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FINANCING INFRASTRUCTURE UNDER BUDGET CONSTRAINTS

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Abstract

In this paper we consider the problem of financing infrastructure when the regulator faces a budget constraint. The optimal budget-constrained mechanism satisfies four properties. The first property is bunching at the top, that is the more efficient firms produce the same quantity irrespective of their costs. The second property is separability of less efficient firms. The third property is that the mechanism is a third best one, that is, the optimal budget constrained output is strictly lower than the second best output for any given type. Finally, if the budget constraint is too strong, then we have full bunching.

Keywords: Regulation, Asymmetric Information, Budget Constraint
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1 Introduction

In this paper we analyze the problem of financing infrastructure when the public authority faces financial constraints and when the cost of the producer is unknown. Financing infrastructure is a major issue in every economy. Building up roads, bridges, railways, water-ways, airways, water supplies, electricity and public services such as health and education is necessary for the development of an economy. The benefits of infrastructure being widely recognized, the main problem is to raise funds to finance it. Infrastructures are either financed completely by public funds (for example, in the case of public goods which we address in this paper), or is financed partially by the private sector. An example of the second case is network industries, where the essential facility is partially financed by the private competitors that use it. However, liberalization and open-access policies do not eliminate the necessity to finance infrastructure. In network industries, the regulators should not only guarantee an open-access of the network to competitors (through adequate access charge) but should also guarantee the financing of infrastructures. If the essential facility cannot be financed completely with the access charge, public funds should be invested to guarantee the continuity and the quality of the service provided. In this context railway tracks is a good example.[1]

Baron and Myerson (1982) and Laffont and Tirole (1993) developed a procedure to finance infrastructure, when the cost of the monopolist firm is unknown to the regulator. The main property of the optimal or second best mechanism is full separability of types, that is, if the monopolist is a high (low) cost type then she produces lower (higher) quantity and recieve a lower (higher) transfer. The cost of this separability of types under asymmetric information is the information rent enjoyed by the lower cost types. In the optimal mechanism all but the lowest cost firm produce less than first best to reduce the information rent. Unfortunately, if infrastructure projects are numerous, public funds are usually scarce. It is reasonable to imagine that the fund provider may be unable to finance the infrastructure at the level required for the optimal mechanism (Baron and Myerson (1982)). The aim of this paper is to construct the optimal mechanism when the available fund is limited. We call this mechanism the *optimal budget-constrained mechanism*. We highlight the differences between the optimal mechanism

[1] In July 2000, the British Government publised its ten year plan, announcing a £60 billion investment package for the railways. Approximately £30 billion would come from the government itself, via the Strategic Rail Authority (SRA)-*Source:* SRA Annual Report 2000-2001.

and the budget-constrained one.

To illustrate our problem, suppose that a local community plans to finance a public good for its citizens. In developing countries, budgets for investment are limited, mainly because the local communities do not have access to credit. This procurement problem is a mechanism design problem if the fund provider does not know the cost conditions under which a firm can supply the public good. Efficiency prescribes that the regulator uses the private information of the firm and that the quantity supplied by a more efficient (or low cost) firm is more than that of a less efficient (or high cost) firm. To achive the same, the regulator uses its money to pay an information rent in order to separate out the more efficient firms from the less efficient ones. However, if public funds are limited, the regulator could be better off if it manage to reduce the monopolist's rent. The optimal budget-constrained mechanism prescribes that the firm supplies a good of lower quantity (compared to the second best quantity) and that the more efficient types of firm produce the same quantity independently of its cost. These two distortions reduces the information rent and the quantity produced by the more efficient firm. However, if the financial constraint is too strong, the budget-constrained mechanism prescribes full bunching. The regulator has two instruments to limit the transfer: bunching for the more efficient firms and under-production. Our comparative static result show that a reduction in available fund reduces the quantities produced by all types of firms and increases the interval in which there is bunching.

In many models of regulation, it is assumed that public subsidies are costly. Transferring one dollar to the firm costs the authority $(1 + \lambda)$ dollars, where λ represents the shadow cost of public funds (see Laffont and Tirole (1993)). This approach is relevant when the public authority itself regulates the monopolist. Our approach applies when the task of regulation is delegated to an agency to which the authority allocates funds (like in the British Government and Strategic Rail Authority example). Thus, our framework is different since we consider the case in which the public authority specifies a maximal level of subsidy that the regulatory agency can use to finance the infrastructure. Finally, we do not model how the highest subsidy is determined by the political authority and consider it as exogenously given.

There are several papers dealing with mechanism design problems under asymmetric information and financial constraint. Laffont and Robert (1996) describe the optimal auction when all the bidders have a financial constraint which is common knowledge. Like in our optimal budget-constraint mechanism, the financial constraint in Laffont and Roberts (1996) reduces the bids of all participants (even those with a low valuation for the good). Che and Gale (2000) extends the result in Laffont and Roberts (1996) by relaxing the assumption that financial constraints are common knowledge. Monteiro and Page Jr. (1998) describe the optimal selling mechanisms for multiproduct monopolists in the presence of budget constrained buyers. To construct the optimal budget-constrained mechanism, we extend the methodology of Thomas (2002). Thomas (2002) considers the incentive problem of a monopolist who faces financially constrained buyers. Surprisingly, in his mechanism, the financial constraint may imply over-consumption (relative to the complete information case) for some types of buyers for whom the separability property holds in the constrained mechanism. In contrast, our mechanism imply underproduction for all types, whenever the available budget is lower than the highest transfer required in the optimal mechanism. Finally, Gautier (2002) considers the regulator's mechanism design problem under financial constraint when there are two types of firm. In Gautier (2002), bunching is an issue only if the financial constraint is sufficiently strong.

The paper is organized in the following way. In Section 2, we develop the benchmark model. In Section 3, we introduce and analyze the mechanism design problem under budget constraint. We conclude our analysis in Section 4. All proofs are relegated in the Appendix.

2 The Model

The utility of the monopolist is $U_m = t - \theta q$ where t is the transfer that she receives from the regulator and θ is her marginal cost and q is the quantity of the public good she produces. The utility function of the regulator is $U_r = S(q) - t$ where $S(q)$ is the consumer's surplus when a quantity q of public good is supplied and t is the transfer to the monopolist. $S(q)$ is assumed to be twice differentiable with $S'(q) > 0$, $S''(q) < 0$ and $S'(0) = \infty$. The regulator's main objective is to select the quantity q to maximize U_r. Since $S'(0) = \infty$, the good is always produced. If the regulator knows the marginal cost θ of the monopolist, then the optimal quantity is $q^f(\theta) = S'^{-1}(\theta)$ and the optimal transfer to the monopolist is $t(\theta) = \theta q^f(\theta)$. The pair $\langle q^f(\theta), t^f(\theta)\rangle$ is the first best outcome.

