PUBLIC DEBT AND LIMITED ALTRUISM: IS RICARDIAN EQUIVALENCE POSSIBLE IF ALTRUISM IS LIMITED ?

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Abstract

There have been many criticisms of Barro's theorem on Ricardian equivalence, but these criticisms apply mainly to a "special form" of Barro's model and there are different possible interpretations of this model. We study another very simple model in which altruism is limited. The effects of public debt are studied under two different types of assumptions. The first type of assumptions is standard: young agents buy the government bonds which are perfect substitutes to the assets of firms. This leads to the "traditional" effects of public debt: it increases current consumptions and decreases future capital stocks. Alternatively, we assume that agents apply a special rule called the "patriarchal rule". If agents expect that their children will apply it, then it is rational for them to apply it. This rule leads to a robust property of Ricardian equivalence.

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1 Introduction

In this paper, we first discuss the Ricardian equivalence property in Barro's model with altruism. After that, we study the effects of public debt in a model with "limited altruism". Altruism is limited in the sense that utility of the parent depends on wealth of the children, not on their utility.

The intuition of Barro's (1974) theorem on Ricardian equivalence is simple and clear: the links between successive generations resulting from altruism entirely offset the postponing of taxes by a public debt. There has been a large debate both theoretical and empirical on the validity of Barro's theorem.[1]

The empirical tests about household behavior and about the implications of the theorem are not conclusive: some of them accept the Ricardian equivalence hypothesis and some others reject it. In their survey on public debt, Elmendorf and Mankiw (1999) conclude. "Despite substantial research in [empirical evidence on the validity of Ricardian equivalence], we believe that the results are ultimately inconclusive" (p. 1654). These authors also give a detailed review of the theoretical discussion.

The theoretical assumptions have also been widely discussed, but those discussions mainly apply to a "special form" of Barro's model which has been developped after Barro's (1974) paper and which is now widely used. A first assumption is that of perfect foresight of the whole future: if this were the case, there would be enough links between all individuals for "everything to be neutral" (Bernheim and Bagwell (1988)). This assumption is not made by Barro and it is not helpful to prove the Ricardian equivalence: if neutrality applies to the expected problem, then it applies to current decisions and that is enough.

An explicit form of Barro's model has been proposed by many authors (Bevan and Stiglitz (1979), Buiter (1979), Carmichael (1982)). They assume perfect foresight on the whole future, equal utility functions for all agents and a linear-separability property of these utilities with respect to the attainable utility of the children. Such very restrictive assumptions permit the study of the dynamics of capital accumulation, but this raises the question of what the OLG analysis adds to what we already know from the infinite-lived consumer approach.

We study another model which is much simpler, assuming some "limited altruism": preferences of an agent depend on his own consumptions and on the wealth of his children.

[1] See the discussion of Barro (1989). See also the surveys of Seater (1993) and of Elmendorf and Mankiw (1999).

This model has nice properties, and assuming perfect foresight of *one period*, the intertemporal equilibrium with operative bequests is easy to study. It is unique and optimal in the set of feasible allocations.

The effects of public debt are studied under two different types of assumptions. We consider first the standard assumptions, which have been made by Barro (1974): young agents buy government bonds which are perfect substitutes to the assets of firms. Then family wealth is modified by the government debt and the study leads to the "traditional" conclusion: an increase in government debt at some period increases the consumption of the old this period and decreases the capital stock in all the following periods. Alternatively, we assume that agents apply a special rule called the "patriarchal rule". The same individual, when he is old, receives the proceeds of government bonds, pays the taxes for himself and for his children, and buys new bonds with what remains. These new bonds are given to the children *inside* the family. There is no gain nor loss by using this rule, and it is rational for the parents to apply it, if they expect that their children will apply it.

Importantly, the Ricardian equivalence property resulting from the patriarchal rule is theoretically very robust and it may be tested on microeconomic data.

In section 2, we present some interpretations of Barro's model. In section 3, we study the model with limited altruism. The neutrality of public debt is analyzed in section 4. We conclude in section 5.

