies certain inequalities between its first three derivatives. Namely, there should exist a constant $M > 0$ such that $|\langle F'''[u]u, u \rangle| \leq M \langle F''u, u \rangle^{3/2}$ and $\langle (F'')^{-1} F', F' \rangle$ is bounded on X. The Hessian $F''(x)$ defines a Riemannian metric $g_{\mu\nu}(x) = \frac{\partial^2 F}{\partial x^\mu \partial x^\nu}(x)$ on X, and thus turns X into a Riemannian manifold.

The minimizers of (2) form the *central path* of the problem:

$$x^*(\tau) = \arg\min_{x \in X} (\tau \langle c, x \rangle + F(x)).$$

This path tends to the minimizer of (1) as $\tau \to \infty$. The computation of $x^*(\tau)$ for a particular τ is accomplished by the Newton algorithm.

Interior-point minimization algorithms yield a sequence of points x_k which approximates a sequence of points $x^*(\tau_k)$ on the central path. The sequence τ_k is chosen in way such that the algorithm remains stable and τ_k tends to infinity as $k \to \infty$. This guarantees that x_k tends to the sought minimizer of (1). For details of the procedure see [6].

For some convex sets X with additional favourable structure, e.g. intersections of an affine subspace with a standard symmetric cone, there exist tricky ways to choose the sequence τ_k such that it quickly tends to ∞. This results in *long-step path-following schemes* [5].

If the convex set X does not possess additional favourable structure, then *short-step path-following schemes* are used. They are based on the idea that the behaviour of F in a certain neighbourhood of a point $x \in X$ is largely determined by the Hessian $F''(x)$ at that point. This neighbourhood can be described as an ellipsoid around x given by $\{y \mid \langle F''(x-y),(x-y)\rangle < 1\}$. This ellipsoid is called the *Dikin ellipsoid* around x. Specifically, it is guaranteed that the Dikin ellipsoid lies entirely in X. Short-step path-following schemes are designed in such a way that at each step of the minimization procedure the next point lies in the Dikin ellipsoid around the previous point [6].

The length of the path traced by such a minimization algorithm, measured in the Riemannian metric defined by the Hessian $F''(x)$, yields an approxiamte lower bound on the computation time. Hence the geodesic length in this metric allows to obtain lower bounds on the complexity of short-step path-following schemes [7],[8]. Therefore the geodesic flow of the metric $g_{\mu\nu}(x) = \frac{\partial^2 F}{\partial x^\mu \partial x^\nu}(x)$ is an important object to study.

Properties of the metric defined by a barrier function on a convex set were studied e.g. in [10],[9].

In this contribution we deal with a particular aspect of these metrics, namely we show how symmetries of the convex set X generate invariants of the geodesic flow of the metric. To this end we consider the geodesic equations as the equations of motion of a dynamical system generated by the Lagrange function $L(x,\dot{x}) = \frac{1}{2}\langle F''\dot{x},\dot{x}\rangle$. By applying Noether's principle to this system, we link the Lie algebra of the symmetry group of X to a set of invariants of the geodesic flow, provided the barrier function F is consistent with the symmetry group. Here by consistency we mean that the symmetries of X are isometries of X considered as Riemannian space.

The theory developed in this contribution sheds light on the structure of the geodesic flow associated with barrier functions on convex sets with symmetries. It allows to lower the dimension of the system of geodesic equations for convex sets X with symmetries and in case of especially rich symmetry groups, to completely integrate it, as will be shown on several concrete examples.

The contribution is structured as follows. In the next section we introduce basic concepts of Riemannian geometry and consider it from the viewpoint of Theoretical Mechanics. In Section 3 we apply this apparatus to the metric associated with barrier functions on convex sets. In Sections 4 and 5 we consider some special cases with rich symmetry groups, namely the convex hulls of surfaces defined by quadratic equations and the symmetric cones, which can be represented as cones of squares for formally real Jordan algebras. In the last section we draw some conclusions and pose some questions for further research.

2 Riemannian geometry and Noether's principle

In this section we present a short introduction to Riemannian geometry and briefly describe Noether's theorem, a concept stemming from Theoretical Mechanics and dealing with symmetries of dynamic systems. We show how the infinitesimal symmetries of a Riemannian manifold, which are described by so-called *Killing vector fields*, generate invariants of its geodesic flow. This provides us with the necessary apparatus to investigate the Riemannian metric induced by a barrier function on a convex set with symmetries.

Riemannian geometry

In this subsection we provide the very basics of Riemannian geometry for further reference. We introduce the metric tensor, the geodesic equation and look at coordinate transformations. Detailed treatments of tensor algebra and Riemannian geometry can be found e.g. in [11].

Let M be a differentiable manifold of dimension n.

Definition 1 *A Riemannian metric on M is a symmetric covariant positive definite tensor field g of second order, i.e. to any point $x \in M$ we have assigned a positive definite quadratic form $g(x)$ on the tangent space T_xM:*

$$g(x) : T_xM \times T_xM \to \mathbf{R}, \qquad g(x) : (u, v) \mapsto g(x)[u, v] = g(x)[v, u].$$

We assume $g(x)$ to be continuously differentiable with respect to x.

Fix a coordinate system $x^1, \ldots, x^n$ on M, then $g(x)$ is described by a symmetric matrix with entries $g_{\mu\nu}(x)$ depending on x, $\mu = 1, \ldots, n$, $\nu = 1, \ldots, n$. Namely, evaluating $g(x)$ on the tangent vectors $\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \in T_xM$, $\mu, \nu \in \{1, \ldots, n\}$, we obtain

$$g(x)\left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right] = g_{\mu\nu}(x).$$

This representation of g will change if we change the coordinate system. Let $y^1, \ldots, y^n$ be another coordinate system on M, $y^k = y^k(x^1, \ldots, x^n)$, and let $g'_{\rho\tau}(y)$ be the entries of the matrix representation of the tensor field g in this coordinate system. Using the relation $\frac{\partial}{\partial y^\tau} = \sum_{\mu=1}^n \frac{\partial x^\mu}{\partial y^\tau}\frac{\partial}{\partial x^\mu}$, we get

$$\begin{aligned} g'_{\rho\tau}(y(x)) &= g(x)[\frac{\partial}{\partial y^\rho}, \frac{\partial}{\partial y^\tau}] = g(x)[\sum_{\mu=1}^n \frac{\partial x^\mu}{\partial y^\rho}\frac{\partial}{\partial x^\mu}, \sum_{\nu=1}^n \frac{\partial x^\nu}{\partial y^\tau}\frac{\partial}{\partial x^\nu}] \\ &= \sum_{\mu,\nu=1}^n \frac{\partial x^\mu}{\partial y^\rho}\frac{\partial x^\nu}{\partial y^\tau} g_{\mu\nu}(x) = \left(\frac{\partial x}{\partial y}\right)^T (g_{\mu\nu}) \left(\frac{\partial x}{\partial y}\right). \end{aligned} \tag{3}$$

Let $x(t)$ be a continuously differentiable curve on M, $t \in [t_0, t_1]$. Its *Riemannian length* is given by

$$l(t_0, t_1) = \int_{t_0}^{t_1} \sqrt{g(x(t))[\dot{x}(t), \dot{x}(t)]}\, dt = \int_{t_0}^{t_1} \sqrt{\sum_{\mu,\nu=1}^{n} g_{\mu\nu}(x(t))\dot{x}^{\mu}(t)\dot{x}^{\nu}(t)\, dt}.$$

For any two points $x_0, x_1 \in M$ that are sufficiently close there exists a unique arc $x(t)$ linking x_0, x_1 which has minimal length. This arc obeys the Euler-Lagrange equations for above functional. These can be rewritten as the following system of differential equations,

$$\ddot{x}^{\mu} + \frac{1}{2} \sum_{\alpha,\beta,\nu=1}^{n} g^{\mu\beta} \left(\frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} + \frac{\partial g_{\beta\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} \right) \dot{x}^{\alpha}\dot{x}^{\nu} = 0, \tag{4}$$

which are called the geodesic equations [11]. Here $(g^{\mu\beta})$ is the inverse of the matrix $(g_{\mu\beta})$.

Noether's theorem

In this subsection we show how a dynamical system is defined by a Lagrange function and how the symmetries of this function induce invariants of the dynamical system. A detailed account on the underlying theory can be found in standard books on Theoretical Mechanics [3].

Let M be a differentiable manifold and TM its tangent bundle, i.e. the set of pairs (x, v), where $x \in M$ and $v \in T_xM$. Suppose $L(x, v)$ is a continuously differentiable function on TM.