We assume that the marginal cost of the monopolist is private information. In this context, we assume that the marginal cost of the monopolist θ belongs to the interval $[\underline{\theta}, \overline{\theta}]$ where $0 < \underline{\theta} < \overline{\theta}$. This interval is assumed to be common knowledge. It is also common knowledge that (i) the marginal cost has a continuous and almost everywhere differentiable density $f(\theta)$ for all

$\theta \in (\underline{\theta}, \overline{\theta})$ and that (ii) $f(\theta) > 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. The regulator's objective is to maximizes $\int_{\underline{\theta}}^{\overline{\theta}} \{S(q(\theta)) - t(\theta)\} f(\theta) d\theta$ subject to incentive compatibility constraint (or IC) and participation constraint (or PC). A direct mechanism $M = \langle q(.), t(.)\rangle$, in this context, simply specifies a type contingent quantity-transfer pair. Here $q : [\underline{\theta}, \overline{\theta}] \to \mathbf{R}_+$ and $t : [\underline{\theta}, \overline{\theta}] \to \mathbf{R}_+$. For simplicity we restrict attention to continuous and almost everywhere differentiable mechanisms. Let $U_m(\theta; \theta') = t(\theta') - \theta q(\theta')$ be the utility of the monopolist under the mechanism M if her true type is θ and if she announces $\theta' \in [\underline{\theta}, \overline{\theta}]$. With slight abuse of notation, let us define $U_m(\theta) \equiv U_m(\theta; \theta)$, for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Given these definitions, incentive compatibility states that $U_m(\theta) \geq U_m(\theta; \theta')$, for all pairs $\{\theta, \theta'\} \in [\underline{\theta}, \overline{\theta}]^2$ and participation constraint states that $U_m(\theta) \geq 0$, for all $\theta \in [\underline{\theta}, \overline{\theta}]$.

It is well known in the literature that the optimal mechanism M satisfies both the incentive compatibility constraint and the participation constraint if and only if the utility of any type $\theta \in [\underline{\theta}, \overline{\theta}]$ is given by $U_m(\theta) = \int_{\theta}^{\overline{\theta}} q(\tau) d\tau$ and the optimal type-contingent quantity $q(\theta)$ is non-increasing in θ.[2]

We will refer to the mechanism design problem of the regulator, stated in the previous paragraph, as the benchmark model. The optimal (or second best) mechanism for the benchmark model is summarized in the next Proposition. Before stating the Proposition we provide two relevant definitions. For any $\theta \in [\underline{\theta}, \overline{\theta}]$, let $L(\theta) \equiv \frac{F(\theta)}{f(\theta)}$ be the hazard rate function where $F(.)$ is the distribution function associated with the density function $f(.)$. For any $\theta \in [\underline{\theta}, \overline{\theta}]$, let $z(\theta) \equiv \theta + L(\theta)$ be the virtual type function.

PROPOSITION 1 The optimal mechanism is $M^b = \langle q^b(.), t^b(.)\rangle$ where

1. $q^b(\theta) = S'^{-1}(z(\theta))$ and
2. $t^b(\theta) = \theta q^b(\theta) + \int_{\theta}^{\overline{\theta}} q^b(\tau) d\tau \;\; \forall \theta \in [\underline{\theta}, \overline{\theta}]$

We omit the proof of Proposition 1 since it is quite well known in the literature (see Baron and Myerson (1982)). It is important to observe that

[2] The reason for the necessity of non-increasingness of optimal quantity follows directly by solving the inequality in the definition of IC for any pair of types. The reason for the necessity of $U_m(\theta) = \int_{\theta}^{\overline{\theta}} q(\tau) d\tau$ is the following. For IC to hold, it is necessary that $U_m'(\theta) = -q(\theta)$ for almost all $\theta \in (\underline{\theta}, \overline{\theta})$. Under the optimal mechanism PC implies that $U_m(\overline{\theta}) = 0$. These two conditions together imply that $U_m(\theta) = \int_{\theta}^{\overline{\theta}} q(\tau) d\tau$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. The sufficiency part is quite straightforward.

for the benchmark model it is necessary that $q^b(\theta)$ is non-increasing in $\theta \in [\underline{\theta}, \overline{\theta}]$. Non-increasingness of quantity is satisfied if and only if the virtual type $z(\theta)$ is non-decreasing in $\theta \in [\underline{\theta}, \overline{\theta}]$. Given that $z(\theta)$ is non-decreasing, we get $q^b(\theta)$ is non-increasing in $\theta \in [\underline{\theta}, \overline{\theta}]$. Moreover, since $L(\theta) > 0$ for all $\theta \in (\underline{\theta}, \overline{\theta}]$ and $z(\theta) > \theta$, we get $q^b(\theta) < q^f(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta}]$ and $q^b(\underline{\theta}) = q^f(\underline{\theta})$. Thus, for all but the lowest cost firm, we have underproduction under second best compared to the first best. For our main problem, to be analyzed in the next section, we take the following assumption which is stronger than non-decreasingness of the virtual type function $z(.)$.

ASSUMPTION 1 For almost all $\theta \in (\underline{\theta}, \overline{\theta})$, $\theta(1 + L'(\theta)) \geq 2L(\theta)$.

For assumption 1, it is necessary that $z(\theta)$ is non-decreasing for all $\theta(\in [\underline{\theta}, \overline{\theta}])$ for which assumption 1 holds. Observe that non-decreasingness of $z(\theta)$ means that $1 + L'(\theta) \geq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. This is clearly a necessary condition for assumption 1 since $\theta > 0$ and $L(\theta) \geq 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. For all density functions with the property that $f'(\theta) \leq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$, assumption 1 is satisfied. Therefore, Exponential Distribution, Pareto Distribution and Uniform Distribution satisfies assumption 1. Under certain parametric restrictions Beta Distributions, Gamma Distributions and Weibull Distributions also satisfy assumption 1. Among the class of distributions with the property that there exists a non-empty interval (a, b) such that $f'(\theta) > 0$ for all $\theta \in (a, b)$, Triangular Distribution satisfy assumption 1. A Normal Distribution satisfies assumption 1 if and only if $\frac{2\sigma^2 x}{(2\sigma^2 + x(\mu - x))} \geq L(x)$ for all $x \in (\underline{\theta}, \mu)$ where the mean $\mu = \frac{(\underline{\theta} + \overline{\theta})}{2}$ and σ is the standard deviation. For Logistic and Laplace Distributions, the sufficiency conditions are $\underline{\theta} \geq \frac{\sqrt{3}}{\pi}\sigma$ and $\underline{\theta} \geq \frac{\sigma}{\sqrt{2}}$ respectively.[3]

3 Budget Constraint

In the benchmark problem we have a fully separating mechanism where the optimal transfer is strictly decreasing in type. Complete separation is feasible if the maximum fund available to the regulator (or the regulator's budget constraint) $\bar{T}$ exceeds the transfer that needs to be provided to the

[3]We obtained the condition for Normal Distribution by taking doubly-truncated Normal Distribution following Hald's (1952) convention. The same double truncation technique was applied to obtain the sufficiency conditions with Logistic and Laplace Distributions. For all these three Distributions we assumed symmetry around the mean $\mu = \frac{\underline{\theta} + \overline{\theta}}{2}$.

lowest type $t^b(\underline{\theta})$.[4] However, if this fund is less than the transfer needed to separate out all types, that is if $\bar{T} < t^b(\underline{\theta})$, then the regulator's problem is

$$[P^*] \quad \max_{\langle q(\theta),t(\theta)\rangle} \int_{\underline{\theta}}^{\overline{\theta}} \{S(q(\theta)) - t(\theta)\} f(\theta)d\theta,$$

subject to

1. $U_m(\theta) \geq U_m(\theta;\theta'),\ \forall\{\theta,\theta'\} \in [\underline{\theta},\overline{\theta}]^2$,
2. $U_m(\theta) \geq 0,\ \forall\theta \in [\underline{\theta},\overline{\theta}]$ and
3. $t(\theta) \leq \bar{T}\ \forall\theta \in [\underline{\theta},\overline{\theta}]$.

