

# Supermodularity and Complementarity in Economics: An Elementary Survey\*

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## Abstract

The literature on supermodular optimization and games is surveyed from the perspective of potential users in economics. This methodology provides a new approach for comparative statics based only on critical assumptions, and allows a general analysis of games with strategic complementarities. The results are presented in a simplified yet rigorous manner, without reference to lattice theory, for the special case of one-dimensional parameter and actions sets, with the emphasis being on wide accessibility. Detailed applications are presented for well-known models of consumer behavior, monopoly pass-through, Bertrand and Cournot competition, strategic R&D, search and matching. Wherever appropriate, useful tricks for applications and comparative comments are inserted.

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# 1 Introduction

This survey provides an overview of the theory of supermodular optimization and games, with a marked emphasis on accessibility, for as broad an audience as possible. Supermodular optimization is a new methodology for conducting comparative statics or sensitivity analysis, i.e. determining how changes in exogenous parameters affect endogenous variables in optimizing models<sup>1</sup>. As such, the use of this methodology is pervasive in economics, and the conclusions thereby derived are often one of the main motivations behind the construction of a model. The main characteristic of this methodology is that it relies essentially on critical assumptions for the desired monotonicity conclusions and dispenses with superfluous assumptions that are often imposed only by the use of the classical method based on the Implicit Function Theorem, including smoothness, interiority and concavity. The main insight is indeed quite simple. If in a maximization problem, the objective reflects a complementarity between an endogenous variable and an exogenous parameter, in the sense that having more of one increases the marginal return to having more of the other, then the optimal value of the former will be increasing in the latter. In the case of multiple endogenous variable, then all of them must also be complements so as to guarantee that their increases are mutually reinforcing. This conclusion follows directly from the underlying complementarity relationship and is thus independent of the aforementioned superfluous assumptions. It thus holds even if there are multiple optimal values of the endogenous variable(s).

Is a new look at complementarity needed? Topkis (1998) quotes Samuelson (1947) as asserting "In my opinion the problem of complementarity has received more attention than is merited by its intrinsic importance", only to correct himself later in Samuelson (1974), by adding "The time is ripe for a fresh modern look at the concept of complementarity. ... The last word has not yet been said on this ancient preoccupation of literary and mathematical economists. The simplest things are often the most complicated to understand fully." It is hoped that this survey will convince the reader of the correctness of Samuelson's latter view.

Another major methodological breakthrough due to this framework of analysis is the theory of supermodular games, better known in economics as games with strategic complementarities. The main characteristic of such game is that they have monotonic reaction "curves", reflecting a complementarity relationship between own actions and rivals' actions.

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<sup>1</sup>This framework of analysis has been developed by Topkis (1968, 1978).

As a consequence of Tarski's fixed-point theorem, the latter property guarantees the existence of pure-strategy Nash equilibrium points. As the latter type of equilibrium is most often the desired concept in economic models, the scope of game-theoretic modeling in economics is thereby substantially enlarged. Another key observation in this respect is that supermodularity is often the relevant notion in the comparative statics of Nash equilibrium points. Furthermore, it will be argued that supermodular games are more conducive to predictable comparative statics properties than games with continuous best-responses, the latter being the other class of games with pure-strategy Nash equilibrium points.

While maintaining rigor in the presentation of the concepts and proofs, some informal aspects in the exposition are adopted, whenever they result in substantial simplification. The major step in achieving such an accessible exposition lies in the restriction of the theory to the case of real action and parameter spaces. While this reduces the scope of the theory and masks its striking elegance, it does nevertheless cover most economic applications of broad interest. The main results from the theory of supermodular games are also thereby simplified and more accessible. The multi-dimensional Euclidean framework is presented in summary form at the end of the survey. General comparative comments are given at various points to provide some sense of the scope, usefulness and limitations of this theory from an applications-oriented perspective. A number of well-known economic applications are covered, including monopoly theory, Cournot and Bertrand competition, a two-stage R&D model, search, matching, and growth theory. Some of these are covered both with the cardinal and the ordinal notions of complementarity, so as to provide some comparative perspective. Various practical tricks in fully exploiting the benefits of this theory are also illustrated via some of the applications presented.

This survey is organized as follows. The next section presents the simplified version of Topkis's Monotonicity Theorem with real decision and parameter spaces, and compares this result with the standard method. Section 3 introduces games with strategic complementarities and their key properties, including the comparative statics of their equilibria. Section 4 presents the ordinal complementarity conditions and Milgrom-Shannon's Theorem. The (Euclidean) multi-dimensional case forms Section 5. Concluding remarks and other aspects of the theory, not covered in this survey, are summarized in Section 6. Last but not least, several illustrative applications and comparative comments are presented all along to bring out the value added of this new approach in an accessible manner.

## 2 Monotone Comparative Statics

This section provides a simplified exposition of Topkis's (1978) framework in the special case where both the parameter and the decision sets are subsets of the reals. A number of economic applications are then presented to illustrate in very familiar settings the relevance and the scope of application of this simplified version of the general theory.

### 2.1 Topkis's Monotonicity Theorem: The Scalar Case

Topkis considered the following parametrized family of constrained optimization problems, where  $A_s \subset A$ , with the intent of deriving sufficient conditions on the objective and constraint set that yield monotone optimal solutions:

$$a^*(s) = \arg \max\{F(s, a) : a \in A_s\}. \quad (1)$$

We take the parameter and action sets,  $S$  and  $A$ , to be subsets of  $R$ , and  $A_s$  a correspondence from  $S$  to  $A$ , with  $A_s$  being the set of feasible actions when the parameter is  $s$ .

A function  $F : S \times A \rightarrow R$  has (strictly) increasing differences in  $(s, a)$  if

$$F(s', a') - F(s', a)(>) \geq F(s, a') - F(s, a), \quad \forall a' > a, s' > s \quad (2)$$

or in other words if the difference  $F(\cdot, a') - F(\cdot, a)$  is an increasing function<sup>2</sup>. This property does not discriminate between the two variables in that (2) is clearly equivalent to

$$F(s', a') - F(s, a')(>) \geq F(s', a) - F(s, a), \quad \forall a' > a, s' > s. \quad (3)$$

For functions on  $R^2$ , increasing differences is equivalent to supermodularity, so the two terms will be used interchangeably.<sup>3</sup>

For smooth functions, supermodularity/increasing differences admit a convenient test.<sup>4</sup>

**Lemma 1** *If  $F$  is twice continuously differentiable, increasing differences is equivalent to  $\partial^2 F(a)/\partial a \partial s \geq 0$ , for all  $a$  and  $s$ .*

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<sup>2</sup>Throughout, a function  $f : S \rightarrow R$  is increasing (strictly increasing) if  $x \geq y \Rightarrow f(x) \geq (>)f(y)$ .

<sup>3</sup>Supermodularity is defined later on, when dealing with multi-dimensional comparative statics.

<sup>4</sup>Furthermore, if  $\partial^2 f(a)/\partial a_i \partial a_j > 0$ , for all  $i \neq j$ , then  $F$  is strictly supermodular. On the other hand, the reverse implication need not hold.

**Proof.** Increasing differences is equivalent to  $F(\cdot, a') - F(\cdot, a)$  is an increasing function (when  $a' > a$ ), which is equivalent to  $\partial[F(s, a') - F(s, a)]/\partial s \geq 0$ , or  $\partial F(s, a')/\partial s \geq \partial F(s, a)/\partial s$ , i.e.  $\partial F(s, a)/\partial s$  is increasing in  $a$ , or  $\partial^2 F(a)/\partial a \partial s \geq 0$ . ■

Increasing differences is interpreted as formalizing the notion of (Edgeworth) complementarity: Having more of one variable increases the marginal returns to having more of the other variable. It turns out that some form of complementarity between endogenous and exogenous variables lies at the heart of any monotone comparative statics conclusion.

A simplified version of Topkis's Monotonicity Theorem is now given. Though a special case of the original result, it is adequate for most applications. It is assumed throughout that  $F$  is continuous (or even just upper semi-continuous) in  $a$  for each  $s$ , so that the max in (1) is always attained. Furthermore, the correspondence  $a^*(s)$  then always admits maximal and minimal (single-valued) selections, denoted  $\bar{a}(s)$  and  $\underline{a}(s)$  respectively.

**Theorem 2** Consider Problem (3) with  $S, A \subset R$  and assume that

(i)  $F$  has increasing differences in  $(s, a)$ , and

(ii)  $A_s = [g(s), h(s)]$  where  $h, g : S \rightarrow R$  are increasing functions with  $g \leq h$ .

Then the maximal and minimal selections of  $a^*(s)$ ,  $\bar{a}(s)$  and  $\underline{a}(s)$ , are increasing functions.

Furthermore, if (i) is strict, then every selection of  $a^*(s)$  is increasing.

**Proof.** By way of contradiction, assume that  $\bar{a}(s)$  is not increasing, so that for some  $s' > s$ ,  $\bar{a}(s') < \bar{a}(s)$ . Then, using (ii) and the facts that  $\bar{a}(s) \in A_s$  and  $\bar{a}(s') \in A_{s'}$ , we have  $g(s) \leq g(s') \leq \bar{a}(s') < \bar{a}(s) \leq h(s) \leq h(s')$ , so that  $\bar{a}(s) \in A_{s'}$  and  $\bar{a}(s') \in A_s$ . Now, using the latter facts along with  $\bar{a}(s) \in a^*(s)$ , (i), and  $\bar{a}(s') \in a^*(s')$ , we have

$$0 \geq F[s', \bar{a}(s)] - F[s', \bar{a}(s')] \geq F[s, \bar{a}(s)] - F[s, \bar{a}(s')] \geq 0 \quad (4)$$

so equality holds throughout. Hence,  $\bar{a}(s) \in a^*(s')$ , a contradiction to the fact that  $\bar{a}(s') = \max\{a^*(s')\}$ , in view of  $\bar{a}(s') < \bar{a}(s)$ . Hence,  $\bar{a}(\cdot)$  is increasing. A similar proof holds for  $\underline{a}(\cdot)$ .