2 On the interpretations of Barro's (1974) model

2.1 Barro's model with altruism towards children

In the standard overlapping-generations model of Allais (1947)[2], Samuelson (1958) and Diamond (1965), there is an infinite number of periods $t = 0, 1, 2, ...$ and each individual lives two periods. In period t live N_t young agents (the members of generation t born at the beginning of period t) and N_{t-1} old agents (the members of generation $t-1$). Each old agent is the parent of n young agents, his children. The fertility rate is constant and the growth factor of N_t is equal to n: $N_{t+1} = nN_t$.

In period t, each young agent inelastically supplies one unit of labor and receives a wage

[2] The study of the overlapping-generations model in the appendix of the book of Allais (1947) has been pointed out by Malinvaud (1987).

income w_t. In Barro's model, he also receives a gift x_t from his parent[3]. He allocates his total wealth between consumption c_t and saving s_t:

$$w_t + x_t \;=\; c_t + s_t. \tag{1}$$

In period $t+1$, when old, this agent does not work; he receives the proceeds of his savings $R_{t+1}s_t$ (the gross interest rate is R_{t+1}), consumes d_{t+1} and makes a gift x_{t+1} to each of his n children:

$$R_{t+1}s_t \;=\; d_{t+1} + nx_{t+1}. \tag{2}$$

There is a non-negativity constraint on the gift:

$$x_{t+1} \;\geq\; 0. \tag{3}$$

Firms are competitive. In each period t, the production of one good is defined by a concave, differentiable, constant return to scale production function F_t of capital K_t and labor L_t (we assume that technical progress is exogenous).

At equilibrium, the capital stock K_t is equal to total saving of the preceding period $N_{t-1}s_{t-1}$, and employment L_t is equal to labor supply N_t. We define the *total output* as the sum of production and capital after depreciation. At equilibrium, this total output Y_t is equal to total income $N_tw_t + N_{t-1}R_ts_{t-1}$ and the gross return on saving is equal to the marginal productivity of capital: $R_t = \partial Y_t/\partial K_t$. Output per unit of labor is

$$Y_t/N_t = w_t + R_tk_t \equiv y_t(k_t) \textit{ with } k_t = K_t/N_t = \frac{1}{n}s_{t-1} \textit{ and } R_t = y_t'(k_t). \tag{4}$$

The equilibrium condition on the good market is equivalent to the resource constraint:

$$N_tc_t + N_{t-1}d_t + K_{t+1} = Y_t = N_ty_t(k_t). \tag{5}$$

Barro (1974) assumes that the utility of a member of generation t depends on his life-cycle consumptions and the attainable utility of each of his children[4]:

[3]This is the most usual formulation of the transmission of wealth, even it is generally called "bequest". The true formulation with bequest, i.e. $R_{t+1}x_t$ received by the old in $t+1$, is the one used by Barro (1974). Assuming perfect capital market, they are equivalent. We use the term of wealth rather than income when it applies to gifts.

[4]Barro (1974) asumes $n = 1$. The same formulation applies to n *identical* children.

$$U_t \; = \; U_t(c_t, d_{t+1}, U^*_{t+1}). \tag{6}$$

The attainable utility U^*_{t+1} is the maximum utility of the child conditional to the gift x_{t+1} he receives from his parent. But this utility also depends on the attainable utility U^*_{t+2}, and by induction on the utilities of all descendants. The optimal choice of the gift x_{t+1} takes into account the infinite chain of the impacts of U^*_{t+i+1} on U_{t+i}. These links have been explicitly studied under the assumption of perfect foresight of all the future in a very simplified form of Barro's model, which assumes intergenerational linear separability.