The optimal budget-constrained mechanism for $[P^*]$ is summarized in the next Theorem.

THEOREM 1 Under Assumption 1, the optimal budget constrained mechanism for $[P^*]$ is $M^* \equiv \langle q^*(.), t^*(.), \tilde{\theta}\rangle$ where

1.
$$q^*(\theta) = \begin{cases} q^*(\tilde{\theta}) & \forall\theta \in [\underline{\theta},\tilde{\theta}) \\ S'^{-1}\left(z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}\right) & \forall\theta \in [\tilde{\theta},\overline{\theta}] \end{cases}$$

with $\Psi(\tilde{\theta}) = \frac{F(\tilde{\theta})^2}{\tilde{\theta}f(\tilde{\theta})-F(\tilde{\theta})} > 0 = \Psi(\underline{\theta})$ for all $\tilde{\theta} \in (\underline{\theta},\overline{\theta}]$,

2.
$$t^*(\theta) = \begin{cases} \bar{T}(\equiv t^*(\tilde{\theta})) & \forall\theta \in [\underline{\theta},\tilde{\theta}) \\ \theta q^*(\theta) + \int_{\theta}^{\overline{\theta}} q^*(\tau)d\tau & \forall\theta \in [\tilde{\theta},\overline{\theta}] \end{cases}$$

3. the optimal cut-off point $\tilde{\theta} \in [\underline{\theta},\overline{\theta}]$ is obtained from $\bar{T} = \tilde{\theta}q^*(\tilde{\theta}) + \int_{\tilde{\theta}}^{\overline{\theta}} q^*(\tau)d\tau$,

provided $\bar{T} > \mathbf{T} \equiv \overline{\theta}S'^{-1}\left(z(\overline{\theta}) + \frac{\Psi(\overline{\theta})}{f(\overline{\theta})}\right)$.

If $\bar{T} \leq \mathbf{T}$ then $M^* = \langle q^*(.), t(.), \tilde{\theta}\rangle$ specifies a full-bunching solution with $\tilde{\theta} = \overline{\theta}$ and $\langle q^*(\theta) = \frac{\bar{T}}{\overline{\theta}}, t^*(\theta) = \bar{T}\rangle$ for all $\theta \in [\underline{\theta},\overline{\theta}]$.

[4] Given that the transfer under second best is strictly decreasing, $t^b(\underline{\theta}) > t^b(\theta)$ for all $\theta \in (\underline{\theta},\overline{\theta}]$, if the budget constraint $\bar{T} \geq t^b(\underline{\theta})$ then the second best mechanism is always feasible.

REMARK 1 Assumption 1 is sufficient to guarantee that for any given cut-off point $\tilde{\theta} \in [\underline{\theta}, \overline{\theta}]$, $z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}$ is non-decreasing in θ for all $\theta \in (\tilde{\theta}, \overline{\theta}]$. Monotonicity of $z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}$ is necessary for the optimal output $q^*(\theta)$ to be non-increasing in $\theta \in [\tilde{\theta}, \overline{\theta}]$. Moreover, assumption 1 also guarantees that $\Psi(\theta)$ is non-decreasing in $\theta \in [\underline{\theta}, \overline{\theta}]$. Assumption 1, is sufficient for the monotonicity of $z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}$ since it implies and is implied by non-decreasingness of $z(\theta) + \frac{\Psi(\theta)}{f(\theta)}$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. If the monotonicity of $z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}$ is violated in the non-bunching interval $(\tilde{\theta}, \overline{\theta}]$, then the analysis can be modified à la Guesnerie and Laffont (1984).

From Theorem 1 it is obvious that the optimal quantity $\frac{\bar{T}}{\overline{\theta}}$ for the full bunching case (that is for $\bar{T} \leq \mathbf{T}$) is strictly lower than any $q^*(\theta)$ for the partial bunching case (that is for $\bar{T} > \mathbf{T}$). Moreover, from Theorem 1 it also follows that if the budget constraint is not binding, that is if $\bar{T} \geq t^b(\underline{\theta})$, then $q^*(\theta) = q^b(\theta)\ \forall \theta \in [\underline{\theta}, \overline{\theta}]$ since $\Psi(\underline{\theta}) = 0$. If, instead, $\bar{T} < t^b(\underline{\theta})$, then the following two Propositions summarize a comparative study between the optimal budget-constrained mechanism M^* and the optimal mechanism M^b under the benchmark model.

PROPOSITION 2 If $\bar{T} < t^b(\underline{\theta})$, then $q^*(\theta) < q^b(\theta)\ \forall \theta \in [\underline{\theta}, \overline{\theta}]$.

PROPOSITION 3 Call $\hat{\theta} = \{\theta \in (\underline{\theta}, \overline{\theta}) \mid t^b(\theta) = \bar{T}\}$. If $\bar{T} < t^b(\underline{\theta})$ then $\hat{\theta} \geq \tilde{\theta}$.

Theorem 1 and its two complementary Propositions (2 and 3) describe the optimal budget-constrained mechanism and compare it with the optimal mechanism. While full separability of types is the main property of the unconstrained mechanism M^b, this property does not hold in the budget-constrained mechanism M^*, at least for the lower cost firms. Hence, with binding budget constraint, the optimal quantity under the budget constrained mechanism is strictly lower than that of the second best mechanism (see Proposition 2). In the budget-constrained mechanism, there is a conflict between the necessity of separability (the IC constraints) and the budget constraint. Separability of types implies increasing information rents for the lower cost firms. With limited resources, it becomes impossible to finance the information rents of the more efficient firms. Hence, there is bunching for the lower cost firms. However, the regulator limits as much as possible the region in which the regulatory contract is a bunching contract (see Proposition 3). For that, the contract offered to the higher cost firms (that is firms for which the budget constraint is non-binding) is distorted compared to the optimal mechanism M^b. Reducing the quantities of the less efficient firms (compared to the optimal mechanism M^b), reduces the information rent, and hence, it is possible to finance separability for a larger fringe of firms. Without any distortions in quantity, the bunching zone would have been $[\underline{\theta}, \hat{\theta}]$, while by imposing optimal distortions in quantity, the bunching zone is reduced to $[\underline{\theta}, \tilde{\theta}]$. This is shown in Figure 1. The optimal budget-constrained mechanism, described in the Theorem 1, takes care of the trade off between the cost of abandoning separability for the more efficient firms and the cost of larger distortions to preserve it. However, the cost of keeping separability for high cost firm may be too high. In that case we have a full bunching solution.