If (i) is strict, the same contradiction argument for any selection  $\tilde{a}(s)$  of  $a^*(\cdot)$  shows that (4) holds with a strict middle inequality, a contradiction (as  $0 > 0$ ). So  $\tilde{a}(\cdot)$  is increasing. ■

In this proof, the contradiction hypothesis, i.e.  $\bar{a}(s') < \bar{a}(s)$ , rules out the possibility that  $A_s \cap A_{s'} = \emptyset$ , or equivalently that  $\sup A_s < \inf A_{s'}$ . Indeed, if the latter were true, then one would necessarily have  $\bar{a}(s') > \bar{a}(s)$ , from feasibility alone.

To rephrase the result, in the one-dimensional case with smoothness, it is sufficient for monotone comparative statics that the objective satisfy  $\partial^2 F(s, a)/\partial a \partial s \geq 0$  and the constraint set be a compact interval whose end points increase in the parameter.

There is an order-dual to Topkis's Theorem<sup>5</sup>, giving obvious dual conditions under which an argmax is decreasing in a parameter.

**Theorem 3** *Consider Problem (3) with  $S, A \subset \mathbb{R}$  and assume that*

- (i)  $F$  has decreasing differences in  $(s, a)$ , and*
  - (ii)  $A_s = [g(s), h(s)]$  where  $h, g : S \rightarrow \mathbb{R}$  are decreasing functions with  $g \leq h$ .*
- Then the maximal and minimal selections of  $a^*(s)$  are decreasing functions. Furthermore, if (i) is strict, then every selection of  $a^*(s)$  is increasing.*

An alternative way to think of Theorem 3 is that it can be derived from Theorem 2 simply by considering the parameter to be  $-s$  instead of  $s$ . Thus  $F(s, a)$  has increasing differences in  $(s, a)$  if and only if  $F(s, a)$  has decreasing differences in  $(-s, a)$ . This connection will prove useful in applications below.

## 2.2 Comparing with Standard Comparative Statics

For comparative purposes, we now review the standard method for comparative statics taught in graduate economics classes. Consider Problem 1 and assume that  $F$  is smooth and the argmax is interior. Then the first-order condition is<sup>6</sup>  $F_2(s, a^*(s)) = 0$ . As this is an identity with respect to  $s$ , under the conditions of the Implicit Function Theorem (here strong concavity of  $F$  in  $a$  for each  $s$ , i.e.  $F_{22} < 0$ ), we can differentiate them with respect to  $s$ . Solving for  $a_1^*$ , one gets (since  $F_{22} < 0$  by the strong concavity of  $F$  in  $a$ )

$$a^{*'}(s) = -\frac{F_{21}(s, a^*(s))}{F_{22}(s, a^*(s))} \geq 0,$$

if we assume  $F_{21} \geq 0$  (i.e. increasing differences of  $F$  in  $(s, a)$ ).

Thus Topkis's theorem dispenses with the need for assumptions of concavity and smoothness of the objective function, interiority of the solution and convexity of the feasible set. In

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<sup>5</sup>Writing in the tradition of the operations research literature, Topkis (1978) actually considered the problem of *minimizing* an objective that has decreasing differences in the action and the parameter. Clearly, to go from his framework to the present one or vice-versa, one simply needs to multiply the objective by  $-1$ .

<sup>6</sup>Throughout, subscripts denoting partials with respect to the corresponding variable.

not imposing interiority of the solution, Topkis's theorem does require the constraint set to have the "ascendancy" property implicit in condition (ii) of Theorem 2. This condition guarantees that the boundaries of the constraint set are increasing functions of the parameter, so that an argmax that lies on parts of the boundary of the feasible set for some parameter values would clearly inherit the desired monotonicity property, even if it switches back and forth between the interior of the feasible set and its boundary. In a nutshell, Topkis's theorem is able to dispense with concavity requirements because if these fail locally, the argmax will be on the boundary, and hence increasing in the parameter by condition (ii).

The traditional comparative statics method only works when there is a unique argmax that varies smoothly with the parameter without ever straying against the constraint. As a result, traditional economic theory inherited an excessive tendency to impose assumptions of concavity of objective functions, Inada-type conditions, and convexity of feasible sets even in cases where these are not readily justifiable on economic grounds. Likewise, the development of theories where important nonconvexities are inherently present, due for instance to some form of increasing returns, was probably hampered or delayed. In other words, the scope and direction of economic research were probably negatively influenced by these important limitations of a purely methodological nature.

Another key advantage of this new method, not covered here, is that it allows the parameter set to be *any partially ordered set*, such as a set of integers<sup>7</sup>, or such abstract objects as a set of probability distributions partially ordered e.g. by first-order stochastic dominance<sup>8</sup>, or a set of sets ordered by inclusion... It also allows the decision set similar latitude.

## 2.3 Economic Applications

We now provide several simple illustrations of Topkis's Theorem drawn from various familiar settings in economics. While all of these examples demonstrate natural ways in which complementarity can arise and be exploited for comparative statics, some of the examples also serve to illustrate useful tricks in using this methodology. In particular, a monotone-

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<sup>7</sup>In deriving the comparative statics properties of Cournot equilibria, Amir and Lambson (2000) show that the simplification of treating the number of firms as a real number—common in the literature, e.g. Seade (1980) — can lead to misleading or erroneous results.

<sup>8</sup>Thus a comparative statics conclusion may well take the form that as social inequality increases (exogenously) according to some nonparametric measure of income distribution, then crime (endogenously) increases, or as an agent's market information improves in some nonparametric way, she invests more, etc...

transformed objective function may satisfy the conditions of Topkis's Theorem under some natural assumptions while obviously preserving the same optimal argmax. Also, the same machinery can be used to perform nonmonotone comparative statics upon a change of the decision variable. The most frequent change of variable is to consider minus the original variable, allowing one to go back and forth between increasing and decreasing outcomes. To check the conditions of the Theorem, the cross-partial test will always be used due to its convenience, although smoothness is actually *not needed* for most of the results presented.

**(a) Consumer theory.** Consider the familiar problem of a consumer maximizing utility  $U(x_1, x_2)$  from the consumption of two goods  $x_1$  and  $x_2$  at given respective prices  $p_1$  and  $p_2$  and income  $m$ . We wish to derive conditions under which  $x_1$  is a normal good, or the demand for  $x_1$  increases in  $m$ . The problem is then  $\max\{U(x_1, x_2) : p_1x_1 + p_2x_2 = m\}$ . If  $U$  is increasing in  $x_2$ , we can solve the constraint for  $x_2$  and plug it into the objective:

$$\max\{U(x_1, (m - p_1x_1)/p_2) : x_1 \in [0, m/p_1]\}. \quad (5)$$

The constraint set  $[0, m/p_1]$  clearly satisfies (ii) of Theorem 2, with  $g$  being the 0 function and  $h$  the function  $m/p_1$ . By Lemma 1, assuming  $U$  is smooth, the objective in (5) has increasing differences in  $(x_1, m)$  iff the familiar condition

$$p_2U_{21}(x_1, x_2) - p_1U_{22}(x_1, x_2) \geq 0, \forall x_1, x_2 \geq 0 \quad (6)$$

holds. It follows from Topkis's Theorem that (6) is sufficient to guarantee that  $x_1$  is a normal good. Observe that this conclusion survives if income is expressed in discrete units, or if demand  $x_1^*(m)$  switches from being interior to being on a boundary as  $m$  changes, or if demand is multi-valued for some income levels (the conclusion that  $x_1$  is a normal good then holds for the extremal selections of  $x_1^*(m)$ ). Furthermore, no concavity-type assumptions on  $U$  are needed! In fact, the only assumptions needed here are (6) and the fact that the utility function is increasing in the consumption of  $x_2$  (in solving the budget constraint).

**(b) Monopoly pass-through.** Consider a monopoly firm with constant unit cost  $c$  charging a price  $p \in [c, \infty)$  and facing a direct demand function  $D(\cdot)$ . The profit function is

$$\Pi(p, c) = (p - c)D(p), \quad p \in [c, \infty).$$

To verify increasing differences of the profit function directly, we have  $\partial^2\Pi(p, c)/\partial p\partial c = -D'(p)$ , which is  $\geq 0$  iff  $D'(p) \leq 0$ . Since the feasible set  $[c, \infty)$  has the form specified in

Theorem 2, assuming  $D'(p) \leq 0$  is sufficient for the conclusion that the extremal selections from the optimal price  $p^*$  are increasing in  $c$ .

We now present an alternative way to arrive at the same conclusion. As the optimal price is invariant under a monotone transformation, we may equivalently consider the objective

$$\log \Pi(p, c) = \log(p - c) + \log D(p) , \quad p \in [c, \infty).$$

As  $\partial^2 \log \Pi(p, c) / \partial p \partial c = (p - c)^{-2} \geq 0$ , the desired monotonicity conclusion follows. Observe here that  $D$  need not be decreasing in own price for this conclusion to hold!<sup>9</sup>

We now present interesting comparative statics results of a nonmonotone sort, simply by making use of a change of variable. Define mark-up (over marginal cost) as  $m \triangleq p - c$ , and write the equivalent objective with this change of variable as

$$\log \tilde{\Pi}(m, c) \triangleq \log(m) + \log D(m + c).$$

The latter objective has decreasing differences in  $(m, c)$  if  $D$  is log-concave (i.e.  $\log D$  is concave)<sup>10</sup>. Since the constraint set is  $m \in [0, \infty)$ , it follows from Topkis's Theorem that the maximal and minimal selections from the optimal mark-up  $m^*(c)$  are decreasing in  $c$ , or equivalently that the extremal selections of  $p^*$  have all their slopes  $\leq 1$  (since  $p^*(c) = m^*(c) + c$ ). Overall then,  $p^*$  has all its slopes in  $[0, 1]$  and is thus continuous and single-valued. Hence, there is always positive but partial pass-through.