2.2 A simplified form of Barro's model

A standard assumption made on the utility function (6) is additive separability and linearity with respect to the attainable utility of each child, i.e. $U_t = u(c_t, d_{t+1}) + \gamma U^*_{t+1}$, and γ is called the degree of altruism[5]. The attainable utility $U^*_{t+1} = U^*_{t+1}(w_{t+1}+x_{t+1}, \cdot)$ depends on the wealth of the child and on all future prices. With perfect foresight, the recursive definition of these functions:

$$U^*_t(w_t + x_t, \cdot) \; = \; \max_{c_t, d_{t+1}, x_{t+1}} \{u(c_t, d_{t+1}) + \gamma U^*_{t+1}(w_{t+1} + x_{t+1}, \cdot)\} \tag{7}$$

is the Bellman equation of the infinite horizon optimal problem: maximize $\sum_{t=0}^{\infty} \gamma^t u(c_t, d_{t+1})$ subject to the budget constraints (1), (2), (3) of all generations[6]. This is "as if" the first parent solves the optimization problems of *all* his descendants for given prices. Equalizing prices to marginal productivities and saving to capital leads to the dynamics of the capital stock in the economy.

This elegant treatment allows to study the conditions under which public debt has no effect on the intertemporal equilibrium, the so called *Ricardian equivalence* property. Barro condition of "operative bequests" is that all gifts are positive, implying the existence of an infinite chain of wealth transmissions linking all generations. It has been studied by Weil (1987), Abel (1987), and recently by Thibault (2000).

[5] γ may depend on the number of children. This has to be explicited when fertility is endogenous (Becker and Barro (1988)).

[6] There is a special study of the behavior of the first old agents who choose x_0 and d_0, when c_{-1} and the initial capital stock k_0 are given (see de la Croix and Michel (2002) and Michel, Thibault and Vidal (2003)).

As suggested by Barro (1974), Ricardian equivalence can also result from transfers made by altruistic children to the parent. This behavior has been studied by Carmichael (1982): the logic of the infinite chain of gifts is reversed, from any date to minus infinity. Barro considers in fact the two possibilities by entering the utility of the previous generation U^*_{t-1} as an additional argument of the utility function (6). This leads to the technical difficulty of interdependence of the utilities. This more sophisticated form of rationality has been studied by Kimball (1987).

2.3 Ricardian equivalence as a consequence of expectations

In his study of the effects of public debt, Barro (1974) assumes that expectations are static at their current values. In fact the assumptions made on expectations do not matter except one: the debt policy does not modify the discounted sum of the taxes (the government respects its intertemporal budget constraint). Solving an expected problem, the utility of an agent is a function of all present and future consumptions of the members of his dynasty, his descendants, and total discounted wealth defines the budget constraint of the dynasty. The neutrality argument of Barro[7] applies to this expected problem and thus the *current* decision of consumption does not depend on the current and expected debt policy. To obtain the Ricardian Equivalence it is even not necessary to solve the expected problem, but for choosing their current consumption the agents of generation t should *expect* the attainable utility of children.

We now turn to another story which looks much simpler.

3 A simplified model

The budget constraints, equations (1), (2) and (3), are the same as in Barro's model. The wealth of each young agent is equal to the sum of his wage income and the gift made to him by his parent:

$$\omega_{t+1} \;=\; w_{t+1} + x_{t+1}.$$

In period $t+1$, there are two sources of income in the family: the gross return on savings $R_{t+1}s_t$ and the total wage income nw_{t+1} of the n children. We define the family wealth as the sum: $W_{t+1} = R_{t+1}s_t + nw_{t+1}$. The budget constraint (2) is equivalent to:

[7] This argument is explicit in Becker's (1974) study of social interactions inside a family with a finite number of members.

$$W_{t+1} \equiv R_{t+1}s_t + nw_{t+1} = d_{t+1} + n\omega_{t+1}. \tag{8}$$

The condition (3) of a non-negative gift is equivalent to: $\omega_{t+1} = w_{t+1} + x_{t+1} \geq w_{t+1}$.

3.1 Limited altruism

Formulation of altruism is a hard problem: utilities of utilities imply a definition of cardinal utilities. In the dynastic model where parents care about the welfare of children, it leads to a recursive definition. At equilibrium, Barro's model leads to a reduced form depending on the wealth of the children.