Let $x_0, x_1 \in M$ be two points on M and let $x(t)$ be a continuosly differentiable curve on M, $t \in [t_0, t_1]$, which links these two points, $x(t_0) = x_0,\ x(t_1) = x_1$. We can define the following functional on the class of such curves:

$$\mathcal{L}(x(t)) = \int_{t_0}^{t_1} L(x(\tau), \dot{x}(\tau))\, d\tau.$$

It assigns to any such curve $x(t)$ a real number.

A necessary condition for a curve $x(t)$ to minimize the functional $\mathcal{L}$ are the Euler-Lagrange equations

$$\frac{\delta}{\delta x} L(x, \dot{x}) = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0. \tag{5}$$

These equations form an implicit system of differential equations of second order. In this way the function L, also called *Lagrangian* or Lagrange function, induces a dynamical system on M.

Let now $\mathcal{G}$ be a Lie group acting on M such that

$$L(x,v) = L(G(x), DG(v)) \quad \forall\, G \in \mathcal{G}, \tag{6}$$

where DG is the differential of the mapping G. This means that the Lagrangian L is invariant with respect to the action of $\mathcal{G}$. Let $g \in \mathbf{g}$ be an element of the Lie algebra of $\mathcal{G}$. Then we can associate the vector field

$$\xi_g(x) = \frac{d}{d\varepsilon} \exp(\varepsilon g)[x]|_{\varepsilon=0} \tag{7}$$

with g. Here for any $\varepsilon \in \mathbf{R}$ the mapping $\exp(\varepsilon g)$ is in $\mathcal{G}$.

Inserting $G = \exp(\varepsilon g)$ into (6) and differentiating with respect to ε at $\varepsilon = 0$, we obtain

$$\left\langle \frac{\partial L(x,v)}{\partial x}, \xi_g \right\rangle + \left\langle \frac{\partial L(x,v)}{\partial v}, D\xi_g[v] \right\rangle = 0. \tag{8}$$

Here $D\xi_g[v]$ is the derivative of ξ_g in the direction v.

It is now easily checked that the quantity $\langle \frac{\partial L}{\partial \dot{x}}, \xi_g \rangle$ remains constant along the trajectories of (5):

$$\begin{aligned} \frac{d}{dt}\left\langle \frac{\partial L}{\partial \dot{x}}, \xi_g \right\rangle &= \left\langle \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right), \xi_g \right\rangle + \left\langle \frac{\partial L}{\partial \dot{x}}, D\xi_g[\dot{x}] \right\rangle \\ &= \left\langle \frac{\partial L}{\partial x}, \xi_g \right\rangle + \left\langle \frac{\partial L}{\partial \dot{x}}, D\xi_g[\dot{x}] \right\rangle = 0. \end{aligned}$$

The same holds for the quantity $L - \langle \frac{\partial L}{\partial \dot{x}}, \dot{x} \rangle$:

$$\frac{d}{dt}\left(L - \left\langle \frac{\partial L}{\partial \dot{x}}, \dot{x} \right\rangle\right) = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\dot{x} - \frac{\partial L}{\partial \dot{x}}\ddot{x} = 0.$$

These results are the subject of Noether's theorem.

Theorem 1 (Noether) *Let the Lagrangian $L(x,\dot{x})$ be invariant under the action of the Lie group $\mathcal{G}$. Then, for any element $g \in \mathbf{g}$ of the Lie algebra of $\mathcal{G}$, the entity $\langle \frac{\partial L}{\partial \dot{x}}, \xi_g \rangle$ is preserved under the system dynamics defined by L. In addition, the quantity $L - \langle \frac{\partial L}{\partial \dot{x}}, \dot{x} \rangle$ is preserved.* □

This theorem is extremely important in Theoretical Physics, as it links symmetries of the Lagrangian, i.e. the underlying physivcal system, to invariants of the system dynamics induced by this Lagrangian.

Remark. The first group of invariants can be interpreted as (generalized) impulses, while the last invariant can be interpreted as energy. Theorem 1 is actually a special case of Noether's theorem, which corresponds to a time-invariant Lagrangian. The energy integral corresponds to the one-dimensional Lie group of transformations $t \mapsto t + const$ and exists independently of the presence of a symmetry group $\mathcal{G}$.

Noether's theorem applied to Riemannian spaces

In this subsection we demonstrate how Noether's theorem links the isometries of a Riemannian manifold to the invariants of its geodesic flow. To this end we define a Lagrange function which induces the geodesic flow. It turns out that by definition it is invariant with respect to the isometry group of the Riemannian manifold. The results of this subsection can be found e.g. in [4].

The geodesic flow of a Riemannian manifold M can be interpreted as dynamical system with Lagrangian

$$L(x,v) = \frac{1}{2} g(x)[v,v] = \frac{1}{2} \sum_{\mu\nu} g_{\mu\nu} v^\mu v^\nu. \tag{9}$$

Indeed, the reader will easily verify that inserting (9) into (5) yields (4). By definition a mapping $G : M \to M$ is an isometry if it preserves the quadratic form g, i.e. the quantity (9) for any vector $v \in T_x M$. Hence the Lagrangian (9) is invariant under the action of the isometry group.

Let a be an element of the Lie algebra of the isometry group.

Definition 2 *The corresponding vector field ξ_a defined by (7) is called a Killing vector field.*

By Noether's theorem, any Killing vector field ξ_a induces an invariant

$$I_a(x,\dot{x}) = \sum_{\mu\nu} g_{\mu\nu} \dot{x}^\mu \xi_a^\nu(x)$$

of the geodesic flow. Here ξ_a^ν is the ν-th component of the vector field ξ_a.

In addition, we have the energy integral

$$L - \left\langle \frac{\partial L}{\partial \dot{x}}, \dot{x} \right\rangle = -\frac{1}{2} \sum_{\mu\nu} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -L.$$

Note that along a geodesic the quantity $\sqrt{\sum_{\mu\nu} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = |\dot{x}|_{g(x)}$ is nothing else than the derivative of the length and thus equal to 1. Hence the energy integral equals $-\frac{1}{2}$.

Let us summarize these results in the following lemma.

Lemma 1 *Let M be a Riemannian manifold with metric $g_{\mu\nu}(x)$. Then the quantity $\sum_{\mu\nu} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = g[\dot{x},\dot{x}] \equiv 1$ is an invariant of the geodesic flow. Let $\mathcal{G}$ be the isometry group of M. Let $a \in \mathbf{g}$ be an element of the Lie algebra of $\mathcal{G}$ and let ξ_a be the corresponding Killing vector field. Then the entity $I_a = \sum_{\mu\nu} g_{\mu\nu}\dot{x}^\mu\xi_a^\nu(x) = g[\dot{x},\xi_a]$ is an invariant of the geodesic flow on M.* $\square$

3 Application to the metric on convex sets

In this section we compute invariants of the geodesic flow on convex sets corresponding to the symmetries of these sets. We investigate when a barrier function is consistent with the automorphism group of the set. Specifically, the metric induced by the universal barrier function is always invariant with respect to the whole automorphism group. The universal barrier function is a barrier function which is naturally linked to the convex set. A precise definition will be given later on.

Let X be a convex set and F a barrier on it. Then the isometry group of X, equipped with the Riemannian metric induced by F, consists of all diffeomorphisms G of X such that $F''(x)[u,v] = F''(G(x))[DG(u), DG(v)]$ for all $x \in X$ and for all $u, v \in T_xX$. This can be written as

$$F''(x) = (DG(x))^T F''(G(x))(DG(x)) \qquad \forall\, x \in X, \tag{10}$$

where $DG(x)$ is the Jacobian of G at x.

The results of the previous section furnish the following theorem.

Theorem 2 *Let X be a convex set, equipped with a Riemannian metric induced by a barrier F. Then the quantity $F''(x)[\dot{x},\dot{x}] \equiv 1$ is an invariant of the geodesic flow. Let $\mathcal{G}$ be the group of diffeomorphisms G of X satisfying (10). Let $g \in \mathbf{g}$ be an element of the Lie algebra of $\mathcal{G}$ and let ξ_g be the corresponding Killing vector field on X. Then the entity $I_g = F''(x)[\dot{x},\xi_g] = \langle \dot{x},\xi_g\rangle_{F''(x)}$ is an invariant of the geodesic flow on X.* $\square$

Remark. The convex structure of X is preserved only under affine transformations. Nevertheless, also nonlinear diffeomorphisms of X can induce invariants of the geodesic flow, as long as they are isometries of the metric defined by F, i.e. satisfy (10). In general, Riemannian geometry does not distinguish between affine and nonlinear transformations. Therefore expressing the derivatives of the barrier F in the invariant language of tensor algebra allows to extend the class of allowable transformations also to nonlinear diffeomorphisms. Whether this will reveal new properties of barrier functions and push forward the corresponding optimization algorithms remains to be investigated.