If  $D$  is log-convex, a similar argument shows that  $\log \tilde{\Pi}(m, c)$  has increasing differences in  $(m, c)$ , so that the extremal selections of  $m^*(c)$  are increasing in  $c$ , or equivalently the extremal selections of  $p^*$  have all their slopes  $\geq 1$ , so that pass-through exceeds 100%<sup>11</sup>.

**(c) Assortative matching.** Becker (1973) considered a model of marriage with  $n$  women and  $n$  men looking to match to form  $n$  couples. Each of the sexes is ranked by productive ability according to the natural order in  $\{1, 2, \dots, n\}$ , with person 1 being the least productive and person  $n$  the most productive. If woman  $i$  and man  $j$  get married, they generate a surplus of  $f(i, j)$  as a couple. A matching is any list of  $n$  couples consisting of one woman and one

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<sup>9</sup>Observe here that taking monotone transformation of a non-supermodular objective function may bring about supermodularity of the transformed objective. We will say more on this point later.

<sup>10</sup>Indeed  $\partial^2 \log D(m + c) / \partial m \partial c = [DD'' - D'^2] / D^2 \geq 0$  iff  $DD'' - D'^2 \geq 0$ , i.e.  $D$  is log-concave.

<sup>11</sup>As examples, consider two of the most widely used demand functions. For linear demand (clearly log-concave), cost pass-through is 50%, and for constant-elasticity demand (log-convex), pass-through is  $\varepsilon(1 + \varepsilon)^{-1} > 1$ . The above results are a generalization of this illustration.

man each. The social planner's problem is to find a matching that maximizes social surplus, defined as the sum  $\sum f(i, j)$  over all couples in a given matching. Becker's main question is to derive conditions on  $f$  that lead to assortative matching, i.e. to the couples being  $(1, 1), (2, 2), \dots, (n, n)$ . He shows that if  $f$  has strictly increasing differences<sup>12</sup>, the optimal matching must be assortative. For otherwise, there would necessarily be two couples  $(i, j)$  and  $(i', j')$  with (say)  $i' > i$  but  $j' < j$  as part of the presumed matching, so that by increasing differences of  $f$ , there would hold  $f(i', j) + f(i, j') > f(i, j) + f(i', j')$ , which contradicts the optimality of the presumed matching, as it says that matching  $i'$  to  $j$  and  $i$  to  $j'$  leads to a higher surplus than  $i$  to  $j$  and  $i'$  to  $j'$ .

**(d)Growth theory.** Consider a two-period version of the standard Solow-Koopmans optimal growth model with possible increasing returns. Assume the utility function  $u$  is such that  $u' > 0$  and  $u'' < 0$ , the production function  $f$  is such that  $f' > 0$  (no restrictions on  $f''$ ) and the discount factor  $\delta \in (0, 1)$ . With  $y_t$  denoting savings, the planner's objective is

$$\max \sum_{t=1}^2 u(x_t - y_t) \quad \text{subject to} \quad x_{t+1} = f(y_t) \text{ and } y_t \in [0, x_t].$$

The two-period value function  $V_2$  satisfies

$$V_2(x) = \max\{u(x - y) + \delta u[f(y)] : y \in [0, x]\}$$

Here, the maximand need not be concave in  $y$ , so  $y^*(x)$  may be multi-valued. Since the maximand has increasing differences in  $(x, y)$  and the constraint set satisfies Condition (iii) of Theorem 2, it follows that the optimal savings correspondence  $y^*(x)$  is increasing in  $x$  (no restrictions are actually needed on  $f$ .)

For an infinite-horizon extension of this example, where the parameter set includes  $\delta$  and the length of the horizon, see Amir, Mirman and Perkins (1991) and Amir (1996).

Many more examples are given in the next section in one of the most important general uses of Topkis's Theorem: When showing the best response map in a supermodular game is increasing in the rivals' strategies (viewing the latter as a parameter.)

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<sup>12</sup>The interpretation of increasing differences in this context is the standard one: the productive abilities of the two members of the couple are complements in production, or that having a higher-ability mate increases a person's marginal productivity of own ability.

### 3 Supermodular Games

A fundamental contribution of the methodology at hand is that it gave rise to a new class of games, introduced by Topkis (1979) and further developed by Vives (1990) and Milgrom and Roberts (1990), which always have pure-strategy equilibrium points in addition to many other nice general properties. The importance of this class of games in economics is underscored by the fact that many standard models of oligopoly and macroeconomic coordination games are supermodular games under quite general –though not universal– conditions on their primitives. The class of supermodular games, also known as games with strategic complementarities, substantially enlarges the scope of application of noncooperative game theory in economics, particularly since it dispenses with the reliance on mixed-strategies to ensure existence of Nash equilibrium, randomized behavior still being a controversial feature of game theory for economists. Recall that the classical approach to existence of pure-strategy Nash equilibrium generally requires the quasi-concavity of each payoff function in own action<sup>13</sup>.

#### 3.1 The Class of Supermodular Games

An  $n$ -player normal-form game is given by a tuple  $(N, A_i, F_i)$  where  $A_i \subset R$ ,  $N = \{1, 2, \dots, n\}$  is the player set, and  $A_i$  and  $F_i : A_1 \times A_2 \times \dots \times A_n \rightarrow R$  are player  $i$ 's action set and payoff function. Such a game is supermodular if for every player,

- (i) the action set is a compact set.
- (ii) Each payoff function has increasing differences in own action and in each rival's action.<sup>14</sup>

The key property of supermodular games is an immediate consequence of Topkis's Theorem: For each player, the best-reply (or reaction) correspondence, mapping the rivals' action vector  $a^{-i}$  vector into the player's set of best-responses, has extremal selections that are increasing in each rival's action. Thus the overall best response mapping, taking the full action vector  $a$  into the set of best responses, also has extremal selections that are increasing (coordinatewise.) That either one of these selections has a fixed point –which is clearly a pure-strategy equilibrium of the game – is a direct consequence of Tarski's (1955) fixed point

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<sup>13</sup>Additionally, since existence is then established via the Brouwer-Kakutani fixed-point Theorem, the strategy spaces have to be convex, and the best-response correspondence upper hemi-continuous (with the latter requiring some form of continuity of the payoff in the actions.)

<sup>14</sup>Each payoff must also be u.s.c. in own action to ensure the best response of each player is well-defined (i.e. that the maximum is attained.)

theorem stated without proof next.

**Theorem 4** *Let  $A$  be a compact Euclidean interval and  $F : A \rightarrow A$  be increasing. Then the set of fixed-points of  $F$  is nonempty and has (coordinate-wise) largest and smallest elements.*

In the one-dimensional case, Tarski's Theorem is easily illustrated graphically, say for a map from  $[0, 1]$  to  $[0, 1]$ . Then, considering the mapping  $G(a) \triangleq F(a) - a$ , one has  $G(0) = F(0) \geq 0$  and  $G(1) = F(1) - 1 \leq 0$ . Clearly the graph of  $G$  can only have upward jumps, so for it to get from above to below the horizontal axis, it must necessarily cross it at some point  $\bar{a}$ , for which then  $G(\bar{a}) = \bar{a}$ .

An  $n$ -player submodular game is defined in a dual way, the only change being that "decreasing differences" replaces "increasing differences" in (ii). By Theorem 3, the reaction correspondences of a submodular game are downward-sloping<sup>15</sup>. As there is no version of Tarski's Theorem for downward-sloping mappings (a fact that is easy to see graphically), submodular games need not have pure-strategy equilibrium points. An interesting exception is the case is the two-player case ( $n = 2$ .) If, in a submodular game, we think of (say) player 2's action vector as being  $-a_2$  instead of  $a_2$ , then the game becomes a supermodular game since decreasing differences in  $(a_1, a_2)$  become increasing differences in  $(a_1, -a_2)$  as can be easily seen.<sup>16</sup>No such argument holds for  $n \geq 3$ .

### 3.2 Coincidence of Solution Concepts

For any normal-form game, let  $U \subset A_1 \times A_2 \times \dots A_n$  denote the set of strategies that survive iterated deletion of strictly dominated strategies. It is well-known that that only pure-strategies in  $U$  can have positive mass at any mixed-strategy or correlated-strategy equilibrium (see e.g. Fudenberg and Tirole, 1991.)

Since the best-response map of a supermodular game is increasing, it is easy to see that there exist (coordinate-wise) largest and smallest Nash equilibria. Denote these by  $\bar{a}$  and  $\underline{a}$ . The following is a simplified version of a key result in Milgrom and Roberts (1990b):

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<sup>15</sup>Thus supermodular games and submodular games correspond to games with strategic complementarity and substitutability, respectively, in the language of Bulow, Geanakoplos and Klemperer (1985).

<sup>16</sup>An alternative argument, due to Vives (1990), is as follows. For a submodular game, the extremal reaction curves  $r_1$  and  $r_2$  are both decreasing, so the composition  $r_1 \circ r_2 : A_1 \rightarrow A_1$  is increasing. By Tarski's theorem, the latter has a fixed point,  $a_1^*$ . Then  $(a_1^*, r_2(a_1^*))$  is easily seen to be a Nash equilibrium.

**Theorem 5** *For any supermodular game,  $\sup U = \bar{a}$  and  $\inf U = \underline{a}$ .*

**Proof.** Let  $\mathbf{0}$  and  $\mathbf{1}$  stand for the smallest and the largest action vectors of all player 1's (say) rivals. We first show for player 1 that the largest best-reply to  $\mathbf{1}$ , denoted  $r_1(\mathbf{1})$ , strictly dominates all larger actions. To this end, by increasing differences of  $F_1$ , for any action vector  $a_{-1}$  of the rivals and any  $a_1 > r_1(\mathbf{1})$ , we have

$$F_1(a_1, a_{-1}) - F_1(r_1(\mathbf{1}), a_{-1}) \leq F_1(a_1, \mathbf{1}) - F_1(r_1(\mathbf{1}), \mathbf{1}) < 0$$

since  $r_1(\mathbf{1})$  is the largest best reply to  $\mathbf{1}$ . Hence,  $r_1(\mathbf{1})$  strictly dominates all actions  $a_1 > r_1(\mathbf{1})$ .