We simplify drastically the problem assuming that the utility function of the parent depends directly on the wealth of the children, not on their utility:

$$U_t = U_t(c_t, d_{t+1}, \omega_{t+1}).$$

This means that the parent has his own opinion on what is good for children, either ignoring or neglecting the preferences of the children.[8] It could be interpreted as the reduced form of some recursive definition when prices are exogenous (the small economy assumption), but this is not robust to changes in policy variables. It is more in the tradition of the joy of giving models[9] in which the gift is an argument of the utility function (Yaari (1964), Blinder (1974)). Changing this argument by introducing the (net) wage income as a perfect substitute of the gift has important consequences as we shall see.

We assume that the function U_t is strictly quasi-concave, differentiable and satisfies the Inada conditions (infinite marginal utility of a zero value for each argument). The dependence on the period means that preferences may differ accross generations. The first order conditions characterizing an interior solution are:

$$\frac{\partial U_t}{\partial c_t} = R_{t+1}\frac{\partial U_t}{\partial d_{t+1}} \quad \text{and} \quad \frac{\partial U_t}{\partial d_{t+1}} = \frac{1}{n}\frac{\partial U_t}{\partial \omega_{t+1}}. \tag{9}$$

[8] This form of altruism has been used to study the dynamics of capital accumulation by Lambrecht, Michel and Thibault (2000) and with education choice of parents by Lambrecht, Michel and Vidal (2001). See the discussion by Becker (1981), in particular p.193.

[9] The link between a joy of giving model and the reduced form of Barro's model has been studied by Abel and Warshawski (1988).

The first condition is the usual arbitrage condition over the life-cycle consumptions. The second condition results from the arbitrage between own consumption when old and the wealth of each child[10].

3.2 Characterization, uniqueness and optimality of the intertemporal equilibrium

We assume here that at any period t, gifts are positive, young agents are identical and have perfect foresight on one period. The family wealth in period $t+1$ is equal to the sum of the return on saving of the parent and the wages of the children. At equilibrium, it is equal to total product per old agent: $W_{t+1} = R_{t+1}s_t + nw_{t+1} = n(R_{t+1}k_{t+1} + w_{t+1}) = ny_{t+1}(k_{t+1})$. This implies:

$$\omega_{t+1} = y_{t+1}(k_{t+1}) - d_{t+1}/n. \tag{10}$$

Condition (10) which is satisfied at all periods, including the initial period at which k_0 and $s_{-1} = nk_0$ are given. Consumption of the young satisfies relation (1) and thus we have at equilibrium:

$$c_t \;=\; \omega_t - s_t \;=\; y_t(k_t) - d_t/n - nk_{t+1}.$$

This equality is equivalent to the resource constraint (5). Since $R_{t+1} = y'_{t+1}(k_{t+1})$, we see that the first order conditions (9) characterizing the behavior of agents are equivalent to those of the problem: maximize with respect to k_{t+1} and d_{t+1}

$$V_t \;=\; U_t(y_t(k_t) - d_t/n - nk_{t+1}, \quad d_{t+1}, \quad y_{t+1}(k_{t+1}) - d_{t+1}/n). \tag{11}$$

This shows that *the intertemporal equilibrium with positive gifts is unique*. It is simply defined by induction as the sequence (k_{t+1}, d_{t+1}) maximizing V_t for given k_t and d_t. At the initial date $t = 0$, k_0 and c_{-1} are given and d_0 maximizes $U_{-1}(c_{-1}, d_0, y_0(k_0) - d_0/n)$.

Moreover the intertemporal equilibrium is *Pareto-optimal* for the sequence of utilities in the set of feasible allocations that satisfy the resource constraint (5). Indeed for any other feasible

[10] These conditions apply when children have not necesseraly the same wage income, if the utility function is symmetric with respect to the children's wealth, $U_t = \frac{1}{n}\sum_{i=1}^{i=n} U_t(c_t, d_{t+1}, \omega^i_{t+1})$. Then it is optimal to allocate the same wealth ω_{t+1} to each child and thus to adapt gifts to the personal wage income of the child.

path (c_t, d_t, k_{t+1}), at the first period at which (k_t, d_t) is not at its equilibrium values, the utility U_{t-1} of generation $t-1$ is lower than its equilibrium level, which is the maximum of V_{t-1}.