Let us now focus on the group $Aut(X)$ of affine diffeomorphisms of X. Let $G \in Aut(X)$. Then the action of G can be described by an affine transformation $x \mapsto Ax + b$ and $DG = A$ is a constant matrix. Note that under these conditions we have $\frac{\partial^2}{\partial x^2}[F(G(x))] = A^T F''(G(x))A$ and therefore (10) simplifies to

$$\frac{\partial^2}{\partial x^2}[F(G(x)) - F(x)] = 0 \qquad \forall\, x \in X. \tag{11}$$

Hence $G \in Aut(X)$ satisfies (10) if and only if the difference $F(G(x)) - F(x)$ is an affine function of x, i.e. there exists a vector α and a constant β such that $F(G(x)) - F(x) = \langle \alpha, x \rangle + \beta$.

Here the quantities α, β depend on $G \in Aut(X)$, or, equivalently, on the pair (A, b) representing the automorphism G. Let $G_1, G_2 \in AutX$ be represented by the pairs $(A_1, b_1), (A_2, b_2)$, respectively. It is then easily checked that $\alpha(G_2 \circ G_1) = \alpha(G_1) + A_1^T \alpha(G_2)$, $\beta(G_2 \circ G_1) = \beta(G_1) + \beta(G_2) + \langle \alpha(G_2), b_1 \rangle$.

One can also consider the infinitesimal condition of invariance. Let g be an element of the Lie algebra of $AutX$. Then the corresponding vector field ξ_g is an affine function of x, i.e. there exist a matrix $\mathbf{A}$ and a vector $\mathbf{b}$ such that $\xi_g(x) = \mathbf{A}x + \mathbf{b}$. In this setting (8) transforms to $F'''[v, v, \mathbf{A}x + \mathbf{b}] + 2\langle F''[v], \mathbf{A}v \rangle = 0$ for all $x \in X, v \in T_x X$, which is equivalent to the system

$$F'''\mathbf{A} = 0, \qquad F'''[\mathbf{b}] + F''\mathbf{A} + \mathbf{A}^T F'' = 0. \tag{12}$$

Let us summarize these results in the following lemma.

Lemma 2 *An affine diffeomorphism G of X belongs to the isometry group of the Riemannian space X if and only if it satisfies (11). An element g of the Lie algebra of $AutX$, represented by the pair $(\mathbf{A}, \mathbf{b})$, belongs to the Lie algebra of the isometry group of X if and only if it satisfies system (12).* □

Convex sets X with a nontrivial Lie algebra of the automorphism group are relatively common. Examples include

- Spheres

- Lorentz cones
- Rays, lines
- Cones
- Parabolic sets $\{(y, x) \mid y > \langle x, x\rangle\}$
- Hyperbolic sets $\{(y, x) \mid y > 0,\ y^2 > |x|^2 + 1\}$
- Cones of positive semidefinite matrices
- Sets which contain the previous ones as a factor
- Affine images of the previous sets

Usually these sets are equipped with barrier functions that are consistent with the whole automorphism group. This property is guaranteed for the universal barrier function.

Definition 3 (Nesterov, Nemirovski [6]) *Let $X \subset \mathbf{R}^n$ be a convex set. For $x \in X$ define the polar set of X centered at x by $X^*(x) = \{y \in \mathbf{R}^n \mid \langle z - x, y\rangle \leq 1\ \forall\, z \in X\}$. Then the function $F(x) = \log Vol_n(X^*(x))$, where Vol_n denotes the volume in $\mathbf{R}^n$, is called the universal barrier function of X.*

Lemma 3 *Let X be a convex set and F its universal barrier function. Then the group $AutX$ of affine diffeomorphisms of X is contained in the isometry group of X, considered as Riemannian space equipped with the metric induced by F.*

Proof. The lemma follows from the fact that every affine diffeomorphism G of X can be extended to a linear automorphism A of the cone C fitted to X. If A is a linear automorphism of a convex cone C, then for the universal barrier function F of the cone we have $F(Ax) - F(x) = -\ln|\det A| = const$ for all $x \in C$ [10]. Hence we have also $F(Gx) - F(x) = -\ln|\det A| = const$ for all $x \in X$. Here A is the linear part of G. $\square$

It becomes evident that Theorem 2 applies to a rich class of convex sets and barrier functions on them, including the standard cones equipped with the standard barrier functions used in Mathematical Programming. The applications of Theorem 2 reach far beyond obtaining first integrals of the geodesic equations for lowering their dimension. The theorem allows to find geodesic submanifolds of X and thus to obtain a picture of the geometric configuration of the geodesics. One can then identify lower-dimensional convex sets which are embedded in X together with the structure induced by their own barriers.

4 Convex hulls of second order algebraic varieties

In this section we consider three convex sets with an especially rich symmetry group, namely the convex hulls of a sphere, a paraboloid and a hyperboloid. It turns out that for these sets the apparatus developed above yields enough invariants of the geodesic flow to completely integrate the geodesic equations and to obtain explicit formulae for the geodesics. Moreover, it allows to identify rich families of geodesic submanifolds of different dimensions. The geodesic equations for the two-dimensional parabolic set were found in [8].

Two-dimensional case

Before passing to the above-mentioned sets in n dimensions, let us consider their 2-dimensional prototypes, namely the convex sets

$$\begin{aligned} D &= \{(x,y)\,|\,x^2+y^2<1\}, \\ P &= \{(x,y)\,|\,y-x^2>0\}, \\ H &= \{(x,y)\,|\,x,y>0,\ xy>1\} \end{aligned}$$

with the respective barriers

$$\begin{aligned} F_D &= -\ln(-x^2-y^2+1), \\ F_P &= -\ln(y-x^2), \\ F_H &= -\ln(xy-1) \end{aligned}$$

and Hessians

$$\begin{aligned} F''_D &= \frac{1}{(1-x^2-y^2)^2}\begin{pmatrix} 2(1+x^2-y^2) & 4xy \\ 4xy & 2(1-x^2+y^2) \end{pmatrix}, \\ F''_P &= \frac{1}{(y-x^2)^2}\begin{pmatrix} 2(y+x^2) & -2x \\ -2x & 1 \end{pmatrix}, \\ F''_H &= \frac{1}{(xy-1)^2}\begin{pmatrix} y^2 & 1 \\ 1 & x^2 \end{pmatrix}. \end{aligned}$$

Let us perform a nonlinar coordinate transformation and introduce the coordinates

$$\begin{aligned} \varphi_d &= \arctan\tfrac{y}{x} & r_d &= \sqrt{x^2+y^2}, \\ \varphi_p &= x & r_p &= y-x^2, \\ \varphi_h &= \ln\tfrac{y}{x} & r_h &= xy-1 \end{aligned}$$

on D, P, H respectively. By (3), in these coordinates the metric on D, P, H takes the form

$$g_d = \begin{pmatrix} \frac{2r_d^2}{1-r_d^2} & 0 \\ 0 & \frac{2(1+r_d^2)}{(1-r_d^2)^2} \end{pmatrix}, \quad g_p = \begin{pmatrix} \frac{2}{r_p} & 0 \\ 0 & \frac{1}{r_p^2} \end{pmatrix}, \quad g_h = \frac{1}{2r_h} \begin{pmatrix} r_h + 1 & 0 \\ 0 & \frac{r_h+2}{r_h(r_h+1)} \end{pmatrix},$$

respectively.

Thus in all three cases the metric tensor is diagonal and depends only on one of the two coordinates. Let us investigate the geodesic flow for a general 2-dimensional Riemannian manifold with a metric of this type. Suppose the manifold is parameterized by the coordinates φ, r and the components of the metric are given by

$$g_{\varphi\varphi} = g_{\varphi\varphi}(r), \quad g_{r\varphi} = 0, \quad g_{rr} = g_{rr}(r).$$

Let us apply Lemma 1. The energy integral yields

$$g_{rr}(\dot{r})^2 + g_{\varphi\varphi}(\dot{\varphi})^2 = 1.$$

In addition we have a one-dimensional isometry group consisting of transformations $\varphi \mapsto \varphi + const$. This group yields the invariant

$$g_{\varphi\varphi}\dot{\varphi} = c,$$

which can physically be interpreted as momentum.