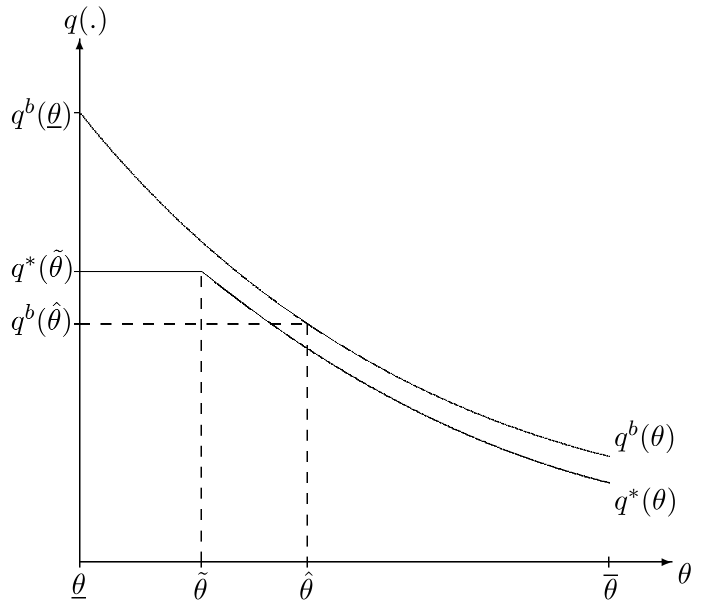

Figure 1: The Optimal Quantities for $\bar{T} = t^b(\underline{\theta})$ and $\bar{T} \in (\mathbf{T}, t^b(\underline{\theta}))$.

In Baron and Myerson (1982), the decision to build up the infrastructure is itself a regulatory instrument. In their optimal mechanism, the infrastructure is built whenever the associated surplus is larger than the cost and this decision does not interfere with the optimal mechanism. Likewise, if, in our problem, $S'(0)$ is finite, then exclusion of the higher cost firms from the mechanism is another instrument that can be used to tackle the problem of budget constraint. In our problem, the decision to build the infrastructure can be incorporated ex-post, together with the cut-off point. Given $S'(0) < \infty$, let θ^{**} denote the highest type for which $q^*(\theta^{**}) > 0$. Then θ^{**} and $\tilde{\theta}$ are determined by the following conditions:

$$S(q^*(\theta^{**})) = t^*(\theta^{**}),$$

$$\bar{T} = \tilde{\theta} q^*(\tilde{\theta}) + \int_{\tilde{\theta}}^{\theta^{**}} q^*(\tau) d\tau.$$

It is obvious that θ^{**} is lower than its corresponding value in the optimal mechanism M^b. Hence, a financial constraint also reduces the probability $F(\theta^{**})$ that the infrastructure is built. Thus, if $S'(0)$ is finite, then it is possible that building infrastructure will be delayed due to inadequate funds.

We conclude the analysis of the budget-constrained mechanism with a comparative static result followed by a simple example. Consider any two budget constraints $\bar{T}_1$ and $\bar{T}_2$ such that $\bar{T}_1 < \bar{T}_2 \leq t^b(\underline{\theta})$. With slight abuse of notation, let $q_i^*(\theta)$ be the type contingent output and $\tilde{\theta}_i$ be the cut-off point, both associated with the budget constraint $\bar{T}_i$ for $i = 1, 2$.

PROPOSITION 4 If $\mathbf{T} \leq \bar{T}_1 < \bar{T}_2 \leq t^b(\underline{\theta})$, then $\Psi(\tilde{\theta}_1) \geq \Psi(\tilde{\theta}_2)$ and $\tilde{\theta}_1 > \tilde{\theta}_2$ which together imply $q_1^*(\theta) \leq q_2^*(\theta) \forall \theta \in [\underline{\theta}, \overline{\theta}]$.

A reduction in available funds reduces the optimal quantities and the cut-off point (provided $\mathbf{T} \leq \bar{T}_1$). This comparative static result is intuitive. Due to the scarcity of resources, the opportunity cost of paying information rents to the more efficient firms increases and hence the regulator prefers to save on these rents to finance the infrastructure with its available resources. This result also explains why for a sufficiently low transfer $\bar{T}(< \mathbf{T})$, the budget-constrained mechanism prescribes full bunching.

EXAMPLE 1 In this example we consider a very simple functional form for the surplus function and assume that $f(.)$ has uniform distribution over $[\underline{\theta}, \overline{\theta}]$. In particular, we take the surplus function to be $S(q) = 2\sqrt{q}$. Under these assumptions the optimal budget-constrained mechanism $M^* = \langle q^*(.), t^*(.), \tilde{\theta} \rangle$ specifies for the partial bunching case the following type contingent quantity-transfer pairs:

$$q^*(\theta) = \begin{cases} \frac{\underline{\theta}^2}{\tilde{\theta}^4} & \forall \theta \in [\underline{\theta}, \tilde{\theta}) \\ \frac{1}{(2(\theta - \tilde{\theta}) + \frac{\tilde{\theta}^2}{\underline{\theta}})^2} & \forall \theta \in [\tilde{\theta}, \overline{\theta}] \end{cases}$$

$$t^*(\theta) = \begin{cases} \bar{T}(\equiv t^*(\tilde{\theta})) & \forall \theta \in [\underline{\theta}, \tilde{\theta}) \\ \theta q^*(\theta) + \int_{\theta}^{\overline{\theta}} q^*(\tau) d\tau & \forall \theta \in [\tilde{\theta}, \overline{\theta}] \end{cases}$$

The optimal cut-off point $\tilde{\theta}$ can be obtained by solving $\bar{T} = \frac{\underline{\theta}^2 (2(\overline{\theta} - \tilde{\theta})\underline{\theta} + \overline{\theta}\tilde{\theta})}{\tilde{\theta}^3 (2(\overline{\theta} - \tilde{\theta})\underline{\theta} + \tilde{\theta}^2)}$. The critical value of the transfer is $\mathbf{T} = \frac{\underline{\theta}^2}{\overline{\theta}^3}$. Therefore, for all $\bar{T} \leq \mathbf{T}$, the optimal budget-constrained mechanism $M^* = \langle q^*(.), t^*(.), \tilde{\theta} \rangle$ gives a full bunching solution where the optimal quantity-transfer pair is $\left(q^*(\theta) = \frac{\bar{T}}{\overline{\theta}}, t^*(\theta) = \bar{T} \right)$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$ and the optimal cut-off point is $\tilde{\theta} = \overline{\theta}$.