This strong property seems surprising, because the equilibrium dynamics may converge to a capital-labor ratio which is in overaccumulation, parents beeing "too generous". In that case, it is possible to increase consumptions by reducing the capital stock, but this also reduces the family wealth and the gift to the children. There is a *wealth effect* on welfare[11] and wealth $W_t = ny_t(k_t)$ is an increasing function of k_t. This effect dominates the decrease in consumption effect when there is over-accumulation of capital.

4 Government debt

Government bonds have one period maturity. The reimbursement with interest payment of the debt B_t in period $t+1$ is $R^B_{t+1}B_t$. It is financed by lumpsum taxes T^1_{t+1} on each young agent and T^2_{t+1} on each old agent, and by issuing a new debt B_{t+1}. Assuming that there are no government spending, the period $t+1$ current account equilibrium condition of the government is $R^B_{t+1}B_t = N_{t+1}T^1_{t+1} + N_t T^2_{t+1} + B_{t+1}$, and in terms of debt per young agent, $b_t = B_t/N_t$:

$$R^B_{t+1}b_t \;=\; nT^1_{t+1} + T^2_{t+1} + nb_{t+1}. \tag{12}$$

A distribution of government bonds, say by helicopter, is equivalent to a negative tax.

4.1 Government bonds as a fraction of family wealth

We assume that government bonds are perfect substitute of the assets of firms. This implies that the returns are equal: $R^B_{t+1} = R_{t+1}$ and that the total saving of the youngs denoted s^b_t is equal to the sum of the government debt and the next period capital stock. At equilibrium in period t, we have: $s^b_t = b_t + nk_{t+1}$.

The first order conditions (9) characterizing the maximum utility are unchanged. The family net wealth in period $t+1$ is $W^b_{t+1} = R_{t+1}s^b_t - T^2_{t+1} + n(w_{t+1} - T^1_{t+1})$. Using the equilibrium conditions: $s^b_t = b_t + nk_{t+1}, R_{t+1}k_{t+1} + nw_{t+1} = ny_{t+1}(k_{t+1})$ and $R_{t+1}b_t - T^2_{t+1} - nT^1_{t+1} = nb_{t+1}$, we obtain: $W^b_{t+1} = ny_{t+1}(k_{t+1}) + nb_{t+1}$. Family wealth is equal to the *sum of the total product and the government debt per family*. Substituting $c_t = y_t(k_t) - d_t/n - nk_{t+1}$ and $\omega_{t+1} =$

[11] Wealth of the children is an argument of the utility of parents. The dependence of utility on wealth in the Ramsey model has been studied by Kurz (1968).

$(W^b_{t+1} - d_{t+1})/n$ in (9) we may characterize the intertemporal equilibrium with public debt as the sequence (k_{t+1}, d_{t+1}) that maximizes

$$V^b_t \;=\; U_t(y_t(k_t) - d_t/n - nk_{t+1}, \quad d_{t+1}, \quad y_{t+1}(k_{t+1}) + b_{t+1} - d_{t+1}/n) \tag{13}$$

for given (k_t, d_t). And d_0 maximizes $U_{-1}(c_{-1}, d_0, y(k_0) + b_0 - d_0/n)$. Government bonds are included in the family wealth.

Effects of public debt on the equilibrium. Assuming normal goods, an increase in b_{t+1} increases the three terms in V^b_t. Consumptions c_t, d_{t+1} and ω_{t+1} increase for a given (k_t, d_t) and thus the capital stock k_{t+1} decreases. This leads to a decrease of the net resource $y_{t+1}(k_{t+1}) - d_{t+1}/n$ available for the youngs in the following period. We have the traditional wealth effect of government bonds (Modigliani 1961). More precisely an increase in public debt at a given period increases consumptions of the generation old at this period and it has a negative effect on the capital stock of all following periods.

Effects on welfare. Given (k_t, d_t), an increase in b_{t+1} increases the welfare of the agents of generation t. It may decrease or not the welfare of the following generation, and it certainly does if b_{t+2} is nil.