A similar argument shows that  $r_1(\mathbf{0})$ , defined as the smallest best reply to  $\mathbf{0}$ , strictly dominates all actions  $a_1 < r_1(\mathbf{0})$ .

Now, the first step of the iteration is to eliminate all player 1's actions  $> r_1(\mathbf{1})$  or smaller than  $r_1(\mathbf{0})$ . Repeat this procedure for each other player, and then again starting with player 1, and so on. As shown on Figure 1, this process will clearly converge to  $[\underline{a}, \bar{a}]$ . ■

Thus,  $\underline{a}$  and  $\bar{a}$  form the same bounds on most solution concepts for normal-form games, including pure, mixed or correlated-strategy equilibrium. When there is a unique pure-strategy equilibrium, more can be said.

**Corollary 6** *A supermodular game with a unique pure-strategy equilibrium,  $a^*$ , is dominance-solvable, and  $a^*$  is also the unique mixed-strategy equilibrium and correlated equilibrium.*

**Proof.** This follows from Theorem 5, since  $\underline{a} = \bar{a} = a^* = U$  here, along with the fact that action profiles outside of  $U$  cannot be part of a mixed or correlated equilibrium. ■

The intuition behind this result can be captured by a simple graphical illustration in the case of two-player games. In Figure 1(a), one round of deletion of strategies that are not best-replies to any strategy of the rival is shown. Continuing this process, starting at each round from the reduced interval of actions left from the previous round, would clearly converge to the unique equilibrium. In Figure 1(b), in view of the multiplicity of pure-strategy equilibria, the same process will clearly converge to the rectangle  $[\underline{a}, \bar{a}]$ .

### 3.3 Comparative Statics of Equilibrium Points

To determine the direction of change of an equilibrium point as an exogenous parameter changes, Topkis's Theorem is not a priori applicable. Consider a parametric family of games

with Player  $i$ 's payoff given by  $F_i(a^i, a^{-i}, s)$  where  $s \in S$  is a parameter. Milgrom and Roberts (1990b) prove the following very useful result in applying the theory of supermodular games:

**Theorem 7** *Assume that*

(i) *for each  $s \in S \subset R$ , the game is supermodular, and*

(ii)  *$F_i$  has increasing differences in  $(a^i, s)$  for each  $a^{-i}$ .*

*Then the extremal equilibria of the game are increasing functions of  $s$ .*

**Proof.** Let  $B_s(\cdot) : A \rightarrow A$  denote the best reply correspondence when the parameter is  $s$ . For each  $s \in S$ , there are smallest and largest Nash equilibria, by (i). Let  $s_1 > s_2$  and  $\bar{a}(s)$  denote the largest Nash equilibrium of the game with parameter  $s$ ,  $s \in S$ . For each  $i$ , player  $i$ 's reaction correspondence  $r_i^s(a^{-i})$  shifts out as  $s$  increases, by (ii) and Topkis's Theorem. Hence,  $B_s(\cdot)$  also shifts out as  $s$  increases. Since  $\bar{a}(s_2)$  is the largest fixed point of  $B(s_2)$  and since  $B_{s_1}(\cdot) \geq B_{s_2}(\cdot)$ ,  $B_{s_1}(\cdot)$  maps  $[\bar{a}(s_2), 1_A]$  into itself. By Tarski's Theorem, the latter restricted map has a fixed point in  $[\bar{a}(s_2), 1_A]$ , which is then clearly  $\geq \bar{a}(s_2)$ .

A similar argument works for the smallest equilibrium  $\underline{a}(s)$ . ■

Thus for a supermodular game with smooth payoffs, the extremal equilibria increase in the parameter if  $\partial^2 F_i / \partial a^i \partial s \geq 0$  for all  $i$ . The intuition behind this result is nicely captured by Figure 2. Indeed, from Topkis's Theorem applied at fixed  $a^{-i}$ , the effect of Assumption (ii) is precisely that player  $i$ 's reaction correspondence shifts up as  $s$  increases. It follows that the maximal and minimal equilibrium points both increase, while the middle equilibrium moves in the opposite direction<sup>17</sup>. The overall increase in the (extremal) equilibrium actions for each player can intuitively be regarded as consisting of two effects. The direct effect is the positive response of the player's action to the increase in the parameter, or the shift in his reaction correspondence. The indirect effect is the positive reaction of the player to the direct-effect increases of the other players's actions.

This theorem has natural applications in a variety of settings in economic theory, some of which are explored below. It confirms that strategic complementarity of a game is a natural property that is easily compatible with clear-cut comparative statics properties.

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<sup>17</sup>Under multiple equilibria, it has long been known that the comparative statics properties of equilibria are intimately linked to their stability properties (in terms of Cournot best-reply dynamics.) This is Samuelson's (1947, 1974) so-called Correspondence Principle. The connection with Theorem 5 is that the extremal equilibria of a supermodular game are always stable. See Echenique (2002) for a recent thorough study.

By contrast, a game with a continuous (single-valued) best-response mapping but without strategic complementarities need not possess clear-cut monotone comparative statics properties, as is easily seen graphically. In other words, outward shifts in the continuous reaction curves need not imply higher equilibrium actions, unless the game is symmetric. Observe that the requisite fact that the reaction curves shift out as the parameter increases is a consequence of the fact that each payoff has increasing differences in own action and in the parameter. Thus supermodularity plays a role in comparing equilibrium points even within the classical paradigm. Generally, with continuous reaction curves, the equilibrium will satisfy continuity in the parameter, a property that guarantees a sense of stability in that equilibria will not vary drastically for small parameter variations. However, in most cases in economics, one is actually interested in qualitative comparative statics and wishes to determine a monotone direction of response of endogenous variables to variations in the exogenous variables. In most exercises of comparative equilibrium points in applications, supermodularity is most often (at least tacitly) behind the analysis.

There is no dual version of Theorem 6 for submodular games, or games with strategic substitutes, even in the two-player case. Indeed, if one only replaces "supermodular" by "submodular" in Theorem 6, the analogous reasoning would tell us that the (then downward-sloping) reaction correspondences also shift outwards as the parameter increases, but here it need not follow that the equilibrium actions will both increase, as can easily be seen from the analogous picture. It all depends on the relative magnitudes of the shifts of the two curves. In this case, a similar intuitive break-up of the overall response into a direct and an indirect effect will reveal that the two effects are conflicting, with the direct effect being the same as the above case and the indirect effect amounting to a decrease in the player's action. The overall effect then depends on the relative strengths of these two effects, whence the ambiguous final outcome.

### 3.4 Other Important Properties

Supermodular games possess other properties of interest, which have not been addressed here. An important property of supermodular games is that general learning schemes, so-called adaptive learning schemes<sup>18</sup>, converge to the interval bounded by the extremal pure-strategy

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<sup>18</sup>This is a very broad class of learning schemes that includes the well-known special cases of Cournot best-reply dynamics and fictitious play.

equilibria, (Topkis, 1979, Vives, 1990 and Milgrom and Roberts, 1990). This property has also been used by Topkis (1979) to devise algorithms for computing equilibria of supermodular games. Echenique (2001b) shows that a supermodular game is also supermodular in mixed strategies when these are ordered with first-order stochastic dominance. Echenique and Edlin (2001) establish that mixed-strategy equilibria of supermodular games are unstable with respect to a large class of dynamics (as in coordination games such as the Battle of the Sexes). For Bayesian games, such as some classes of auctions, an extension of the property of strategic complementarity is quite natural (Vives 1990 and Athey 2001). Finally, Echenique (2001a) shows that the class of supermodular games can be substantially enlarged if the partial order on the strategy sets is endogenous, as opposed to allowing only exogenous orders with some natural interpretation.

### 3.5 Examples of Supermodular Games

We now provide several examples of well-known models in economic theory that are supermodular games under natural conditions.

**(a) Coordination games.** The standard  $2 \times 2$  Battle of the Sexes given below is a supermodular game since  $2 + 1 > 0 + 0$ . More generally, large classes of coordination-type games are typically supermodular games.

0, 0	2, 1
1, 2	0, 0

**(b) Search.** In Diamond's (1980) model, agent  $i$  expands effort  $a^i \in [0, 1]$  searching for trading partners, and has a payoff function given by (with  $s > 0$  being a parameter characterizing the search environment):

$$F_i(a^i, a^{-i}) = sa^i \sum_{j \neq i} a^j - C_i(a^i).$$

Since  $\partial^2 F_i(a^i, a^{-i}) / \partial a^i \partial a^j = s > 0$  for all  $i \neq j$ , the game is supermodular, for any cost functions. Furthermore, since  $\partial^2 F_i / \partial a^i \partial s \geq 0$ , it follows from Theorem 5 that the largest and the smallest equilibria are increasing in  $s$  (which is a measure of the ease of search.)

**(c) Bertrand oligopoly.** In price competition with differentiated products and linear costs, firm  $i$ 's profit function, when it charges price  $p^i$  and its rivals' price vector is  $p^{-i}$ , is

$$F_i(p^i, p^{-i}) = (p^i - c_i)D_i(p^i, p^{-i}),$$

where  $c_i$  is its unit cost and  $D_i$  its demand function<sup>19</sup>. We can clearly restrict consideration to the price set  $[c_i, \infty)$  for firm  $i$  since prices in  $[0, c_i)$  are dominated strategies. Since monotone transformations leave the best-reply structure unchanged, and

$$\log F_i(p^i, p^{-i}) = \log(p^i - c_i) + \log D_i(p^i, p^{-i}),$$

it follows (Milgrom and Roberts, 1990b) that the game is log-supermodular if  $\log D_i$  has increasing differences in  $(p^i, p^{-i})$  or equivalently, using the cross-partial test, if for all  $j \neq i$ ,

$$D_i(p^i, p^j) \frac{\partial^2 D_i(p^i, p^j)}{\partial p^i \partial p^j} - \frac{\partial D_i(p^i, p^j)}{\partial p^j} \frac{\partial D_i(p^i, p^j)}{\partial p^i} \geq 0.$$

The latter property has an intuitive and precise interpretation: firm  $i$ 's price elasticity of demand is increasing in the rivals' prices. Most demand functions used in industrial economics satisfy this condition. Examples are provided in [Vives 2000 pp. 151-154] and Milgrom and Roberts (1990b), who show that they lead to a unique pure-strategy equilibrium, so that by Corollary 6, that the Bertrand oligopoly is dominance-solvable.