4 Conclusions

In this paper, we have analyzed the problem of financing infrastructure under asymmetric information, when there is financial constraint. The optimal budget-constrained mechanism broadly differs from the optimal mechanism in three respects. Firstly, there is an overall reduction in the quantity of the public good provided. This makes the budget-constrained solution a third best one. Hence the efficiency at the top does not hold. Secondly, the budget-constrained mechanism exhibits a substantial amount of bunching. Our comparative static results show that as the available resources decreases, more and more types of firms are bunched. Finally, if the available resources are too small, there is full bunching. This result is not surprising. If the available resources are too small, the regulator prefers to use less efficient regulation and invest the money in infrastructure building rather than in financing separability.

Given that the budget-constrained solution is a third-best one, there are welfare losses. The welfare is reduced because each type of firm produces a lower quantity of the public good and hence the consumer surplus is lower. Moreover, the welfare is also reduced because there is bunching for the more efficient firms. To satisfy the wealth constraint, the regulator gives up separability for the more efficient firms. From our comparative static result it is obvious that welfare loss is decreasing in available resources ($\bar{T}$).

5 Appendix

PROOF OF THEOREM 1: For an almost everywhere differentiable mechanism, incentive compatibility (or IC) implies that truth-telling is a best response of the monopolist, that is $\left\{\frac{\delta}{\delta\theta'}[U_m(\theta;\theta')]\right\}_{\theta'=\theta} = 0$ almost everywhere. This condition implies that $t'(\theta) = \theta q'(\theta)$ almost everywhere. From IC we also know that $q(\theta)$ must be non-increasing in θ and hence $t'(\theta) \leq 0$ almost everywhere in $(\underline{\theta}, \overline{\theta})$. For the optimization program $[P^*]$, let $\tilde{\theta}$ be the first type for which the budget constraint is not binding. Therefore, for all $\theta \in [\underline{\theta}, \tilde{\theta})$, the budget constraint is binding and for all $\theta \in [\tilde{\theta}, \overline{\theta}]$, the budget constraint is not binding (or free).[5] This means that $t'(\theta) = 0$ for all $\theta \in (\underline{\theta}, \tilde{\theta})$ and $t'(\theta) \leq 0$ for all $\theta \in (\tilde{\theta}, \overline{\theta})$. From IC and PC we also know that $U'_m(\theta) = -q(\theta) < 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$ and optimality of the mechanism guarantees that $U_m(\overline{\theta}) = 0$. Finally, non-increasingness of $q(\theta)$ implies that $q'(\theta) = 0$ for all $\theta \in (\underline{\theta}, \tilde{\theta})$, $q'(\theta) \leq 0$ for all $\theta \in (\tilde{\theta}, \overline{\theta})$ and since $t(.)$ is not differentiable at $\tilde{\theta}$,

$$q(\tilde{\theta}^-) \geq q(\tilde{\theta}^+) \tag{1}$$

The regulator's optimization problem $[P^*]$ can now be divided into two sub-problems $[P_1^*]$ and $[P_2^*]$ for the intervals $[\underline{\theta}, \tilde{\theta})$ and $[\tilde{\theta}, \overline{\theta}]$ respectively.

$[P_1^*]$ $\quad \max \int_{\underline{\theta}}^{\tilde{\theta}} \{S(q_1(\theta)) - U_m(\theta) - \theta q_1(\theta)\} f(\theta) d\theta$ subject to

1. $U'_m(\theta) = -q_1(\theta)$,
2. $\bar{T} - U_m(\theta) - \theta q_1(\theta) = 0$,
3. $U_m(\underline{\theta})$ free, $\tilde{\theta}$ and $U_m(\tilde{\theta})$ given, and
4. $q_1(\theta) \equiv q(\theta)$.

$[P_2^*]$ $\quad \max \int_{\tilde{\theta}}^{\overline{\theta}} \{S(q_2(\theta)) - U_m(\theta) - \theta q_2(\theta)\} f(\theta) d\theta$ subject to

1. $U'_m(\theta) = -q_2(\theta)$,

[5] Observe that we are assuming that it is possible to find type contingent quantity-transfer pairs which allows for partial bunching and partial separability. In otherwords, we are trying to find the optimal constrained mechanism for the case when the available fund $\bar{T}$ is above some critical level **T** which allows for partial bunching and partial separation. The solution to this program will provide the exact amount of this critical level **T**.

2. $U_m(\overline{\theta}) = 0$,

3. $\tilde{\theta}$ and $U_m(\tilde{\theta})$ given, and

4. $q_2(\theta) \equiv q(\theta)$.

$[P_1^*]$ and $[P_2^*]$ are two optimal control problems. In both these sub-problems $q(.)$ is the control variable and $U_m(.)$ is the state variable. Finally, $\tilde{\theta}$ is the optimal cut-off point that links the two problems.

The Hamiltonian function associated with $[P_i^*]$ is $H_i(\theta) = \{S(q_i(\theta)) - U_m(\theta) - \theta q_i(\theta)\} f(\theta) - \lambda_i(\theta) q_i(\theta)$ for $i = 1, 2$. Here $\lambda_i(\theta)$ is the co-state (or auxiliary) variable associated with the Hamiltonian $H_i(\theta)$ for the type θ. The Lagrangian associated with the sub-problem $[P_1^*]$ is $L_1(\theta) = H_1(\theta) + \mu(\theta)[\bar{T} - U_m(\theta) - \theta q_1(\theta)]$ where $\mu(\theta)$ is the Lagrangian multiplier associated with the type θ.

The necessary conditions for $[P_1^*]$ are

$[P_1^*(1)]$ $\frac{\delta L_1(\theta)}{\delta q_1(\theta)} = \{S'(q_1(\theta)) - \theta\} f(\theta) - \lambda_1(\theta) - \theta\mu(\theta) = 0$,

$[P_1^*(2)]$ $\lambda_1'(\theta) = -\frac{\delta L_1(\theta)}{\delta U_m(\theta)} = f(\theta) + \mu(\theta)$,

$[P_1^*(3)]$ $\lambda_1(\tilde{\theta})$ is free,

$[P_1^*(4)]$ $\lambda_1(\underline{\theta}) = 0$,

$[P_1^*(5)]$ $\mu(\theta) \geq 0$ and

$[P_1^*(6)]$ $\bar{T} - U_m(\theta) - \theta q_1(\theta) = 0$.

Similarly, the necessary conditions for $[P_2^*]$ are

$[P_2^*(1)]$ $\frac{\delta H_2(\theta)}{\delta q_2(\theta)} = \{S'(q_2(\theta)) - \theta\} f(\theta) - \lambda_2(\theta) = 0$,

$[P_2^*(2)]$ $\lambda_2'(\theta) = -\frac{\delta H_2(\theta)}{\delta U_m(\theta)} = f(\theta)$,

$[P_2^*(3)]$ $\lambda_2(\tilde{\theta})$ is free and

$[P_2^*(4)]$ $\lambda_2(\overline{\theta})$ is free.