To study welfare in the long run, we consider stationary long run values of debt b per young, production $y(k)$ and utility:

$$V^b \;=\; U(y(k) - d/n - nk, d, y(k) + b - d/n).$$

Assuming that $U(c, d, w)$ is additively separable, we show in the Appendix that, if there is over-accumulation of capital, public debt has a *positive effect on welfare*. This result is similar to the effect of public debt in the Diamond (1965) model. In the model with limited altruism, it is a consequence of the two effects on net production and on wealth: k decreases but both $y(k) - nk$ and $y(k) + b$ increase.

In this model with short-sighted agents (limited altruism), the real wealth effect of government bonds results from the assumption that these bonds constitute a mean of savings and a mean of transmetting wealth to children.

4.2 Ricardian equivalence in the economy with limited altruism

We now consider a different treatment of government bonds which are no more perfect substitutes of the assets of firms. We assume that in each period, it is the same individual, the old agent in period $t+1$, who receives the gross return $R^B_{t+1}b_t$ of his bonds b_t, pays the taxes both for himself T^2_{t+1} and for his children nT^1_{t+1}, and buys new bonds with the remain, b_{t+1} per child, which he gives to them[12].

This behavior rule implies: $nb_{t+1} = R^B_{t+1}b_t - T^2_{t+1} - nT^1_{t+1}$. It is equivalent to the government equilibrium condition (11), and implies that all new government bonds are bought by the old agents. The young agents do not intervene on the bonds market. They only receive some government bonds from their parent *inside the family* and will apply the same rule when old. We call it the "patriarchal rule".

With this "patriarchal" treatment of government bonds by the old agents, there is no gain nor loss in the relation between the family and the government (assuming no tax collecting cost). The other decisions are unchanged and Ricardian equivalence prevails[13].

This rule is rational if the parent expects that his children will apply it. Children will have no gain nor loss from the government bonds given to them. With this expectation, the definition of net wealth of children with limited altruism does not include government bonds.

Is this rule realistic? More precisely: can the behavior of private agents be "as if" the patriarchal behavior rule applies ? Empirically, one could study this behavior on microeconomic data and test if the current account equilibrium conditions of the government holds at the level of families. This would be a nice test of Ricardian equivalence.

As a theoretical approach, it is compatible with Barro's neutrality theorem. In fact, our result of neutrality is more robust than his for several reasons.

- The argument is based on the *current* government revenue constraint, not on its intertemporal budget constraint (a Ponzi debt is not excluded).
- Heterogeneity which is a hard problem can be easily studied in our simple model. As noted before (see note 9), heterogeneity inside the family can be neutralized by the parent.

[12] If bonds are not anonymous, they are bought at the name of each child.

[13] With government spendings, g_{t+1} per young, if the old heads buy all government bonds, the net family wealth is $ny_{t+1}(k_{t+1}) - ng_{t+1}$ in period $t+1$.

Heterogeneity of families does not matter as long as there are no redistributive effects of taxes (the patriarchal rule is independent of preferences).

- Distortionary taxes do not invalidate Ricardian Equivalence. Consider for exemple a tax rate τ_{t+1} on the return on firms' assets. The tax $\tau_{t+1}R_{t+1}s_t$ is paid through the patriarchal rule of government bonds, whatever the size of savings s_t. Thus the real return on savings is $R_{t+1}s_t$: the type of the tax does not matter. Note also that the interest rate on government bonds may be different, $R^B_{t+1} \neq R_{t+1}$.

5 Conclusion

In this paper we have first emphasized that the restrictive assumptions often made are not necessary to prove Ricardian equivalence. Indeed, neither perfect foresight nor separability and linearity with respect to the attainable utility of children are necessary to obtain the absence of effect of public debt on the expected solution and thus on current decisions.

We then studied the effects of public debt in the economy with limited altruism, assuming that the utility of the parent depends on the net wealth of the children. The conclusion strongly depends on the assumption made on the definition of the net wealth of children which parents take into account.