Combining, we obtain

$$\dot{r} = \operatorname{sgn}(\dot{r})\sqrt{\frac{g_{\varphi\varphi} - c^2}{g_{\varphi\varphi}g_{rr}}} \;\Rightarrow\; \frac{ds}{dr} = \operatorname{sgn}(\dot{r})\sqrt{\frac{g_{\varphi\varphi}g_{rr}}{g_{\varphi\varphi} - c^2}}, \; \frac{d\varphi}{dr} = \operatorname{sgn}(\dot{r})c\sqrt{\frac{g_{rr}}{g_{\varphi\varphi}^2 - c^2 g_{\varphi\varphi}}},$$

where s parameterizes the length of the geodesic. Finally we have

$$\begin{aligned} s(r) &= \operatorname{sgn}(\dot{r}) \int_{r_0}^{r} \sqrt{\frac{g_{\varphi\varphi}(\rho)g_{rr}(\rho)}{g_{\varphi\varphi}(\rho) - c^2}}\, d\rho + s(r_0), \\ \varphi(r) &= \operatorname{sgn}(\dot{r})c \int_{r_0}^{r} \sqrt{\frac{g_{rr}(\rho)}{g_{\varphi\varphi}^2(\rho) - c^2 g_{\varphi\varphi}(\rho)}}\, d\rho + \varphi(r_0). \end{aligned}$$

If we insert the expressions for $g_{\varphi\varphi}, g_{rr}$ for the three considered cases and resolve the appearing integrals, we obtain the following results.

The disc D

On the disc D we obtain a special family of radial geodesics for $c = 0$, namely $\varphi_d = const$,

$$s(r_d) = sgn(\dot{r}_d)\left[\ln\frac{1+r_d+\sqrt{2}\sqrt{1+r_d^2}}{1-r_d+\sqrt{2}\sqrt{1+r_d^2}} + \ln\frac{1+r_d}{1-r_d} + \frac{\sqrt{2}}{2}\ln\frac{\sqrt{1+r_d^2}-r_d}{\sqrt{1+r_d^2}+r_d}\right] + s(0).$$

These are exactly those geodesics which pass through the centre of the disc. For $c \neq 0$ we obtain

$$\begin{aligned}\varphi_d(r_d) &= \frac{sgn(c\dot{r}_d)}{2}\left[\frac{1}{\sqrt{\frac{2}{c^2}+1}}\ln\left(\frac{\frac{1}{c^2}+(\frac{2}{c^2}+1)\rho^2}{\sqrt{\frac{2}{c^2}+1}} + \sqrt{(1+\rho^2)\left(\left(\frac{2}{c^2}+1\right)\rho^2-1\right)}\right)\right.\\ &\left.\quad + \arctan\left(\frac{-1+\frac{1}{c^2}\rho^2}{\sqrt{(1+\rho^2)((\frac{2}{c^2}+1)\rho^2-1)}}\right)\right]_{r_0}^{r_d} + \varphi_d(r_0).\end{aligned}$$

These geodesics cannot pass through the centre. Moreover, it is easy to check that φ_d is monotone and $\dot{r}_d = 0$ yields $\ddot{r}_d > 0$. Hence any of these geodesics consists of two branches, where $\dot{r}_d > 0$ and $\dot{r}_d < 0$, respectively. At the junction of these branches r_d reaches a unique minimum on the given geodesic. Let us put r_0 equal to this minimum. Then we have $\dot{r}_d(r_0) = 0$, which yields $c = sgn(\dot{\varphi}_d)\sqrt{\frac{2r_0^2}{1-r_0^2}}$. Inserting, we obtain

$$\begin{aligned}\varphi_d(r_d) = \frac{sgn(\dot{r}_d\dot{\varphi}_d)}{2}&\left\{r_0\ln\left(\frac{\frac{1}{r_0}-r_0+2\frac{r_d^2}{r_0}+2\sqrt{(1+r_d^2)(\frac{r_d^2}{r_0^2}-1)}}{\frac{1}{r_0}+r_0}\right)+\right.\\ &\left.+\arctan\left(\frac{-1+\frac{r_d^2}{2}\left(\frac{1}{r_0^2}-1\right)}{\sqrt{(1+r_d^2)(\frac{r_d^2}{r_0^2}-1)}}\right)+\frac{\pi}{2}\right\} + \varphi_d(r_0),\end{aligned}$$

$$s(r_d) = sgn(\dot{r}_d)\left\{\sqrt{\frac{1-r_0^2}{2}}\ln\frac{\sqrt{\frac{r_d^2+1}{r_d^2-r_0^2}}-1}{\sqrt{\frac{r_d^2+1}{r_d^2-r_0^2}}+1} + \ln\frac{\sqrt{\frac{r_d^2+1}{r_d^2-r_0^2}}+\sqrt{\frac{2}{1-r_0^2}}}{\sqrt{\frac{r_d^2+1}{r_d^2-r_0^2}}-\sqrt{\frac{2}{1-r_0^2}}}\right\} + s(r_0).$$

In any case, for $r_d \to 1$ the length of the geodesic tends to infinity.

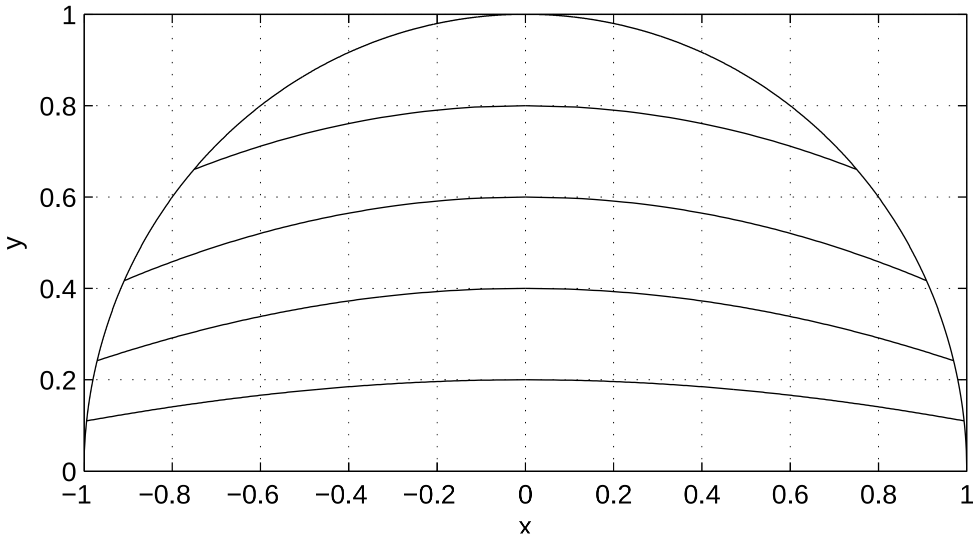

Figure 1: Geodesics on the disc D

The parabolic set P

On the parabolic set P the special family of geodesics corresponding to $c = 0$ is given by

$$x = const, \quad y(s) = x^2 + (y(0) - x^2)e^{sgn(\dot{y})s}.$$

These are vertical lines whose length tends to infinity as they approach the boundary of P or escape to $y \to +\infty$. For $c \neq 0$ we obtain

$$y(x) = \frac{1}{2}(x - x_0)(x + 3x_0) + \frac{2}{c}sgn(\dot{r}_p)(x - x_0)\sqrt{1 - \frac{c^2}{2}(y(x_0) - x_0^2)} + y(x_0).$$

Thus the geodesics are parabolas given by $y(x) = \frac{1}{2}x^2 + ax + b$, where a, b are constants. If such a parabola intersects P, then it intersects the boundary of P in two points. At these points we have $r_p = 0$. Moreover, $r_p(x) = -\frac{1}{2}x^2 + ax + b$. Therefore $r_p(x)$ reaches a unique maximum on any such geodesic. This maximum is given by $r_{\max} = \frac{2}{c^2}$. With this notation we obtain

$$s(r_p) = sgn(\dot{r}_p) \ln \frac{1 - \sqrt{1 - \frac{r_p}{r_{\max}}}}{1 + \sqrt{1 - \frac{r_p}{r_{\max}}}} + s(r_{\max}).$$

This formula allows us to compute explicitly the length of a geodesic linking two points $(x_1, y_1), (x_2, y_2) \in P$. We have

$$\begin{aligned}\Delta s &= \ln \frac{2(y_1+y_2)-(x_1+x_2)^2 + \sqrt{(2(y_1+y_2)-(x_1+x_2)^2)^2 - 16(y_1-x_1^2)(y_2-x_2^2)}}{2(y_1+y_2)-(x_1+x_2)^2 - \sqrt{(2(y_1+y_2)-(x_1+x_2)^2)^2 - 16(y_1-x_1^2)(y_2-x_2^2)}} \\ &= \ln \frac{\left[2(y_1+y_2)-(x_1+x_2)^2+\sqrt{(2(y_1+y_2)-(x_1+x_2)^2)^2-16(y_1-x_1^2)(y_2-x_2^2)}\right]^2}{16(y_1-x_1^2)(y_2-x_2^2)}\end{aligned}$$

If one of the points tends to the boundary of P, then the length of the geodesic arc tends to infinity.