From $[P_1^*(2)]$ we get

$$\lambda_1(\theta) = F(\theta) + \Psi(\theta) + k_1 \tag{2}$$

where $\Psi(\theta) = \int_{\underline{\theta}}^{\theta} \mu(\tau) d\tau$ and k_1 is the constant of integration.[6] Since $\Psi(\underline{\theta}) = F(\underline{\theta}) = 0$ and since $\lambda_1(\underline{\theta}) = 0$ from $[P_1^*(4)]$, we get $k_1 = 0$. Therefore, from (2) we get

[6] It is important to note that $\Psi'(\theta) = \mu(\theta)$. This fact will be used later to determine the functional form of $\Psi(\theta)$.

$$\lambda_1(\theta) = F(\theta) + \Psi(\theta) \tag{3}$$

From $[P_2^*(2)]$ we get

$$\lambda_2(\theta) = F(\theta) + k_2 \tag{4}$$

where k_2 is the constant of integration. Since $\tilde{\theta}$ is the optimal cut-off point for the program $[P^*]$, we get $\lambda_1(\tilde{\theta}) = \lambda_2(\tilde{\theta})$. Then from conditions (3) and (4) we get $k_2 = \Psi(\tilde{\theta})$ and hence

$$\lambda_2(\theta) = F(\theta) + \Psi(\tilde{\theta}) \tag{5}$$

Substituting (3) in $[P_1^*(1)]$ and then simplifying it, using $q(\theta) = q(\tilde{\theta})$ for all $\forall\theta \in [\underline{\theta}, \tilde{\theta})$, we get

$$S'(q_1(\tilde{\theta})) = \theta + \frac{F(\theta) + \Psi(\theta) + \theta\mu(\theta)}{f(\theta)} \tag{6}$$

for all $\theta \in [\underline{\theta}, \tilde{\theta})$.

Similarly, substituting (5) in $[P_2^*(1)]$ and then simplifying it we get $\forall\theta \in [\tilde{\theta}, \overline{\theta}]$

$$S'(q_2(\theta)) = \theta + \frac{F(\theta) + \Psi(\tilde{\theta})}{f(\theta)} \tag{7}$$

To show that $q(.)$ is continuous at the cut-off point $\tilde{\theta}$, we must show that the left hand side of $[P_1^*(1)]$ and $[P_2^*(1)]$ are the same at $\tilde{\theta}$, that is $\{S'(q_1(\tilde{\theta})) - \tilde{\theta}\}f(\tilde{\theta}) - \lambda_1(\tilde{\theta}) - \tilde{\theta}\mu(\tilde{\theta}) = \{S'(q_2(\tilde{\theta})) - \tilde{\theta}\}f(\tilde{\theta}) - \lambda_2(\tilde{\theta})$. Using $\lambda_1(\tilde{\theta}) = \lambda_2(\tilde{\theta})$ we get

$$S'(q_1(\tilde{\theta})) - S'(q_2(\tilde{\theta})) = \frac{\tilde{\theta}\mu(\tilde{\theta})}{f(\tilde{\theta})} \tag{8}$$

If $\mu(\tilde{\theta}) > 0$, then the right hand side of condition (8) is positive. This means that $S'(q_1(\tilde{\theta})) > S'(q_2(\tilde{\theta}))$ and hence by strict concavity of $S(.)$ we get $q_1(\tilde{\theta})) < q_2(\tilde{\theta})$. This violates condition (1). Therefore, it must be the case that $\mu(\tilde{\theta}) = 0$ and hence $q_1(\tilde{\theta})) = q_2(\tilde{\theta})$.

Therefore, the optimal budget constraint mechanism M^* for the partial bunching case satisfies the following three conditions:

(p1) $S'(q^*(\tilde{\theta})) = \theta + \frac{F(\theta)+\Psi(\theta)+\theta\mu(\theta)}{f(\theta)}$

for all $\theta \in [\underline{\theta}, \tilde{\theta})$, $\mu(\tilde{\theta}) = 0$, (p2) $S'(q^*(\theta)) = \theta + \frac{F(\theta)+\Psi(\tilde{\theta})}{f(\theta)}$ for all $\theta \in [\tilde{\theta}, \overline{\theta}]$ and

(p3) $\bar{T} = U_m(\tilde{\theta}) + \tilde{\theta} q^*(\tilde{\theta})$

Here (p1) follows from condition (6), (p2) follows from condition (7) and (p3) is obtained from $[P_1^*(6)]$ which gives us the optimal cut-off point $\tilde{\theta}$.

We now determine $\Psi(\tilde{\theta})$. Integrating condition (6) after substituting $\frac{d}{d\theta}[\theta F(\theta)] = \theta f(\theta) + F(\theta)$, $\frac{d}{d\theta}[\theta \Psi(\theta)] = \theta \mu(\theta) + \Psi(\theta)$ and $S'(q(\tilde{\theta})) \equiv c(\tilde{\theta})$ we get

$$\theta F(\theta) + \theta \Psi(\theta) = c(\tilde{\theta}) F(\theta) + k_3 \tag{9}$$

for all $\theta \in [\underline{\theta}, \tilde{\theta})$. Here k_3 is the constant of integration. Using $F(\underline{\theta}) = \Psi(\underline{\theta}) = 0$ in condition (9) we get $k_3 = 0$. Substituting $k_3 = 0$ in condition (9) and then simplifying it we get

$$\Psi(\theta) = \left(\frac{c(\tilde{\theta}) - \theta}{\theta} \right) F(\theta) \tag{10}$$

Differentiating (10) with respect to θ and then using $\mu(\tilde{\theta}) = 0$ we get $S'(q(\tilde{\theta})) \equiv c(\tilde{\theta}) = \left(\frac{\tilde{\theta}^2 f(\tilde{\theta})}{\tilde{\theta} f(\tilde{\theta}) - F(\tilde{\theta})} \right)$. Substituting $\theta = \tilde{\theta}$ and $c(\tilde{\theta})$ in condition (10) we get $\Psi(\tilde{\theta}) = \left(\frac{\{F(\tilde{\theta})\}^2}{\tilde{\theta} f(\tilde{\theta}) - F(\tilde{\theta})} \right)$. Since $\Psi(\tilde{\theta}) = \int_{\underline{\theta}}^{\tilde{\theta}} \mu(\tau) d\tau$ and the Lagrangian multiplier $\mu(\theta) \geq 0$ for all $\theta \in [\underline{\theta}, \tilde{\theta})$, it is necessary that $\Psi(\tilde{\theta}) \geq 0$. It is obvious that $\Psi(\underline{\theta}) = 0$ since $F(\underline{\theta}) = 0$. Therefore, to show that $\Psi(\theta) \geq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$ it is now more than enough to show that $\Psi'(\theta) \geq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. By differentiating $\Psi(\theta)$ with respect to $\theta \in (\underline{\theta}, \overline{\theta})$ and then setting it to be non-negative we get (b) $\theta(1 + L'(\theta)) \geq 2L(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. Condition (b) is identical to assumption 1. Hence, $\Psi'(\theta) \geq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$ which implies that $\Psi(\theta) \geq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. This however means that $\Psi(\theta) = \left(\frac{\{F(\theta)\}^2}{\theta f(\theta) - F(\theta)} \right) > 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$ since our assumption that $f(\theta) > 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$ implies that $F(\theta) > 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$ and since for $\Psi(\theta)$ to be non-negative it is always necessary that $\theta f(\theta) - F(\theta) > 0$. Thus, conditions (p1), (p2) and (p3) together with $\Psi(\tilde{\theta}) = \left(\frac{\{F(\tilde{\theta})\}^2}{\tilde{\theta} f(\tilde{\theta}) - F(\tilde{\theta})} \right) > 0$ gives us the conditions in Theorem 1 when partial bunching is optimal.