Next, we illustrate the application of the comparative-equilibria result. Since  $\log(p^i - c_i)$  clearly has increasing differences<sup>20</sup> in  $(p^i, c)$  where  $c$  is the vector of unit costs, it follows from Theorem 6 that if  $\log D_i$  has increasing differences, the largest and smallest equilibrium prices are nondecreasing functions of  $c$ , i.e. that higher costs from *any subset* of the firms result in all firms' prices being higher, and are thus always passed on to consumers<sup>21</sup>.

**(d) Cournot duopoly.** For homogeneous-good quantity competition, if firm  $i$  produces output level  $q_i$  at a cost  $C_i(q_i)$  and  $P$  is the inverse demand function, firm  $i$  has profit

$$F_i(q_1, q_2) = q_i P(q_1 + q_2) - C_i(q_i).$$

Since with  $P' \leq 0$ , we have

$$\partial^2 F_i(q_1, q_2) / \partial q_1 \partial q_2 = P'(q_1 + q_2) + q_i P''(q_1 + q_2) \leq 0 \text{ for all } q_1, q_2 \geq 0$$

if and only if

$$P'(z) + zP''(z) \leq 0 \text{ for all } z \geq 0,$$

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<sup>19</sup>It is assumed that firms satisfy whatever demand is forthcoming, i.e that there is no rationing.

<sup>20</sup>Indeed,  $\partial^2 F_i(p) / \partial p^i \partial c_i = \partial^2 \log(p^i - c_i) / \partial p^i \partial c_i = 1 / (p^i - c_i)^2 \geq 0$ , and  $\partial^2 F_i(p) / \partial p^i \partial c_j = 0$  for  $j \neq i$ .

<sup>21</sup>A firm whose unit cost increases will increase its price via the direct effect of responding to its own cost increase and via the indirect effect of responding to rivals' cost increases. A firm whose unit cost remains constant will still increase its price but only via the latter effect.

it follows that the game is submodular if the latter condition holds (see Novshek, 1985 and Amir, 1996). This conclusion is easily seen to be valid even in the  $n$ -firm case, for all  $n^{22}$ .

Furthermore, for  $n = 2$ , if one thinks of (say) firm 2's decision as being  $-q_2$  instead of  $q_2$ , then under the above condition on demand, we clearly have  $\partial^2 F_i(q_1, q_2)/\partial q_1 \partial(-q_2) \geq 0, i = 1, 2$ , so that the Cournot duopoly is then a supermodular game. This "order-reversing" trick works for any two-player submodular game, but does not work for three or more firms, i.e. in general, a submodular  $n$ -player game is also a supermodular game if and only if  $n = 2$ .

**(f) R&D competition.** Consider the following variant, due to Amir and Wooders (1999, 2000), of the standard two-stage game of R&D with one-way spillovers and product market competition. The latter is modelled in reduced-form by assuming a unique equilibrium in the second-stage with equilibrium payoff to firm 1 (say) given by  $\Pi(c_1, c_2)$ , where  $c_1$  and  $c_2$  are the post-R&D constant unit costs for firms 1 and 2. Denoting their first-period decisions, their R&D cost reductions, by  $x_1$  and  $x_2$ , their post-spillover cost reductions are respectively (assuming w.l.o.g. that  $x_1 \geq x_2$ ):  $X_1 = x_1$ , and  $X_2 = x_1$  with probability  $\beta$  and  $x_2$  with probability  $(1 - \beta)$ . Thus  $\beta$  here is the probability that a full spillover takes place, only from the firm ahead in R&D to the laggard. With  $c$  and  $f$  denoting the common initial unit cost and the R&D cost function respectively, firm 1's pre-spillover expected payoff is

$$\begin{cases} \beta\Pi(c - x_1, c - x_1) + (1 - \beta)\Pi(c - x_1, c - x_2) - f(x_1), & \text{if } x_1 \geq x_2 \\ \beta\Pi(c - x_2, c - x_2) + (1 - \beta)\Pi(c - x_1, c - x_2) - f(x_1), & \text{if } x_1 \leq x_2. \end{cases}$$

Assuming  $\Pi$  is strictly submodular,  $\Pi_1 < 0, \Pi_2 > 0$ , and  $\Pi(x, x)$  is decreasing<sup>23</sup> in  $x$ , Amir and Wooders (2000) show that the above payoff is submodular<sup>24</sup> in  $(x_1, x_2)$ , so that the last paragraph of Part (d) applies to the present *duopoly* as well. They also prove that (i) this game always has asymmetric equilibria, but no symmetric equilibria (though the firms are ex-ante identical)<sup>25</sup>, and (ii) although firm  $i$ 's profit is submodular in  $(x_i, \beta)$ , so that optimal

<sup>22</sup>Exploiting in a crucial way the special structure of the Cournot model, that a firm's profit only depends on its output and on the aggregate output of the rivals, Novshek (1985) showed that a submodular Cournot oligopoly always has a pure-strategy Cournot equilibrium, even though the game is then not supermodular.

<sup>23</sup>These three assumptions say respectively that a firm's equilibrium profit from the product market decreases with own cost, increases with rival's cost, as well as with a common cost increase.

<sup>24</sup>Observe that the convenient cross partial test cannot be used here as the payoff has a kink along the diagonal  $x_1 = x_2$ . So one must use the definition (2).

<sup>25</sup>The arguments used in this model constitute a general approach for constructing strategic games with a priori identical players but only asymmetric equilibria, i.e. models with strategic endogenous heterogeneity.

reactions shift inwards as  $\beta$  increases, the equilibrium R&D levels need not both decrease in  $\beta$  (see example in Amir and Wooders 1999).

## 4 Ordinal Complementarity Conditions

The properties of supermodularity and increasing differences are clearly of a cardinal nature. They need not be preserved by monotone transformations of the objective function. Yet the monotonicity of the argmax clearly survives such transformations, a property whose usefulness has already been illustrated in our applications above. For instance, in the monopoly example of Section 2, it was shown that while increasing differences of profits requires the assumption that demand be nondecreasing in price, increasing differences of log-profits holds without any assumption. This illustrates the possibility that applying a judicious monotone transformation to an objective function may change the comparative statics analysis. The question then naturally arises as to whether an ordinal version of the property of increasing differences could be developed (that would be invariant under monotone transformations.)

### 4.1 The Milgrom-Shannon Theorem

Milgrom and Shannon (1994) provided an affirmative answer to the above question, and proposed the following property as ordinal analog of increasing differences:

A function  $F : S \times A \rightarrow R$  has the single-crossing property in  $(s; a)$  if  $\forall a > a', s > s'$ ,

$$F(s', a) - F(s', a') \geq 0 \Rightarrow F(s, a) - F(s, a') \geq 0 \text{ and}$$

$$F(s', a) - F(s', a') > 0 \Rightarrow F(s, a) - F(s, a') > 0.$$

The single-crossing property is strict if  $F(s', a) - F(s', a') \geq 0 \Rightarrow F(s, a) - F(s, a') > 0$ .

Unlike the property of increasing differences, the single-crossing property discriminates between the two variables (action and parameter). In other words, satisfying the single-crossing property in  $(s; a)$  need not imply the same property in  $(a; s)$ .

It is straightforward to verify, directly from the definitions, that

(i) the single-crossing property is more general than increasing differences, in that a function that satisfies the latter necessarily satisfies the former.

(ii) if  $F$  satisfies the single-crossing property, then for any strictly increasing transformation  $g$ ,  $g \circ F$  will also satisfy the single-crossing property (so that this is an ordinal property).

The economic interpretation of this ordinal property is natural, but in terms of a limited form of complementarity. Thus, the single-crossing property says that if a given increase in the decision variable is profitable when the parameter is low, the same increase will continue to be profitable when the parameter is high. Unlike increasing differences though, it does not require the latter profitability level to exceed the former.

An important generalization<sup>26</sup> of Topkis's result is due to Milgrom and Shannon (1994).

**Theorem 8** *Consider Problem (3) with  $S, A \subset R$  and assume that*

*(i)  $F$  has the single-crossing property in  $(s; a)$ , and*

*(ii)  $A_s = [g(s), h(s)]$  where  $h, g : S \rightarrow R$  are increasing functions with  $g \leq h$ .*

*Then the maximal and minimal selections of  $a^*(s), \bar{a}(s)$  and  $\underline{a}(s)$ , are increasing functions.*

*Furthermore, if (i) is strict, then every selection of  $a^*(s)$  is increasing.*

**Proof.** Repeat the contradiction argument of Theorem 2 up to (3), replacing it by

$$F[s, \bar{a}(s)] - F[s, \bar{a}(s')] \geq 0 \Rightarrow F[s', \bar{a}(s)] - F[s', \bar{a}(s')] \geq 0$$

by the single-crossing property. This implies that  $\bar{a}(s) \in a^*(s')$ . Together with the contradiction hypothesis  $\bar{a}(s') < \bar{a}(s)$ , this is a contradiction to  $\bar{a}(\cdot)$  being the maximal selection of  $a^*(s)$ . A similar change works for the case of strict single-crossing property. ■

While the scope of this theory of monotone comparative statics is considerably enlarged by this result, as will be argued below via numerous examples, new difficulties appear with the verification of the underlying ordinal conditions in practical applications of the result. In particular, no differential characterization of these properties is available for smooth functions. Besides using the definition directly, which is often quite involved even for relatively simple problems, Milgrom and Shannon (1994) developed the following relatively practical test based on the well-known Spence-Mirrlees condition (defined by (7) below.)