The hyperbolic set H

Finally, resolving the integrals for the hyperbolic set H, we obtain

$$s(r_h) = s(r_0)+sgn(\dot r_h)\begin{cases}\left[\ln \frac{1-t}{1+t} - \frac{1}{\sqrt{2-4c^2}} \ln \frac{t\sqrt{2-4c^2}-1}{t\sqrt{2-4c^2}+1}\right]_{t(r_0)}^{t(r_h)}, & c^2 < \frac{1}{4}, \\ \left[\ln r_h\right]_{r_0}^{r_h}, & c^2 = \frac{1}{4}, \\ \left[\ln \frac{t-1}{t+1} - \frac{1}{\sqrt{2-4c^2}} \ln \frac{1-t\sqrt{2-4c^2}}{1+t\sqrt{2-4c^2}}\right]_{t(r_0)}^{t(r_h)}, & \frac{1}{4} < c^2 < \frac{1}{2}, \\ \left[\sqrt{2}\sqrt{r_h+2} + \ln \frac{\sqrt{r_h+2}-\sqrt{2}}{\sqrt{r_h+2}+\sqrt{2}}\right]_{r_0}^{r_h}, & c^2 = \frac{1}{2}, \\ \left[\ln \frac{t-1}{t+1} + \frac{2}{\sqrt{4c^2-2}} \arctan(t\sqrt{4c^2-2})\right]_{t(r_0)}^{t(r_h)}, & c^2 > \frac{1}{2},\end{cases}$$

where $t = \sqrt{\frac{r_h+2}{(2-4c^2)r_h+2}}$. For $c^2 \le \frac{1}{2}$ the function $r_h(s)$ is monotone. For $c^2 > \frac{1}{2}$ we have $\dot r_h = 0$ at $r_h = \frac{1}{2c^2-1}$. It can be easily checked that for $r_h = \frac{1}{2c^2-1}$ we also have $\ddot r_h < 0$. Hence r_h assumes a unique maximum on geodesics with $c^2 > \frac{1}{2}$ and these geodesics consist of two branches with different signs of $\dot r_h$.

Computing φ_h as function of r_h and lumping the constants of integration together, we obtain

$$c_1 y = (\sqrt{1+xy} - sgn(\dot r_h c))e^{sgn(\dot r_h c)\sqrt{1+xy}}, \qquad c^2 = \frac{1}{2};$$

$$x = const, \qquad c = \frac{1}{2}; \qquad\qquad y = const, \qquad c = -\frac{1}{2};$$

and

$$\frac{\ln \frac{y}{x} + c_1}{sgn(\dot r_h)} = c\begin{cases}2c \ln \frac{\sqrt{1+(1-2c^2)r}-\sqrt{2}c\sqrt{r+2}}{\sqrt{1+(1-2c^2)r}+\sqrt{2}c\sqrt{r+2}} + \sqrt{2}\sqrt{1-2c^2}\ln \frac{\sqrt{1-2c^2}\sqrt{r+2}+\sqrt{1+(1-2c^2)r}}{\sqrt{1-2c^2}\sqrt{r+2}-\sqrt{1+(1-2c^2)r}} \\ \frac{1}{c}\ln \frac{\sqrt{2}c\sqrt{r+2}-\sqrt{1+(1-2c^2)r}}{\sqrt{2}c\sqrt{r+2}+\sqrt{1+(1-2c^2)r}} - \frac{2\sqrt{2}}{\sqrt{2c^2-1}} \arctan \sqrt{\frac{1+(1-2c^2)r}{(r+2)(2c^2-1)}}\end{cases}$$

if $c^2 \in (0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{1}{2})$ or $c^2 > \frac{1}{2}$ respectively. Hence we have two types of geodesics.

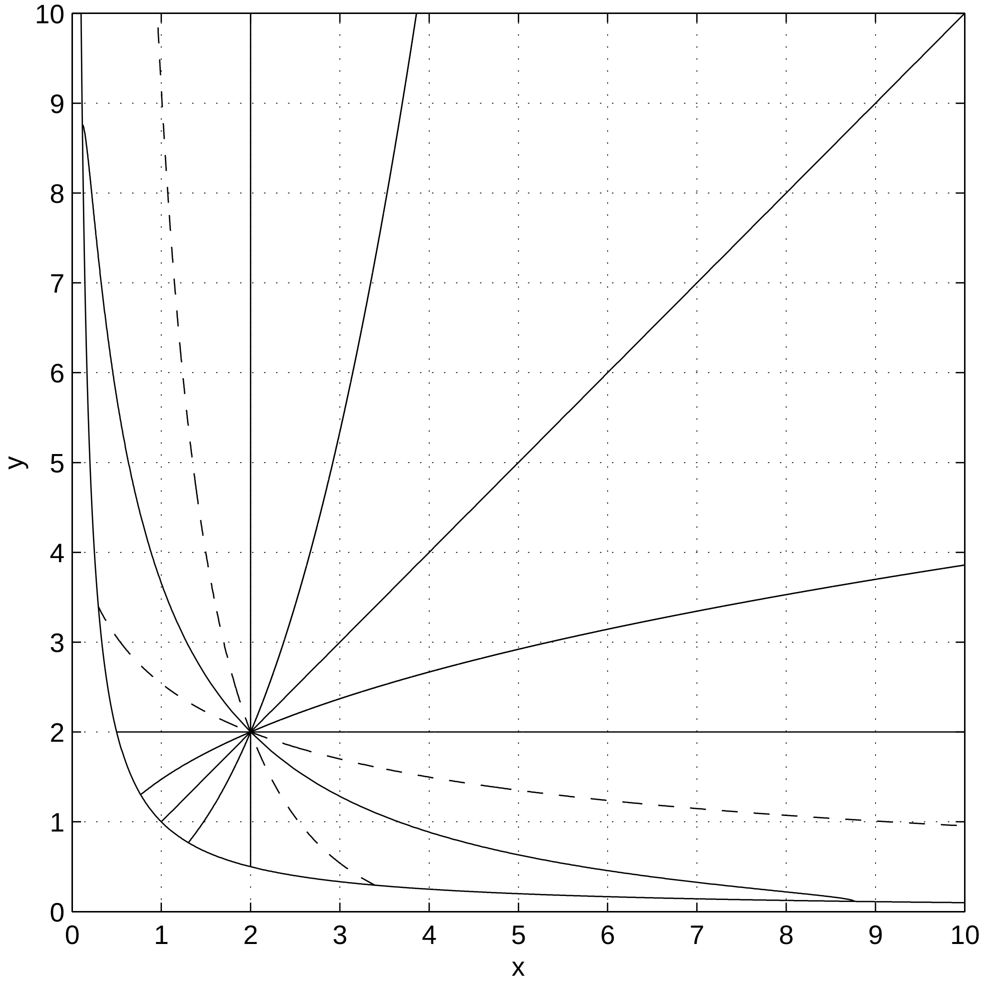

Figure 2: Geodesics on the hyperbolic set H

For $c^2 \leq \frac{1}{2}$ the geodesics tend on one end to the boundary of H and on the other end to infinity. For $c^2 > \frac{1}{2}$ the geodesics tend with both ends to the boundary of H. The geodesics with $c^2 = \frac{1}{2}$ separate these two types. In Figure 2 they are drawn as dashed lines. All geodesics have infinite length on both sides.

Multidimensional case

In this subsection we show that any geodesic on the $(n+1)$-dimensional analogs of the sets D, P, H can be embedded in a two-dimensional affine geodesic submanifold, which is isomorphic to D, P, H, respectively. Thus the solution for the two-dimensional case given in the previous subsections is sufficient to integrate the geodesic flow on the multi-dimensional bodies.

Let us consider the convex hull of the n-dimensional sphere, $D^{n+1} = \{x \in \mathbf{R}^{n+1} \mid |x| < 1\}$.