- Under the standard assumption that government bonds are perfect substitute to the assets of firms (and thus are bought by the young agents), the family wealth includes government bonds. Then public debt initially increases consumption and decreases capital accumulation in all the following periods.

- Under the assumption of a "patriarchal treatment" by parents, the young agents do not intervene on the market of government bonds. Then government bonds are treated separately and there is a robust property of neutrality.

Our conclusions could be tested on microeconomic data. The patriarchal rule applies if the current government revenue constraint is satisfied at the family level.

APPENDIX

We study the effect of public debt in the long run, assuming that the utility function of the head is additively separable:

$$U_t = u_1(c_t) + u_2(d_{t+1}) + u_3(\omega_{t+1}).$$

For $i = 1, 2, 3, u_i$ satisfies: $u_i' > 0$ and $u_i'' < 0$.

The first order conditions are at equilibrium with positive gifts:

$$u_2' = \frac{1}{n} u_3' \text{ and } u_1' = y_{t+1}'(k_{t+1}) u_2'. \quad \text{(A1)}$$

When government bonds are a fraction of family wealth, we have:

$$c_t = y_t(k_t) - d_t/n - nk_{t+1} \text{ and } \omega_{t+1} + \frac{1}{n} d_{t+1} = y_{t+1}(k_{t+1}) + b_{t+1}. \quad \text{(A2)}$$

We study the effects of b_{t+1} and of $z_t \equiv y_t(k_t) - d_t/n$ on the optimal choice of $c_t, d_{t+1}, \omega_{t+1}$. To simplify the notations, we delete the time index.

Effects of $b_{t+1} = b$. Differentiation of (A1) leads to

$$u_2'' \frac{\partial d}{\partial b} = \frac{1}{n} u_3'' \frac{\partial \omega}{\partial b} \text{ and } u_1'' \frac{\partial c}{\partial b} = y' u_2'' \frac{\partial d}{\partial b} + u_2' y'' \frac{\partial k}{\partial b}.$$

Using $nk = z - c$, we also have: $\frac{\partial k}{\partial b} = -\frac{1}{n} \frac{\partial c}{\partial b}$ and $(u_1'' + \frac{1}{n} u_2' y'') \frac{\partial c}{\partial b} = y' u_2'' \frac{\partial d}{\partial b}$. Thus the three derivatives of c, d and ω with respect to b have the same sign. We also have from (A2):

$$\frac{\partial \omega}{\partial b} + \frac{1}{n} \frac{\partial d}{\partial b} = y'(k) \frac{\partial k}{\partial b} + 1 \text{ and } \frac{\partial \omega}{\partial b} + \frac{1}{n} \frac{\partial d}{\partial b} + \frac{y'}{n} \frac{\partial c}{\partial b} = 1. \quad \text{(A3)}$$

All the three derivatives are positive.

Effects of $z_t = z$. Differentiating (A1) and (A2) with respect to z implies:

$$u_2'' \frac{\partial d}{\partial z} = \frac{1}{n} u_3'' \frac{\partial \omega}{\partial z} \text{ and } u_1'' \frac{\partial c}{\partial z} = y' u_2'' \frac{\partial d}{\partial z} + u_2' \, y'' \frac{\partial k}{\partial z}$$

$$\frac{\partial c}{\partial z} = 1 - n \frac{\partial k}{\partial z} \text{ and } \frac{\partial \omega}{\partial z} + \frac{1}{n} \frac{\partial d}{\partial z} = y' \frac{\partial k}{\partial z} = \frac{y'}{n} (1 - \frac{\partial c}{\partial z}) \quad \text{(A4)}$$

$\frac{\partial d}{\partial z}$ and $\frac{\partial \omega}{\partial z}$ have the same sign which is necessarily positive. Indeed, $\frac{\partial d}{\partial z} \leq 0$ and $\frac{\partial \omega}{\partial z} \leq 0$ imply: $\frac{\partial k}{\partial z} \leq 0, \frac{\partial c}{\partial z} \geq 1$, and $u_1'' \frac{\partial c}{\partial z} = y' u_2'' \frac{\partial d}{\partial z} + u_2' y'' \frac{\partial k}{\partial z} \geq 0$, a contradiction. Then we have: $\frac{\partial k}{\partial z} > 0$ and $u_1'' \frac{\partial c}{\partial z} = y' u_2'' \frac{\partial d}{\partial z} + u_2' \, y'' \frac{\partial k}{\partial z} < 0$. Thus the three derivatives are positive.