We will show that given $\Psi(\tilde{\theta}) = \left(\frac{\{F(\tilde{\theta})\}^2}{\tilde{\theta} f(\tilde{\theta}) - F(\tilde{\theta})} \right)$, $q^*(\theta)$ is non-increasing in $\theta \in [\underline{\theta}, \overline{\theta}]$. Observe first that from condition (p1) it follows that $q^*(\theta) = q^*(\tilde{\theta})$ for all $\theta \in [\underline{\theta}, \tilde{\theta})$. To show that $q^*(\theta)$ is non-increasing in $\theta \in [\tilde{\theta}, \overline{\theta}]$ we have to show that $\bar{z}(\theta) \equiv z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}$ is non-decreasing in $\theta \in [\tilde{\theta}, \overline{\theta}]$. Differentiating

$\bar{z}(\theta)$ with respect to $\theta \in (\tilde{\theta}, \overline{\theta})$ and then setting it to be non-negative we get (c) $\frac{f'(\theta)(F(\theta)+\Psi(\tilde{\theta}))}{f^2(\theta)} \leq 2$. To show that condition (c) is true, it is more than enough to show that $\frac{f'(\theta)(F(\theta)+\Psi(\theta))}{f^2(\theta)} \leq 2$ since $\Psi'(\theta) \geq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$ implies that $\frac{f'(\theta)(F(\theta)+\Psi(\tilde{\theta}))}{f^2(\theta)} \leq \frac{f'(\theta)(F(\theta)+\Psi(\theta))}{f^2(\theta)}$ for all $\theta \in (\tilde{\theta}, \overline{\theta})$. From assumption 1 we know that for all $\theta \in (\underline{\theta}, \overline{\theta})$,

$$\theta(1 + L'(\theta)) \geq 2L(\theta)$$

$$\text{or } \theta\left(2 - \frac{f'(\theta)F(\theta)}{f^2(\theta)}\right) \geq 2L(\theta)$$

$$\text{or } \frac{\theta}{L(\theta)}\left(2 - \frac{f'(\theta)F(\theta)}{f^2(\theta)}\right) \geq 2$$

$$\text{or } 2\left(\frac{\theta - L(\theta)}{L(\theta)}\right) \geq \left(\frac{\theta}{L(\theta)}\right)\left(\frac{f'(\theta)F(\theta)}{f^2(\theta)}\right)$$

$$\text{or } 2\left(\frac{F(\theta)}{\Psi(\theta)}\right) \geq \left(\frac{(F(\theta)+\Psi(\theta)}{\Psi(\theta)}\right)\left(\frac{f'(\theta)F(\theta)}{f^2(\theta)}\right) \text{ (since } \Psi(\theta) = \left(\frac{L(\theta)F(\theta)}{\theta - L(\theta)}\right) \text{ for all } \theta)$$

$$\text{or } 2 \geq \left(\frac{f'(\theta)(F(\theta)+\Psi(\theta))}{f^2(\theta)}\right)$$

Thus from assumption 1, we get $\frac{f'(\theta)(F(\theta)+\Psi(\theta))}{f^2(\theta)} \leq 2$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. Therefore, condition (c) holds. This proves that $\bar{z}(\theta) \equiv z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}$ is non-decreasing in $\theta \in [\tilde{\theta}, \overline{\theta}]$ and hence $q^*(\theta)$ is non-increasing in $\theta \in [\tilde{\theta}, \overline{\theta}]$. Observe that $\tilde{\theta} = \overline{\theta}$, corresponds to the transfer $\mathbf{T} = \overline{\theta} S'^{-1}\left(\overline{\theta} + \frac{F(\overline{\theta})+\Psi(\overline{\theta})}{f(\overline{\theta})}\right) > 0$. Therefore, for all $\bar{T} > \mathbf{T}$ the optimal mechanism is a partial bunching one. Finally, since the Hamiltonian H_2 is concave in $q(.)$ and linear in $U_m(.)$, the necessary conditions are also sufficient for $[P_2^*]$. The necessary conditions are also sufficient for $[P_1^*]$ since the Lagrangian $L_1(\theta)$ is concave in (q, U_m) for all $\theta \in [\underline{\theta}, \tilde{\theta})$ (see Chiang (1992)).

If $\bar{T} \leq \mathbf{T}$, then a partial bunching contract is not feasible. Hence, for $\bar{T} \leq \mathbf{T}$, the optimal solution is a full-bunching one implying $q^*(\theta) = \bar{q}$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Given that the mechanism is optimal, from IC and PC it follows that $U_m(\theta) = \int_\theta^{\overline{\theta}} q^*(\tau)d\tau$ and hence we get $\bar{T} - \theta\bar{q} = (\overline{\theta} - \theta)\bar{q}$. Therefore, $\bar{T} = \overline{\theta}\bar{q}$. Thus, in the full bunching case $\tilde{\theta} = \overline{\theta}$ and $q^*(\theta) = \frac{\bar{T}}{\overline{\theta}}$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. ■

PROOF OF 2: Consider first the partial bunching case, that is consider $\mathbf{T} < \bar{T} < t^b(\underline{\theta})$. Observe first that the number $\Psi(\tilde{\theta})$ is strictly positive. This implies that for all $\theta \in [\tilde{\theta}, \overline{\theta}]$, $S'^{-1}\left(z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}\right) < S'^{-1}(z(\theta))$. Therefore, $q^*(\theta) < q^b(\theta)$ for all $\theta \in [\tilde{\theta}, \overline{\theta}]$. Moreover, for all $\theta \in [\underline{\theta}, \tilde{\theta})$, $q^*(\theta) = q^*(\tilde{\theta}) < q^b(\theta)$ since $q^b(\theta) \geq q^b(\tilde{\theta})$ for all $\theta \in [\underline{\theta}, \tilde{\theta})$. For the full bunching case, that is for $\bar{T} \leq \mathbf{T} < t^b(\underline{\theta})$, it is obvious that the optimal fixed quantity $\frac{\bar{T}}{\overline{\theta}}$ is strictly smaller than any $q^*(\theta)$ for the partial bunching case. Hence the result follows.