Lemma 4 *For any geodesic on D^{n+1}, equipped with the barrier $F(x) = -\ln(1-|x|^2)$, there exists a 2-dimensional linear subspace of $\mathbf{R}^{n+1}$ which contains this geodesic.*

Proof. It is easy to check that the Hessian of the barrier F is given by

$$F''(x) = \frac{2I_{n+1}}{1-|x|^2} + 4\frac{xx^T}{(1-|x|^2)^2}.$$

Let us consider the isometry group of D^{n+1} equipped with the corresponding metric. It is the orthogonal group O^{n+1}. The Lie algebra of this group is given by the space of skew-symmetric $(n+1)\times(n+1)$ matrices. Let A be an element of this Lie algebra. The corresponding Killing vector field is given by $\xi_A(x) = Ax$. This yields the invariant

$$I_A = \dot{x}^T F''(x) Ax = \langle x\dot{x}^T F''(x), A\rangle.$$

Since we can plug in any skew-symmetric matrix A, we obtain the matrix-valued invariant

$$I = \frac{1}{2}(x\dot{x}^T F''(x) - F''\dot{x}x^T) = \frac{x\dot{x}^T - \dot{x}x^T}{1-|x|^2}.$$

Consider a point x in D and a geodesic with velocity $\dot{x}$ going through this point. These define the vector-valued quantities $\frac{x}{1-|x|^2}$, $\dot{x}$. Now let us consider x as a point in the Euclidean space $\mathbf{R}^{n+1}$. By the canonical structure of $\mathbf{R}^{n+1}$, the components of the quantities $\frac{x}{1-|x|^2}$, $\dot{x}$ define two cotangent vectors at x. Now the (ij)-th component of I is given by

$$\left(\frac{x}{1-|x|^2}\right)_i (\dot{x})_j - \left(\frac{x}{1-|x|^2}\right)_j (\dot{x})_i.$$

This means that the components of I form exactly the outer product of these two cotangent vectors, which is a 2-form. With a slight abuse of notation we can hence write $I = \frac{x}{1-|x|^2} \wedge \dot{x}$.

Now let us consider two cases.

i) Suppose that $I = 0$ on the considered geodesic. Then at any point of the geodesic the vectors x and $\dot{x}$ are linearly dependent. Since we have $\dot{x} \neq 0$ everywhere, this means that the geodesic is part of a line going through the origin, i.e. it can be embedded in a linear subspace of dimension 1.

ii) Suppose $I \neq 0$ on the considered geodesic. Then at any point of the geodesic the form $\frac{x}{1-|x|^2} \wedge \dot{x}$ remains constant along the geodesic. Therefore the vectors x and $\dot{x}$ span the same two-dimensional linear subspace of $\mathbf{R}^{n+1}$ at any point of the considered geodesic. It follows that the geodesic is completely contained in this subspace, which completes the proof. $\Box$

Clearly the intersection of D^{n+1} with any two-dimensional linear subspace is isomorphic to D. As a by-product we obtain a family of geodesic submanifolds described by the following lemma.

Lemma 5 *The intersection of D^{n+1} with any linear subspace of $\mathbf{R}^{n+1}$ is a geodesic submanifold of D^{n+1}. This submanifold is isometric to D^k, where k is the dimension of the linear subspace.* $\Box$

Let us consider the convex hull of the n-dimensional paraboloid, $P^{n+1} = \{(x,y) \in \mathbf{R}^n \times \mathbf{R} \,|\, y > |x|^2\}$.

Lemma 6 *For any geodesic on P^{n+1}, equipped with the barrier $F(x,y) = -\ln(y-|x|^2)$, there exists a 2-dimensional affine subspace of $\mathbf{R}^{n+1}$ which contains this geodesic and is isomorphic to P.*

Proof. The Hessian of the barrier F is given by

$$F''(x,y) = \frac{1}{(y-|x|^2)^2} \begin{pmatrix} 4xx^T & -2x \\ -2x^T & 1 \end{pmatrix} + \frac{1}{y-|x|^2} \begin{pmatrix} 2I_n & 0 \\ 0 & 0 \end{pmatrix}.$$

We shall investigate the isometry group of P^{n+1} equipped with the corresponding metric. First we derive the automorphism group of P^{n+1}.

Let $x \mapsto a_2 y + Ax + b$, $y \mapsto a_0 y + a_1^T x + b_0$, $a_0, b_0 \in \mathbf{R}$, $a_1, a_2, b \in \mathbf{R}^n$, $A \in \mathbf{R}^{n\times n}$ be an affine transformation of $\mathbf{R}^{n+1} = \mathbf{R}^n \times \mathbf{R}$. It is easy to check that it preserves P^{n+1} if and only if $a_0 > 0$, $a_2 = 0$, $A^T A = a_0 I_n$, $a_1 = 2A^T b$, $b_0 = b^T b$. Hence the automorphism group of P^{n+1} has $\frac{n(n+1)}{2} + 1$ dimensions. For any such automorphism G we have

$$F(G(x,y)) = -\ln(a_0 y + 2b^T Ax + b^T b - |Ax+b|^2) = -\ln(a_0) + F(x,y).$$

Therefore G satisfies (11) and is an isometry of P^{n+1}. Hence $AutP^{n+1}$ is contained in the isometry group of P^{n+1}.

The Lie algebra of $AutP^{n+1}$ consists of all affine transformations $x \mapsto Ax + b$, $y \mapsto a_0 y + 2b^T x$ such that $A + A^T = 2a_0 I_n$. The corresponding Killing vector fields are given by $\xi_{a_0,A,b}(x,y) = (Ax + b, a_0 y + 2b^T x)$. The elements of the Lie algebra given by $(a_0, A, b) = (1, \frac{1}{2}I_n, 0)$, $(a_0, A, b) = (0, S, 0)$, S any skew-symmetric matrix, and $(a_0, A, b) = (0, 0, b)$ yield the invariants

$$\frac{1}{y - |x|^2}(\dot{y} - \dot{x}^T x), \qquad \frac{2}{y - |x|^2}(x\dot{x}^T - \dot{x}x^T), \qquad \frac{2}{y - |x|^2}\dot{x}$$

of the geodesic flow on P^{n+1}. Let now $c \in \mathbf{R}^n$ be an arbitrary constant vector. Combining the last two invariants appropriately, we obtain the new invariant $I_c = \frac{1}{y-|x|^2}((x - c)\dot{x}^T - \dot{x}(x - c)^T$.

Let now $x(s)$ be a geodesic and let $x_0 = x(0)$ be a point on this geodesic. If we set $c = x_0$, then the invariant I_c is zero at $x(0)$ and hence is zero at any point on the considered geodesic. It follows that the vectors $\dot{x}$ and $x - x_0$ are linearly dependent everywhere on this geodesic. But then $x(s) - x_0$ lies in a 1-dimensional linear subspace of $\mathbf{R}^n$. Hence the geodesic can be embedded in an at most 2-dimensional affine subspace with offset $(x_0, 0)$, spanned by the vectors $(\dot{x}(0), 0)$ and $(0, 1)$.

It remains to show that the intersection of P^{n+1} with an affine subspace with offset $(x_0, 0)$ and spanned by the vectors $(v, 0)$, $(0, 1)$, $v \neq 0$, is isomorphic to the two-dimensional parabolic set P. Indeed, the intersection is given by the set of points $\tilde{P} = \{(x, y) \,|\, y > |x|^2, \ \exists\, \alpha \in \mathbf{R} : \ x = x_0 + \alpha v\}$. We can introduce the coordinates $\tilde{x} = (\alpha - \alpha_0)|v|$, $\tilde{y} = y - |x_0 + \alpha_0 v|^2$ on $\tilde{P}$, where $\alpha_0 = -\frac{x_0^T v}{|v|^2}$. Then we have $\tilde{P} = \{(\tilde{x}, \tilde{y}) \,|\, \tilde{y} > \tilde{x}^2\}$ and $F(\tilde{x}, \tilde{y}) = -\ln(\tilde{y} - \tilde{x}^2)$. Hence $\tilde{P}$ is an affine image of P and the restriction of the barrier $F = -\ln(y - |x|^2)$ to $\tilde{P}$ coincides with the barrier induced by the canonical barrier on P. This completes the proof. $\square$

We obtain the following lemma on geodesic submanifolds of P^{n+1}.

Lemma 7 *The intersection of P^{n+1} with any affine subspace of $\mathbf{R}^{n+1}$ containing a line of the form $\{(x, y) \,|\, x = const, \ y \in \mathbf{R}\}$ is a geodesic submanifold of P^{n+1}. This submanifold is isometric to P^k, where k is the dimension of the affine subspace.* $\square$

Let us consider the convex hull of the n-dimensional hyperboloid, $H^{n+1} = \{(x, y) \in \mathbf{R}^n \times \mathbf{R} \,|\, y > 0, \ y^2 > |x|^2 + 1\}$.

Lemma 8 *For any geodesic on the set H^{n+1}, equipped with the barrier $F(x, y) = -\ln(y^2 - |x|^2 - 1)$, there exists a 2-dimensional linear subspace of $\mathbf{R}^{n+1}$ which contains this geodesic and is isomorphic to H.*

Proof. The Hessian of the barrier F is given by

$$F''(x,y) = \frac{4}{(y^2-|x|^2-1)^2}\left(\begin{array}{cc} xx^T & -yx \\ -yx^T & y^2 \end{array}\right) + \frac{2}{y^2-|x|^2-1}\left(\begin{array}{cc} I_n & 0 \\ 0 & -1 \end{array}\right).$$

Let us investigate the isometry group of H^{n+1} equipped with the corresponding metric. First we derive the automorphism group of H^{n+1}.