**Theorem 9** *Let  $F : R^3 \rightarrow R$  be continuously differentiable and  $F_2(a, b, s) \neq 0$ .  $F(a, h(a), s)$  satisfies the single-crossing property in  $(a; s)$  for all functions  $h : R \rightarrow R$  if and only if*

$$F_1(a, b, s) / |F_2(a, b, s)| \text{ is increasing in } s. \tag{7}$$

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<sup>26</sup>The version we state here is a special case of the original theorem, which captures its essential aspects, in particular as far as economic applications are concerned.

The proof is omitted (not insightful). This test is used in applications by verifying (7) and concluding that the objective  $F(a, h(a), s)$  satisfies the single-crossing property in  $(a; s)$  for a suitable choice of function  $h$  (often the identity function, see examples below.)

## 4.2 Some Economic Applications

In order to facilitate comparison with the cardinal complementarity conditions, we derive assumptions on the primitives of some of the models from our previous examples that imply that the ordinal complementarity conditions hold. These examples confirm indeed that the scope of the theory has been significantly expanded by the ordinal complementarity notion. The scope of games with strategic complementarity has also be broadened since Theorems 5-7 continue to hold if supermodularity is replaced by the single-crossing property<sup>27</sup>.

**(a) Cournot duopoly.** The payoff to firm 1 (say) is:  $\Pi_1(q_1, q_2) = q_1P(q_1 + q_2) - C_1(q_1)$ . To show that firm 1's reaction correspondence is downward-sloping, let  $F(a, b, s) = bP(a + s) - C(b)$ . Then  $F_1(a, b, s) = bP'(a + s)$ ,  $F_2(a, b, s) = P(a + s) - C'(a)$  and

$$\frac{\partial F_1(a, b, s) / |F_2(a, b, s)|}{\partial s} = b \frac{[P(a + s) - C'(b)]P''(a + s) - P'^2(a + s)}{[P(a + s) - C'(b)]^2}.$$

Then (7) holds here if  $[P(b + s) - C'(a)]P''(b + s) - P'^2(b + s) \leq 0$ , for which a sufficient condition is  $P(\cdot)P''(\cdot) - P'^2(\cdot) \leq 0$ , or the log-concavity of  $P(\cdot)$ . Taking  $h$  as the identity function in Theorem 9, we conclude that  $F(a, a, s) = aP(a + s) - C(a)$  has the single-crossing property in  $(a; s)$ , or that  $\Pi_i(q_1, q_2)$  has the single-crossing property in  $(q_1; q_2)$ . So the Cournot oligopoly is a game of strategic substitutes under very general conditions. Indeed any concave demand function is log-concave. Not being too convex is all that log-concavity requires. The limit case is  $P(q) = e^{-q}$ , which is convex and log-linear.

Amir (1996b) shows the same result using the definition of the single-crossing property directly. The proof is more involved but the result is a bit more general since the regularity conditions of Theorem 9 are not needed (see Appendix for the proof.)

A partial dual is also of interest: If the cost function is identically zero, then firm  $i$ 's profit function satisfies the strict single-crossing property in  $(q_i, q_{-i})$  if  $P$  is log-convex, in which case the Cournot oligopoly is a game of strategic *complements*! More precisely, profits are then easily seen to be log-supermodular since log-convexity of  $P(\cdot)$  is equivalent to the

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<sup>27</sup>One property of such games that does not extend to the ordinal realm is the fact that such games also have strategic complementarity in mixed strategies (Echenique, 2001b.)

log-supermodularity of  $P(q_i + q_{-i})$ , and  $\log F_i(q_i, -q_{-i}) = \log q_i + \log P(q_i + q_{-i})$ , with the first term being trivially supermodular. Hence, the  $n$ -firm oligopoly is a quasi-supermodular game (with the original or natural output sets.) Log-convexity requires a strong form of convexity, the limit case being also  $P(q) = e^{-q}$ , and the typical example being the hyperbolic demand, e.g.  $P(q) = 1/q^\alpha$ . For details, see Amir (1996b).

**(b) Monopoly pass-through.** As seen in Section 2,  $\Pi$  has increasing differences if  $D' \leq 0$  while  $\log \Pi$  always has increasing differences. Since  $\exp$  is a monotone transformation and  $\exp(\log)$  is the identity function, it follows that  $\Pi$  also satisfies the single-crossing property for all  $D$  functions, since the latter property is preserved by monotone transformations.

**(c) Consumer theory.** Consider the problem in 2.3.(a). To ensure, using Theorem 9, that the objective in (5) satisfies the single-crossing property in  $(x_1, m)$ , so that good 1 is a normal good, let  $F(a, b, s) = U(b, (m - p_1 a)/p_2)$ . Then  $F_1(a, b, s) = (-p_1/p_2)U_1(b, (m - p_1 a)/p_2)$ ,  $F_2(a, b, s) = U_2(b, (m - p_1 a)/p_2)$ , and (7) holds here if and only if

$$U_2 U_{21} - U_1 U_{22} \geq 0, \forall x_1, x_2 \geq 0, \quad (8)$$

or equivalently, if the ratio of marginal utilities  $U_1(x_1, x_2)/U_2(x_1, x_2)$  is increasing<sup>28</sup> in  $x_2$ . From Theorem 9, with  $h$  as the identity function, we conclude that good 1 is normal.

As a further comparative illustration, consider the specific example where  $U(x_1, x_2) = x_1^2 x_2^2$ . Evaluating the various partials, we have  $U_1 = 2x_1 x_2^2$ ,  $U_2 = 2x_1^2 x_2$ ,  $U_{21} = 4x_1 x_2$ , and  $U_{22} = 2x_1^2$ . It is easily seen by inspection that  $U$  is strictly quasi-concave in  $(x_1, x_2)$ , though convex in each variable separately. Using the sufficient condition (6), we can conclude that  $x_1$  is a normal good if  $p_2 U_{21} - p_1 U_{22} = 4x_1 x_2 p_2 - 2x_1^2 p_1 \geq 0$  or if  $2x_2 p_2 - x_1 p_1 \geq 0$ , which clearly fails to hold globally (e.g. for sufficiently small values of  $x_1$ ). Hence, we cannot conclude from (6) that  $x_1$  is a normal good at all price vectors.

However, using condition (8), we have  $U_2 U_{21} - U_1 U_{22} = 8x_1^3 x_2^2 - 4x_1^3 x_2^2 = 4x_1^3 x_2^2 \geq 0$ , so that we can directly conclude that  $x_1$  is a normal good at all price vectors. To verify, an elementary computation yields the familiar Cobb-Douglas demand functions  $x_1^* = m/2p_1$  and  $x_2^* = m/2p_2$ , so that both goods are indeed normal.<sup>29</sup>

<sup>28</sup>Since the first-order condition here is the familiar  $p_1/p_2 = U_1/U_2$ , one can obtain the sufficient condition for ordinal complementarity from its cardinal analog by substituting the first-order relation into the latter. While this connection fails to hold as a general result, it does provide a good intuitive and informal way of relating the two sets of conditions.

<sup>29</sup>It is important to observe that while (6) is necessary and sufficient conditions for cardinal complemen-

**(d) Bertrand competition.** Consider the model of price competition of 3.5(c), but with each firm's cost function being convex. Milgrom and Shannon (1994) show that firm  $i$ 's profit function then satisfies the single-crossing property in  $(p_i, p_{-i})$ . A sense of the difficulty in establishing the single-crossing property can be gained by looking at this proof.

**(e) Nonrenewable Resource extraction.** Consider the two-period problem of resource extraction with stock-dependent utility, no discount and with the notation of 2.3(d):

$$\max \sum_{t=1}^2 \log[x_t(x_t - y_t) + 1] \quad \text{subject to} \quad x_{t+1} = y_t \text{ and } 0 \leq y_t \leq x_t.$$

Since the one-period value function is clearly  $V_1(x) = \log(y^2 + 1)$ , the two-period value function  $V_2$  satisfies

$$V_2(x) = \max\{\log[x(x - y) + 1] + \log(y^2 + 1) : y \in [0, x]\} \quad (9)$$

The optimal savings correspondence is (for details, see Amir 1996, p. 138):

$$y^*(x) = \begin{cases} \{0, 1\} & , \text{ if } 0 \leq x \leq 1, \\ \frac{1}{3}[x + \frac{1}{x} + \frac{1}{x}(x^4 - x^2 + 1)^{1/2}] & , \text{ if } x \geq 1. \end{cases}$$

The maximal and minimal selections of  $y^*(x)$  are both increasing in  $x$ . Yet, the objective in (9) is *submodular* in  $(x, y)$  if  $x \leq 1$ ! Indeed, the cross partial of the objective is equal to  $x(x - 1)/[x(x - y) + 1]^2$  which is  $\leq 0$  if and only if  $x \leq 1$ . So the objective does not satisfy the single-crossing property for  $x \leq 1$ . The feasible set  $[0, x]$  satisfies our condition.

This example establishes in particular that the assumptions in Milgrom-Shannon's Theorem are not necessary for the monotonicity conclusion. This is an important point to clarify since the Milgrom-Shannon Theorem is actually stated with necessary and sufficient conditions for monotonicity of the argmax with respect to both an ordinary parameter (such as  $s$  here) and the constraint set itself. Thus this example shows nicely that the conditions of the Theorem are not necessary if monotonicity is only with respect to the parameter, as in the present treatment, given a constraint set satisfying Theorem 2 (ii).

There are other studies, mostly in industrial organization, in addition to those already mentioned, which make use of the theory of supermodularity. These include Bagwell and tarity, (8) is sufficient, but not necessary, for ordinal complementarity. Either condition may hold while the other fails to hold for particular examples. In other words, we can also provide the reverse illustration to the one just provided. For Bertrand oligopoly, this is done in Amir and Grilo (2000).