■

PROOF OF 3: Observe first that by definition $\hat{\theta}q^b(\hat{\theta}) + \int_{\hat{\theta}}^{\overline{\theta}} q^b(\tau)d\tau = \bar{T} = \tilde{\theta}q^*(\tilde{\theta}) + \int_{\tilde{\theta}}^{\overline{\theta}} q^*(\tau)d\tau$. Using this observation, we prove the proposition by contradiction. We first assume that $\hat{\theta} < \tilde{\theta}$. Then define $h(\theta) = q^b(\theta) - q^*(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta}]$. Given Proposition 2, $h(\theta) > 0$ for all $\theta \in (\underline{\theta}, \overline{\theta}]$. Using the observation we get

$$\hat{\theta}(h(\hat{\theta}) + q^*(\hat{\theta})) - \tilde{\theta}q^*(\tilde{\theta}) + \int_{\tilde{\theta}}^{\overline{\theta}} h(\tau)d\tau + \int_{\hat{\theta}}^{\tilde{\theta}} q^b(\tau)d\tau = 0 \tag{11}$$

Since by assumption $\hat{\theta} < \tilde{\theta}$, from the optimal budget-constrained mechanism M^* we get $q^*(\hat{\theta}) = q^*(\tilde{\theta})$. Substituting $q^*(\hat{\theta}) = q^*(\tilde{\theta})$ in (11) we get

$$\hat{\theta}h(\hat{\theta}) - (\tilde{\theta} - \hat{\theta})q^*(\tilde{\theta}) + \int_{\tilde{\theta}}^{\overline{\theta}} h(\tau)d\tau + \int_{\hat{\theta}}^{\tilde{\theta}} q^b(\tau)d\tau = 0 \tag{12}$$

Since $\int_{\hat{\theta}}^{\tilde{\theta}} q^b(\tau)d\tau \geq (\tilde{\theta} - \hat{\theta})q^*(\tilde{\theta})$, the left hand side of (12) is strictly positive. Hence we have a contradiction. ■

PROOF OF 4: From the optimal mechanism in Theorem 1, we know that $\bar{T}_i = \tilde{\theta}_i q_i^*(\tilde{\theta}_i) + \int_{\tilde{\theta}_i}^{\overline{\theta}} q_i^*(\tau)d\tau$ for $i = 1, 2$. Given that $\Psi'(\theta) \geq 0$, we have the following possibilities:

1. $\tilde{\theta}_1 > \tilde{\theta}_2$ and $\Psi(\tilde{\theta}_1) \geq \Psi(\tilde{\theta}_2)$ and
2. $\tilde{\theta}_1 \leq \tilde{\theta}_2$ and $\Psi(\tilde{\theta}_1) \leq \Psi(\tilde{\theta}_2)$.

We now show that condition (2) is incompatible with $\bar{T}_1 < \bar{T}_2$. Observe that $\bar{T}_1 < \bar{T}_2$ implies that

$$\tilde{\theta}_1 q_1^*(\tilde{\theta}_1) - \tilde{\theta}_2 q_2^*(\tilde{\theta}_2) + \int_{\tilde{\theta}_2}^{\overline{\theta}} (q_1^*(\tau) - q_2^*(\tau))d\tau + \int_{\tilde{\theta}_1}^{\tilde{\theta}_2} q_1^*(\tau)d\tau < 0 \tag{13}$$

If $\tilde{\theta}_1 q_1^*(\tilde{\theta}_1) \geq \tilde{\theta}_2 q_2^*(\tilde{\theta}_2)$, then we already have a contradiction to condition (13) since $\int_{\tilde{\theta}_2}^{\overline{\theta}} (q_1^*(\tau) - q_2^*(\tau))d\tau \geq 0$ and $\int_{\tilde{\theta}_1}^{\tilde{\theta}_2} q_1^*(\tau)d\tau > 0$. Therefore, for condition (13) to be true it is necessary that $\tilde{\theta}_1 q_1^*(\tilde{\theta}_1) < \tilde{\theta}_2 q_2^*(\tilde{\theta}_2)$. Moreover, for condition (13) to hold it is also necessary that $\tilde{\theta}_2 q_2^*(\tilde{\theta}_2) - \tilde{\theta}_1 q_1^*(\tilde{\theta}_1) > \int_{\tilde{\theta}_1}^{\tilde{\theta}_2} q_1^*(\tau) d\tau$. We now show that this condition is not true. Observe first that $\int_{\tilde{\theta}_1}^{\tilde{\theta}_2} q_1^*(\tau) d\tau > (\tilde{\theta}_2 - \tilde{\theta}_1) q_1^*(\tilde{\theta}_2)$ since $q_1^*(\theta) > q_1^*(\tilde{\theta}_2)$ for all $\theta \in [\tilde{\theta}_1, \tilde{\theta}_2)$. Observe next that $(\tilde{\theta}_2 - \tilde{\theta}_1) q_1^*(\tilde{\theta}_2) \geq \tilde{\theta}_2 q_2^*(\tilde{\theta}_2) - \tilde{\theta}_1 q_1^*(\tilde{\theta}_1)$. These two observations together imply that $\tilde{\theta}_2 q_2^*(\tilde{\theta}_2) - \tilde{\theta}_1 q_1^*(\tilde{\theta}_1) < \int_{\tilde{\theta}_1}^{\tilde{\theta}_2} q_1^*(\tau) d\tau$. Thus, for this case we get $\bar{T}_1 > \bar{T}_2$ which is a contradiction.

Thus, we have proved that only condition (1) is compatible with $\bar{T}_1 < \bar{T}_2$. Hence for all $\theta \in [\underline{\theta}, \overline{\theta}]$, $q_1^*(\theta) < q_2^*(\theta)$. ■

6 References

Baron, D. P. and Myerson, R. B. (1982). Regulating a Monopolist with Unknown Costs, *Econometrica* 50(**4**), 911-930.

Chiang, A. C. (1992). Elements of Dynamic Optimization, McGraw-Hill Book Company, Singapore.

Che, Y. K. and Gale, I. (2000). The Optimal Mechanism for Selling to Budget-Constrained Consumers, *Journal of Economic Theory* 92(**2**), 198-233.

Gautier, A. (2002). Regulation under Financial Constraints, Discussion Paper, Bonn Graduate School of Economics, University of Bonn, Germany.

Guesnerie R. and Laffont, J.-J. (1984). A Complete Solution to a Class of Principal-Agent Problems with an Application to the Control of a Self-managed Firm, *Journal of Public Economics* 25, 329-369.

Hald, A. (1952). Statistical Theory with Engineering Applications, John Wiley, New York.

Laffont, J.-J., and Roberts, J. (1996). Optimal Auction with Financially Constrained Buyers, *Economics Letters* 52, 181-186.

Laffont, J.-J., and Tirole, J.-J. (1993). A Theory of Incentives in Procurement and Regulation, MIT Press, Cambridge.

Monteiro, P. K., and Page Jr., F. H. (1998). Optimal Selling Mechanisms for Multiproduct Monopolists: Incentive Compatibility in the Presence of Budget Constraints, *Journal of Mathematical Economics* 30, 473-520.

Thomas, L. (2002). Non-linear Pricing with Budget Constraint, *Economics Letters* 75, 257-263.