Let $y \mapsto a_0y+a_1^Tx+b_0$, $x \mapsto a_2y+Ax+b$, $a_0,b_0 \in \mathbf{R}$, $a_1,a_2,b \in \mathbf{R}^n$, $A \in \mathbf{R}^{n\times n}$ be an affine transformation of $\mathbf{R}^{n+1}$. It is easy to check that it preserves H^{n+1} if and only if $a_0 > |a_2|$, $(a_0-|a_2|^2)I_n + a_1a_1^T = A^TA$, $a_0^2-|a_2|^2+b_0^2-b^Tb=1$, $a_0a_1 = A^Ta_2$, $a_0b_0 = a_2^Tb$, $b_0a_1 = A^Tb$. It follows that A must be regular since $A^TA \succeq (a_0^2-|a_2|^2)I$. By combining the other equalities we obtain $b_0 = b = 0$, $a_1 = \frac{1}{a_0}A^Ta_2$, $a_0^2-|a_2|^2=1$, $A^TA - \frac{1}{a_0^2}A^Ta_2a_2^TA = I$.

Hence the automorphism group of H^{n+1} consists only of linear mappings and has $\frac{n(n+1)}{2}$ dimensions. For any such automorphism G we have

$$\begin{aligned} F(G(x,y)) &= -\ln(a_0^2y^2+2a_0ya_1^Tx+x^Ta_1a_1^Tx-y^2|a_2|^2-2ya_2^TAx-x^TA^TAx-1) \\ &= -\ln(y^2-|x|^2-1) = F(x,y). \end{aligned}$$

Therefore G satisfies (11) and is an isometry of H^{n+1}.

It is readily seen that the Lie algebra of $AutH^{n+1}$ consists of all linear transformations of $\mathbf{R}^n \times \mathbf{R}$ given by $x \mapsto ay+Ax$, $y \mapsto a^Tx$, where $A = -A^T$ is any skew-symmetric matrix and $a \in \mathbf{R}^n$ is any vector. The Killing vector fields corresponding to a and A are given by $\xi_a(x,y) = (ay, a^Tx)$ and $\xi_A(x,y) = (Ax, 0)$. They yield the invariants

$$\frac{2}{y^2-|x|^2-1}(-\dot{y}x+y\dot{x}), \qquad \frac{2}{y^2-|x|^2-1}(x\dot{x}^T-\dot{x}x^T)$$

of the geodesic flow on H^{n+1}. Combining these invariants, we get that the 2-form

$$\frac{1}{y^2-|x|^2-1}\left(\begin{array}{c} x \\ y \end{array}\right) \wedge \left(\begin{array}{c} \dot{x} \\ \dot{y} \end{array}\right)$$

remains constant along any geodesic. Here the wedge product should be interpreted in the same way as indicated in the previous subsection. By a reasoning similar to that in the proof of Lemma 4 we have that the geodesic is completely contained in the at most 2-dimensional linear subspace spanned by the vectors $(x(0),y(0))$ and $(\dot{x}(0),\dot{y}(0))$.

It remains to show that the intersection of H^{n+1} with any 2-dimensional linear subspace is isomorphic to the two-dimensional hyperbolic set H. Indeed, let $\tilde{H}$

be a non-empty intersection of H^{n+1} with a 2-dimensional linear subspace. Then there exists a vector $(x, y) \in \tilde{H}$ with $y > 0$, $y^2 > |x|^2 + 1$. It follows that we can choose a basis $\{(x_1, y_1), (x_2, y_2)\}$ of the linear subspace generating $\tilde{H}$ such that without restriction of generality $y_1 = 0$, $y_2 > 0$, $|x_1|^2 = 1$, $x_2^T x_1 = 0$, $y_2^2 - |x_2|^2 = 1$. We can then represent $\tilde{H}$ as $\tilde{H} = \{(x, y) \,|\, \exists \alpha, \beta \in \mathbf{R} : (x, y) = \alpha(x_1, y_1) + \beta(x_2, y_2), y > 0, y^2 - |x|^2 > 1\} = \{(x, y) = \alpha(x_1, y_1) + \beta(x_2, y_2) \,|\, \beta > 0, \beta^2 - \alpha^2 > 1\}$. The restriction of the barrier F to $\tilde{H}$ is given by

$$F(x, y) = -\ln((\alpha y_1 + \beta y_2)^2 - |\alpha x_1 + \beta x_2|^2 - 1) = -\ln(\beta^2 - \alpha^2 - 1).$$

Note that the 2-dimensional hyperbolic set H can be represented as the set of points $H = \{(\alpha, \beta) \,|\, \beta > 0, \beta^2 - \alpha^2 > 1\}$ with barrier $-\ln(\beta^2 - \alpha^2 - 1)$. Thus $\tilde{H}$ is isometric to H. This completes the proof. □

We obtain the following lemma on geodesic submanifolds of H^{n+1}.

Lemma 9 *The intersection of H^{n+1} with any linear subspace of $\mathbf{R}^{n+1}$ is a geodesic submanifold of H^{n+1}. This submanifold is isometric to H^k, where k is the dimension of the linear subspace.* □

5 Symmetric cones

In this section we apply the results obtained in Section 3 to the standard symmetric cones used in Mathematical Programming. These cones have a symmetry group rich enough to allow a complete integration of the geodesic flow. In addition, we consider a general symmetric cone generated by a formally real Jordan algebra. The geodesic flow on the cone of squares of a formally real Jordan algebra can also be integrated departing from the structure of the Jordan algebra. A complete derivation was recently performed by Michel Baes. The geodesics on the standard symmetric cones were also computed in [8]. Departing from the structure of the Jordan algebra one can also find geodesic submanifolds of symmetric cones corresponding to sets of elements with the same Jordan frame, as was also shown by M. Baes [1].

Cone of positive semidefinite matrices

Consider the cone $S_+^n = \{X \in Sym(n) \,|\, X \succeq 0\}$ of positive semidefinite matrices with the standard barrier function $F(X) = -\ln|X|$. The symmetry group of this cone is isomorphic to $\mathcal{G} = GL(n)$ and acts as $X \mapsto S^T X S$ for any $S \in GL(n)$. The Lie algebra of this group is the space of all $n \times n$ matrices, $\mathbf{g} = M(n, n)$.

The Killing field $\xi_A(X) = XA + A^T X$ corresponding to a matrix $A \in M(n,n)$ generates the invariant

$$\begin{aligned} I_A &= \frac{\partial^2 F}{\partial X^\mu \partial X^\nu} \dot{X}^\mu (XA + A^T X)^\nu = \langle X^{-1}\dot{X}X^{-1}, XA + A^T X \rangle \\ &= 2\langle \dot{X}X^{-1}, A \rangle. \end{aligned}$$

Since we can plug in any A, we have the following lemma.

Lemma 10 *The matrix $\dot{X}X^{-1}$ is constant on any geodesic of the metric defined by the barrier $F(X) = -\ln|X|$ on the positive semidefinite cone.* □

This allows us to integrate the geodesic equations. We obtain

$$X(s) = \exp(\dot{X}(0)X^{-1}(0)s)X(0)$$

with s being the parameter of the length.

Positive orthant

Consider the positive orthant $R^n_+ = \{x \mid x_i \geq 0\}$ with the standard barrier function $F = -\sum_i \ln x_i$. Its symmetry group is given by $\mathcal{G} = \mathbf{R}^n_+$ and acts like $c \in \mathbf{R}^n_+ : x_i \mapsto c_i x_i$. It is easily seen that these transformations satisfy (11) and are also isometries. The Lie algebra of this group is $\mathbf{g} = \mathbf{R}^n$, the Killing vector field corresponding to $a \in \mathbf{R}^n$ is $\xi_a(x) = vec(a_i x_i)$. This yields the invariants $I_a = \sum_i a_i \frac{\dot{x}_i}{x_i}$.

Since we can plug in any a, we get that $\frac{\dot{x}_i}{x_i}$ is constant on any geodesic for any i. This allows to integrate the geodesic flow completely. Let $x(s)$ be a geodesic, let the invariant $\frac{\dot{x}_i}{x_i}$ have the value c_i on this geodesic. Then we have $x_i(s) = x_i(0)e^{c_i s}$.