Ramey (1994), Friedman and Simon (1997), Athey and Schmutzler (2001), Peitz (2000). The analysis of endogenous timing in two-player games according to a natural scheme proposed by Hamilton and Slutsky (1990) is also best approached with the tools of supermodularity (see e.g. Amir and Grilo, 1999). More applications are given in Milgrom and Shannon (1994), Shannon (1995), Topkis (1998) and Vives (2000).

## 5 The Multi-Dimensional Case

Our simplified treatment is extended here to the multi-dimensional case. This presentation still falls short of the full generality of the theory. We begin with the requisite preliminaries. Let  $R^n$  denote  $n$ -dimensional Euclidean space. Given  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$  in  $R^n$ , denote by  $x \vee y$  and  $x \wedge y$  the coordinate-wise sup and inf of  $x$  and  $y$ ,

$$x \vee y = (\max\{x_1, y_1\}, \dots, \max\{x_n, y_n\}), \text{ and } x \wedge y = (\min\{x_1, y_1\}, \dots, \min\{x_n, y_n\}).$$

The inequality  $x \geq y$  means  $x_i \geq y_i$  for all  $i$  (coordinate-wise order), while  $x > y$  means  $x \geq y$  and  $x_i > y_i$  for at least one  $i$ . A rectangle in  $R^n$  is a cartesian product of  $n$  real intervals. Thus a compact rectangle in  $R^n$  is any set of the form  $X_{i=1}^{i=n}[a_i, b_i]$  for some reals  $a_i, b_i$ . Let  $S$  (parameter set) and  $A$  (action set) be rectangles in  $R^l$  and  $R^m$ , respectively.

### 5.1 Cardinal Complementarity Conditions

This part is based on Topkis (1978). A function  $F : A \rightarrow R$  is (strictly) supermodular if

$$F(a \wedge a') + F(a \vee a')(>) \geq F(a) + F(a'), \forall a, a' \in A. \quad (10)$$

If  $F$  is smooth,  $F$  is supermodular iff  $\partial^2 F(a)/\partial a_i \partial a_j \geq 0, \forall i \neq j$ , i.e. iff all the *non-diagonal* terms of the Hessian matrix of  $F$  are positive (no restrictions on the diagonal terms.)

A function  $F : S \times A \rightarrow R$  has (strictly) increasing differences in  $(s, a)$  if

$$F(s, a) - F(s, a')(>) \geq F(s', a) - F(s', a'), \forall a > a', s > s' \quad (11)$$

or in other words if the difference  $F(\cdot, a) - F(\cdot, a')$  is an increasing function. If  $F$  is smooth, (2) is equivalent to  $\partial^2 F(a)/\partial a_i \partial s_j \geq 0$ , for all  $i, j$ . Note that there are no restrictions here on partials of the form  $\partial^2 F(a)/\partial a_i \partial a_j$  or  $\partial^2 F(a)/\partial s_i \partial s_j$ .

For the scalar case ( $l = m = 1$ ) treated earlier, supermodularity and increasing differences are easily seen to be equivalent for  $f$ , as both are then characterized by  $\partial^2 F(s, a)/\partial a \partial s \geq 0$ . Supermodularity of a function  $f$  of  $n$  variables is equivalent to increasing differences or supermodularity of  $F$  with respect to all pairs of variables. This is a simple test, as it reduces to checking all pairwise relations only. Thus the complementarity interpretation of supermodularity and increasing differences extend to the multi-dimensional case.

A simplified multi-dimensional version of Topkis's Monotonicity Theorem is now given. This is still a special case of the original result, particularly with regard to the structure of  $A_s$ . The proof is a simple extension of the proof of Theorem 2.

**Theorem 10** *Assume*

- (i)  $F$  is supermodular in  $a$  for each fixed  $s$ ,
  - (ii)  $F$  has increasing differences in  $(s, a)$ , and
  - (iii)  $A_s = X_{i=1}^{i=m}[g_i(s), h_i(s)]$  where  $h_i, g_i : S \rightarrow R$  are increasing functions with  $g_i \leq h_i$ .
- Then the maximal and minimal selections<sup>30</sup> of  $a^*(s)$  are increasing functions. Furthermore, if (ii) is strict, then every selection of  $a^*(s)$  is increasing.

To relate this result to Theorem 2, observe that the added assumption here, supermodularity of  $F$  in  $a$ , is always trivially satisfied if  $a$  is scalar, whence its omission in Theorem 2. If  $a$  is a vector, it is clear intuitively that such an assumption is needed to ensure that the direct increases in the coordinates of  $a$  triggered by an increase in  $s$  are mutually reinforcing<sup>31</sup>.

A smooth version of Topkis's Theorem, based on Lemma 1 and the remarks below (11).

**Theorem 11** *Theorem 10 continues to hold if (i) and (ii) are replaced respectively by  $\partial^2 F(a)/\partial a_i \partial a_j, \forall i \neq j$  and  $\partial^2 F(a)/\partial a_i \partial s_j \geq 0, \forall i, j$ , with (iii) kept as is.*

This smooth version makes transparent the fact that there are no restrictions on partials of the form  $\partial^2 F(a)/\partial s_i \partial s_j$ . A sufficient condition for (i)–(ii) is to require  $F$  to be supermodular in the vector  $(a, s)$ , which would unnecessarily require the latter partials to be  $\geq 0$ .

Theorem 10 admits a natural dual for decreasing argmax's.

**Theorem 12** *If "increasing" is replaced by "decreasing" in Theorem 8 (ii)–(iii), with all else kept as is, the two given conclusions hold with "increasing" replaced by "decreasing".*

<sup>30</sup>These are always well-defined single-valued functions under Condition (i) of the Theorem.

<sup>31</sup>Without this assumption, one cannot guarantee a monotone comparative statics conclusion for all the coordinates of the vector  $a$ . See Samuelson (1974) for his coffee-tea "paradox".

Of the other results from Topkis (1978), the following has proved useful in a number of applications (see e.g. Amir, Mirman and Perkins 1991 or Amir 1996a).

**Theorem 13** *Let  $a = (a^j, a^{-j}) \in A$  and assume  $F : A \rightarrow R$  is supermodular in  $a$ , then  $V(a^{-j}) \triangleq \max_{a^j} F(a)$  is supermodular in  $a^{-j}$ .*

In words, this result says that maximization of a supermodular objective with respect to a subset of the variables preserves supermodularity with respect to the remaining variables. The use of the above results is illustrated in the following model.

**Monopoly with R&D and advertizing.** Consider a monopolist with three choice variables: price  $p$ , advertizing  $a$ , and investment in process R&D  $x$ . The market demand function is given by  $D(p, a, \theta)$  where  $\theta \in [0, 1]$  is a parameter capturing business cycle effects, with higher  $\theta$  meaning better macroeconomic conditions. Assume  $D$  satisfies<sup>32</sup>

$$D_p < 0, D_a > 0, D_\theta > 0, D_a + pD_{ap} < 0, D_\theta + pD_{p\theta} < 0, D_{a\theta} > 0. \quad (12)$$

The first three parts of (12) express respectively the natural assumptions that sales decrease with price and increase with advertizing and with (upswings in) the business cycle. The fourth part says that revenue has increasing differences in  $(p, a)$ , which requires a strongly submodular demand in  $(p, a)$  here as  $D_a > 0$ , and is a rather restrictive condition. A similar comment applies to the fifth part of (12). The sixth part says that the extra sales generated by higher advertizing increase with the business cycle, a reasonable property.

The firm has a constant unit cost  $C(x, \beta)$ , depending on its process R&D investment  $x$  and on an inter-industry spillover level  $\beta \in [0, 1]$ . We assume that

$$C_x < 0, C_\beta < 0 \text{ and } C_{\beta x} \leq 0. \quad (13)$$

The last condition may be interpreted as saying that the marginal cost reduction is increasing in the level of spillovers. The overall profit function of the firm is

$$\Pi(p, a, x; \theta, \beta) = [p - C(x, \beta)]D(p, a, \theta) - x$$

We first show that the optimal decision vector  $(-p^*, a^*, x^*)$  is increasing in  $(\theta, \beta)$ . To this end, observe that, in view of Assumptions (12)-(13),  $\tilde{\Pi}$  is supermodular in  $(-p, a, x)$  as

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<sup>32</sup>Observe that the analysis below will not rely on any concavity or interiority assumptions whatsoever. Furthermore, while the smoothness assumptions are convenient to express complementarity conditions in intuitive ways, they are not needed in any crucial sense at all for the underlying analysis and results.

$$\tilde{\Pi}_{(-p)a} = -\tilde{\Pi}_{pa} = -D_a - pD_{ap} > 0, \tilde{\Pi}_{(-p)x} = -\tilde{\Pi}_{px} = C_x D_p > 0, \text{ and } \tilde{\Pi}_{ax} = -C_x D_a > 0$$

and  $\tilde{\Pi}$  has increasing differences in  $[(-p, a, x), (\theta, \beta)]$  since

$$\begin{aligned} \tilde{\Pi}_{(-p)\theta} &= -\tilde{\Pi}_{p\theta} = -D_\theta - pD_{p\theta} > 0, \tilde{\Pi}_{(-p)\beta} = -\tilde{\Pi}_{p\beta} = C_\beta D_p \geq 0; \tilde{\Pi}_{a\beta} = -C_\beta D_a > 0, \\ \tilde{\Pi}_{a\theta} &= (p - C)D_{a\theta} > 0; \text{ and } \tilde{\Pi}_{x\theta} = -C_x D_\theta > 0, \tilde{\Pi}_{x\beta} = -C_{x\beta} D > 0. \end{aligned}$$

This establishes that as  $\theta$  and  $\beta$  increase, the optimal advertizing and R&D investment levels increase while the optimal price decreases (in the sense that all the selections from the three decisions are monotonic as indicated.)