Study of the equilibrium dynamics.

We have: $z_{t+1} = y_{t+1}(k_{t+1}) - d_{t+1}/n = \omega_{t+1} - b_{t+1}$ and $\omega_{t+1} = \varphi_{t+1}(z_t, b_{t+1})$ is an increasing function of z_t and of b_{t+1} satisfying: $\partial \varphi_{t+1}/\partial z_{t+1} > 0$ and $0 < \partial \varphi_{t+1}/\partial b_{t+1} < 1$ (from (A3)).

We assume stationary values of $b_{t+1} = b$ and $y_{t+1}(\cdot) = y(\cdot)$. Then the dynamics of z_t are: $z_{t+1} = \varphi(z_t, b) - b$ with $\varphi_1' = \partial \varphi/\partial z > 0$. Thus the dynamics of z_t are monotonic and the sequence (z_t) converges to a steady state $z^*(b)$. We assume: $z^*(b) > 0$, and satisfying the stability condition: $\varphi_1'(z^*(b), b) < 1$. This implies

$$\frac{dz^*(b)}{db} \; = \; \frac{\varphi_2' - 1}{1 - \varphi_1'} < 0 \;\text{ since }\; \varphi_2' \; = \; \partial \varphi/\partial b < 1.$$

We also have: $nk^*(b) = z^*(b) - c^*(b)$ and $c^*(b) = \psi(z^*(b), b)$ with $0 < \psi_1' < 1$ and $\psi_2' > 0$ (from (A4)). Thus

$$n\frac{dk^*(b)}{db} \; = \; \frac{dz^*(b)}{db}(1 - \psi_1') - \psi_2' < 0. \tag{A5}$$

Lemma 1 *If $d\omega^*(b)/db$ is non positive, then $k^*(b)$ is a under-accumulation.*

Proof. Assume $\frac{d\omega^*}{db} \leq 0$. From (A1): $u_2'' \frac{dd^*}{db} = \frac{1}{n} u_3'' \frac{d\omega^*}{db}$ and $\frac{dd^*}{db} \leq 0$, and $u_1'' \frac{dc^*}{db} = y' u_2'' \frac{dd^*}{db} + y'' u_2' \frac{dk^*}{db} > 0$. We have thus: $\frac{dc^*}{db} < 0$ and using the resource equilibrium condition:

$$(y'(k^*(b)) - n)\frac{dk^*}{db} \; = \; \frac{d}{db}(y(k^*(b)) - nk^*(b)) \; = \; \frac{dc^*}{db} + \frac{1}{n}\frac{dd^*}{db} \; < \; 0.$$

This implies with (A5) under-accumulation of capital: $y'(k^*(b)) > n$. Q.E.D.

Assume now over-accumulation at $k^*(b) : y'(k^*(b)) < n$. This implies: $\frac{d\omega^*}{db} > 0$ and the steady state utility satisfies:

$$\frac{dU^*}{db} \; = \; u_1' \, \frac{dc^*}{db} + u_2' \, \frac{dd^*}{db} + u_3' \, \frac{d\omega^*}{db} \; = \; u_2' \, (y' \frac{dc^*}{db} + \frac{dd^*}{db} + n\frac{d\omega^*}{db}).$$

Differentiating $\omega^* + \frac{1}{n} d^* = y(k^*) + b = y(\frac{\omega^* - b - c^*}{n}) - b$ leads to:

$$\frac{1}{n}(y'\frac{dc^*}{db} + \frac{dd^*}{db} + n\frac{d\omega^*}{db}) \;=\; 1 - \frac{1}{n}y'(k^*) + \frac{1}{n}y'(k^*)\frac{d\omega^*}{db}$$

which is positive. Thus $dU^*/db > 0$.

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