Lorentz cone

Consider the Lorentz cone $L_n = \{(t,x) \in \mathbf{R} \times \mathbf{R}^{n-1} \mid t \geq |x|\}$ with the standard barrier function $F(t,x) = -\ln(t^2 - x^2)$. Its symmetry group is given by

$$\mathcal{G} = \left\{ \mathbf{U} = \begin{pmatrix} u_0 & u_1^T \\ u_2 & U \end{pmatrix} \mid \exists c > 0 : \ \mathbf{U}^T \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix} \mathbf{U} = c \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix} \right\}.$$

and acts like

$$\begin{pmatrix} t \\ x \end{pmatrix} \mapsto \mathbf{U} \begin{pmatrix} t \\ x \end{pmatrix}.$$

The Lie algebra of this group is given by

$$\mathbf{g} = \left\{ \mathbf{A} = \begin{pmatrix} \alpha_0 & \alpha^T \\ \alpha & A \end{pmatrix} \mid 2\alpha_0 I_{n-1} = A + A^T \right\}.$$

The corresponding Killing fields are

$$\xi_{\mathbf{A}}(t,x) = \mathbf{A} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \alpha_0 t + \alpha^T x \\ t\alpha + Ax \end{pmatrix}.$$

Hence we obtain the general invariant

$$I_{\mathbf{A}} = \frac{2}{t^2 - x^2}(\alpha_0 \dot{t} t - \dot{t}\alpha^T x - 2\alpha_0 \dot{x}^T x + t\alpha^T \dot{x} + \dot{x}^T A x).$$

Writing this in matrix form, we obtain the invariants

$$I_1 = \frac{t\dot{x} - \dot{t}x}{t^2 - x^2}; \qquad I_2 = \frac{x\dot{x}^T - \dot{x}x^T}{t^2 - x^2}; \qquad I_3 = \frac{\dot{t}t - \dot{x}^T x}{t^2 - x^2}.$$

The first two invariants can be joined to a skew-symmetric matrix which defines a 2-form, in the same way as explained above. Let us summarize this in the following lemma.

Lemma 11 *The 2-form*

$$\Omega = \frac{1}{t^2 - x^2} \begin{pmatrix} t \\ x \end{pmatrix} \wedge \begin{pmatrix} \dot{t} \\ \dot{x} \end{pmatrix}$$

and the scalar $\frac{\dot{t}t - \dot{x}^T x}{t^2 - x^2}$ *are constant on any geodesic of the metric defined by the barrier* $F(X) = -\ln(t^2 - x^2)$ *on the Lorentz cone.* □

These invariants allow us to integrate the geodesic flow on the Lorentz cone. The invariance of the 2-form Ω implies that the intersection of the Lorentz cone with any one-dimensional or two-dimensional linear subspace is a geodesic submanifold. But these intersections are easily shown to be isometric to the positive orthant in $\mathbf{R}$ or $\mathbf{R}^2$. Hence the geodesic flow on the Lorentz cone reduces to the geodesic flow on these orthants.

Cone of squares of a general Jordan algebra

A well studied object which is of great interest in Convex Optimization is the cone of squares of a formally real Jordan algebra, which possesses many advantageous properties. Details on the theory of Jordan algebras and its cones of squares can be found in [2]. In this section we provide brief definitions of a Jordan algebra and associated objects. We single out a subgroup of the automorphism group of the cone of squares which furnishes invariants with an especially compact description.

Definition 4 *A commutative algebra $\mathcal{A}$ is called Jordan algebra if for any elements $x, y \in \mathcal{A}$ we have $x^2(xy) = x(x^2y)$.*

To any $z \in \mathcal{A}$ we can associate linear mappings L_z, Q_z defined by

$$L_z x = zx, \qquad Q_z x = 2z(zx) - z^2 x \qquad \forall\ x \in \mathcal{A}.$$

A special subclass of Jordan algebras is constituted by the *formally real* Jordan algebras. The cone $\mathcal{K} = \{x^2 \,|\, x \in \mathcal{A}\}$ of squares of a formally real Jordan algebra is a convex set, which can be equipped with the standard barrier function $F(x^2) = -\ln\det x^2$. The determinant of an element of a formally real Jordan algebra is well-defined. A definition can be found e.g. in [2]. The cones of squares of formally real Jordan algebras comprise all previous examples of this section.

A classical result [2] characterizes the automorphism group $Aut\mathcal{K}$ of the cone $\mathcal{K}$ as follows. Any linear diffeomorphism of the cone of squares $\mathcal{K}$ of a formally real Jordan algebra $\mathcal{A}$ can be decomposed into a product of the quadratic mapping Q_z for some $z \in \mathcal{A}$ and an automorphism of $\mathcal{A}$. Moreover, any linear diffeomorphism of $\mathcal{K}$ is an isometry of $\mathcal{K}$ if $\mathcal{K}$ is considered as Riemannian manifold equipped with the metric induced by F.

Corollary 1 *The Lie algebra of the isometry group of $\mathcal{K}$ is the product of the Lie algebra of the automorphism group of $\mathcal{A}$ and the linear space $\{L_a \,|\, a \in \mathcal{A}\}$.* □

The element L_a of the Lie algebra generates the Killing vector field $\xi_a(x) = ax$ and the invariant

$$I_a = \frac{\partial^2 F}{\partial x^\mu \partial x^\nu} \dot{x}^\mu (ax)^\nu.$$

These invariants can be joined into the vector valued invariant

$$I_Q = \dot{x}^T \frac{\partial^2 F}{\partial x^2} L_x.$$

Lemma 12 *Let $\mathcal{K}$ be the cone of squares of a formally real Jordan algebra $\mathcal{A}$ equipped with the barrier function $F(x^2) = -\ln\det x^2$. Let L_x be the matrix of the linear operator $z \mapsto xz$, $x, z \in \mathcal{A}$. Then the vector $\dot{x}^T \frac{\partial^2 F}{\partial x^2} L_x$ is an invariant of the geodesic flow on $\mathcal{K}$.* □

6 Conclusions

In this contribution we have shown how to exploit the symmetries of a convex set and its barrier function for obtaining information about the geodesic flow on this set. Namely, when considering the geodesic equations as Hamiltonian dynamical system, the theorem of Noether furnishes an invariant of this system from any element of the Lie algebra of its isometry group. These invariants allow to lower the dimension of the geodesic equations by passing to geodesic submanifolds.

We have computed these invariants for several special convex sets, namely the convex hull of a sphere, a paraboloid and a hyperboloid and for the standard symmetric cones. All these sets have a rich symmetry group which allows to completely integrate the geodesic flow.

In order to yield an invariant of the geodesic flow it is necessary for a symmetry group of a convex set to be consistent with the barrier function on this set. It is a nearly trivial fact that the universal barrier function of a convex set is consistent with the full automorphism group of this set. Hence the geodesic flow generated by the universal barrier function enjoys the full set of invariants.

Open questions for further research would be the following. Let a convex set be given. Is the infimum of the self-concordance parameter over all barrier functions on this set equal to the infimum over those barrier functions which are consistent with the symmetry group of the set? Given a barrier function on a convex set with self-concordance parameter ν, does there exist and how to obtain a barrier function with self-concordance parameter ν that is consistent with the symmetry group of the set?

References

[1] Michel Baes. Geodesic submanifolds in jordan algebras. Private communication, CORE, 2003.

[2] Hel Braun and Max Koecher. *Jordan-Algebren*, volume 128 of *A Series of Comprehensive Studies in Mathematics*. Springer, Berlin, New York, 1966.

[3] L. D. Landau and Lifshitz E. M. *Course of Theoretical Physics: Mechanics*. Butterworth-Heinemann, Boston, 1976.

[4] Yozo Matsushima. *Differentiable manifolds*, volume 9 of *Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1972. Translated from the Japanese by E. T. Kobayashi.

[5] Yuri Nesterov. Long-step strategies in interior-point primal-dual methods. *Mathematical Programming*, 76:47–94, 1996.

[6] Yuri Nesterov and Arkadi Nemirovski. *Interior Point Polynomial Methods in Convex Programming: Theory and Algorithms*. SIAM, Philadelphia, 1994.

[7] Yuri Nesterov and Arkadi Nemirovski. Central path and riemannian distances. Discussion paper 2003/30, CORE, Louvain-la-Neuve, 2003.

[8] Yuri E. Nesterov and Michael J. Todd. On the riemannian geometry defined by self-concordant barriers and interior-point methods. *Foundations of Computational Science*, 2:333–361, 2002.

[9] M. J. Todd. On adjusting parameters in homotopy methods for linear programming. Technical Report 1170, Ithaca, NY 14853, 1996.

[10] E. B. Vinberg. The theory of convex homogeneous cones. *Transactions of Moscow Mathematical Society*, 12:340–403, 1963.

[11] Rolf Walter. *Differentialgeometrie*. Bibliographisches Institut, Mannheim-Vienna-Zurich, 1978.