Next, we address a somewhat different issue: relating short run and long responses to a given change. Suppose that in the short run, technology or R&D investment is fixed. We can still conclude from our previous analysis that the optimal price and advertizing will then react as before to changes in  $\theta$  and  $\beta$ . A question of interest is: Are these reactions greater in the short run or in the long-run? The answer is that they are greater in the long-run since, in the long run, a given change in  $\theta$  and  $\beta$  has all the short run effects plus the direct effect of increasing R&D investment  $x$ , which in turn has the known indirect effects of increasing  $(-p, a)$ . This is a simple expression of LeChatelier's principle, Milgrom and Roberts (1995).

Lastly, we show the optimal profit function  $\Pi^*(\theta, \beta) \triangleq \arg \max\{\Pi(p, a, x; \theta, \beta) : p, a, x \geq 0\}$  is supermodular in  $(\theta, \beta)$ . To this end, we invoke Theorem 13 as follows.  $\Pi(p, a, x; \theta, \beta)$  is supermodular in  $(p, a, x; \theta, \beta)$  in view of (12)-(13) and the fact that  $\Pi_{\theta\beta} = -C_\beta D_\theta > 0$ . (In other words, given the latter computation, the cross-partial of  $\Pi$  with respect to any two of the five variables is  $\geq 0$ .) Hence, since maximization preserves supermodularity,  $\Pi^*(\theta, \beta)$  is supermodular in  $(\theta, \beta)$ . The interpretation is that the marginal profitability of higher R&D investment increases with (upswings in) the business cycle, again a reasonable property.

## 5.2 Ordinal Complementarity Conditions

The ordinal versions of the multi-dimensional complementarity conditions, due to Milgrom and Shannon (1994), are as follows (with the same spaces and notation as above).

A function  $f : A \rightarrow R$  is quasi-supermodular if  $\forall a, a' \in A$ ,

$$F(a) \geq F(a \wedge a') \Rightarrow F(a \vee a') \geq F(a') \text{ and } F(a) > F(a \wedge a') \Rightarrow F(a \vee a') > F(a') \quad (14)$$

A function  $f : S \times A \rightarrow R$  has the single-crossing property in  $(a; s)$  if  $\forall a' > a, s' > s$ ,

$$\begin{aligned} F(s, a') - F(s, a) \geq 0 &\Rightarrow F(s', a') - F(s', a) \geq 0 \\ \text{and } F(s, a') - F(s, a) > 0 &\Rightarrow F(s', a') - F(s', a) > 0. \end{aligned} \tag{15}$$

The single-crossing property is strict if  $F(s, a') - F(s, a) \geq 0 \Rightarrow F(s', a') - F(s', a) > 0$ .

Recall that the single-crossing property is not symmetric with respect to the two variables, in that it may hold for  $(a; s)$  and not for  $(s; a)$ . On the other hand, quasi-supermodularity in  $a$  treats all coordinates of  $a$  symmetrically, and implies the single-crossing property in both  $(a_i; a_j)$  and in  $(a_j; a_i)$  for every  $i \neq j$ .

A simplified version of the main result of Milgrom and Shannon (1994) is:

**Theorem 14** *The conclusions of Theorem 10 hold if supermodularity is replaced by quasi-supermodularity and (strict) increasing differences by the (strict) single-crossing property.*

### 5.3 A Comparative Discussion

Milgrom and Shannon (1994) also show that all the above results (including in Section 3.3) for supermodular games also hold for quasi-supermodular games (i.e. games satisfying the ordinal complementarity conditions.) Indeed, these results actually rely on the monotonicity of the best-response correspondences only, for which the ordinal conditions suffice.

Besides the advantage of being much easier to verify in actual applications, the cardinal complementarity conditions sometimes lead to some desirable properties that are not satisfied under the ordinal conditions. In particular, the following two differences between the two notions are often of critical importance in many applications: (i) the single-crossing condition is not preserved by addition, and (ii) Theorem 13 need not hold if supermodularity is replaced by quasi-supermodularity. Thus maximization of a quasi-supermodular objective with respect to a subset of the variables need not give rise to a quasi-supermodular value function (with respect to the remaining variables.)

Although the ordinal conditions are more general than the cardinal conditions, in a typical application with the objective function being a sum or other composite of primitive functions, the respective minimally sufficient conditions on these primitives that imply the two types of conditions are generally not comparable. For a detailed investigation of this point centering on Cournot oligopoly, see Amir (2004).

There are some interesting analogies between supermodularity and quasi-supermodularity on the one hand and concavity and quasi-concavity on the other hand. Indeed, in both cases,

- (i) the ordinal condition is more general than the cardinal condition.
- (ii) the cardinal property survives addition while the ordinal property need not. This is a key difference in many economic applications since objective functions often take the form of a sum of some benefit terms and some cost terms<sup>33</sup>. In such cases, the cardinal notions often turn out to be critically needed.
- (iii) the cardinal property is conveniently verified via a set of inequalities on the Hessian matrix while the ordinal property requires a more difficult test.
- (iv) the cardinal property is preserved by partial maximization (Theorem 13), while the ordinal property need not be.

## 6 Other Aspects of the Theory

Several other relevant aspects of the theory under consideration have not been covered in the present survey. In this section, we provide a nonexhaustive list of these aspects and a brief summary. Further details may be found in the given references.

(a)**Parametric optimization under uncertainty.** Athey (2001, 2002) develops several different general results dealing with monotone comparative statics and preservation of complementarity conditions for objective functions that are represented as the integral of a multivariate function against a transition probability (or a family of probability distributions.) This work provides a unifying framework that encompasses most results on comparative statics under uncertainty in economics, as well as a new existence result for Bayesian games (such as auctions) in monotone strategies.

(b)**Critical sufficient conditions.** As both Topkis's Theorem and Milgrom-Shannon's Theorem provide sufficient conditions for increasing optimal argmax's, it is desirable to investigate how close to necessary these conditions are. The idea is to consider families of problems obtained by varying some model primitives such as a set of parameters. A condition is critical (with respect to a family of models) if whenever it fails, the monotonicity conclusion will fail for some specific model from the designated family (see Milgrom and Shannon 1994

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<sup>33</sup>In dynamic optimization problems, the objective to which Topkis's Theorem is applied is the right-hand side of the Bellman equation, a sum of today's reward and the value of the remainder (see e.g. 2.3(d))

and Milgrom and Roberts 1994). For the model of Bertrand competition with constant unit costs, allowing the latter parameter to vary – thereby defining a firm – is a natural way of specifying the relevant family: See Topkis (1995) and Amir and Grilo (2003).

(c)**Strict monotonicity of argmax's.** In some applications, it may be desirable to extend the conclusion of (weakly) increasing argmax's to one of strict monotonicity. This requires two new assumptions: (i) smoothness of the objective function, (ii) a strengthening of the assumption of increasing differences to this property: The partial of the objective function with respect to the decision must be strictly (not weakly as in Topkis's work) increasing in the parameter: See Amir (1996a), Edlin and Shannon (1998), or Topkis (1998 p. 71).

(d)**Macroeconomics.** Models of Keynesian-type coordination failure can be usefully analyzed using the theory of supermodular games: See Cooper and John (1988) and Cooper (1999).

(e)**Institutional complementarity.** A rather different application of this framework is to study general socio-economic systems using the underlying notions of complementarity as interpretative and descriptive properties in a much more precise manner than is typically done in related work. This general area of application is rather different from the material covered in the present survey in that, most often, no specific models of the issues under consideration are developed. Rather, the framework of economic complementarities is invoked to explain the observed presence of certain clusters in firms' strategies, institutional arrangements and other aspects of socio-economic organization that are mutually reinforcing: see Milgrom and Roberts (1990, 1994b, 1995a) and Hall and Soskice (2001).

## 7 Appendix: Proofs

We provide the proofs of the various results stated in the main text, in order of appearance.

**Lemma 15** *In Cournot oligopoly, assume that  $P(\cdot)$  is strictly decreasing and log-concave and each firm's cost function is strictly increasing and lower semi-continuous. Then the profit function satisfies the single-crossing property in  $(q_i, q_{-i})$ .*

**Proof.** We need to show that for any  $q'_i > q_i$  and  $q'_{-i} > q_{-i}$ ,

$$q'_i P(q'_i + q_{-i}) - C_i(q'_i) \leq q_i P(q_i + q_{-i}) - C_i(q_i) \Rightarrow q'_i P(q'_i + q'_{-i}) - C_i(q'_i) < q_i P(q_i + q'_{-i}) - C_i(q_i). \quad (16)$$

Log-concavity of  $P(\cdot)$  is equivalent to log-submodularity of  $P(q_i + q_{-i})$  in  $(q_i, q_{-i})$ , i.e.

$$\log P(q'_i + q'_{-i}) + \log P(q_i + q_{-i}) \leq \log P(q'_i + q_{-i}) + \log P(q_i + q'_{-i}),$$

or

$$P(q'_i + q'_{-i})P(q_i + q_{-i}) \leq P(q'_i + q_{-i})P(q_i + q'_{-i}). \quad (17)$$

With the LHS of (15) as starting point,

$$q'_i P(q'_i + q_{-i}) - C_i(q'_i) \leq q_i P(q_i + q_{-i}) - C_i(q_i) \quad (18)$$

$$\leq q_i \frac{P(q'_i + q_{-i})P(q_i + q'_{-i})}{P(q'_i + q'_{-i})} - C_i(q_i), \text{ by (16)} \quad (19)$$

Multiplying across by  $P(q'_i + q'_{-i})/P(q'_i + q_{-i})$ ,

$$q'_i P(q'_i + q'_{-i}) - \frac{P(q'_i + q'_{-i})}{P(q'_i + q_{-i})} C_i(q'_i) \leq q_i P(q_i + q'_{-i}) - \frac{P(q'_i + q'_{-i})}{P(q'_i + q_{-i})} C_i(q_i).$$

Since  $C_i(q'_i) > C_i(q_i)$  and  $P(q'_i + q'_{-i}) < P(q'_i + q_{-i})$ , it follows that

$$q'_i P(q'_i + q'_{-i}) - C_i(q'_i) < q_i P(q_i + q'_{-i}) - C_i(q_i). \quad (20)$$

Since (15) holds, the proof is complete. ■

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