(UN)CONDITIONAL DISTRIBUTION OF COMPENSATING VARIATION IN DISCRETE CHOICE MODELS[1]

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ABSTRACT

For a large class of additive random utility discrete choice models with income effects, we compute the probability distribution of the compensating variation. We show that the cumulative distribution function only depends on the choice probabilities. Our results are used to compute the distribution of equivalent variation. The moments of the compensating variation are a one-dimensional integral of the choice probabilities. Using the expected compensating variation, we extend Shephard's Lemma to the probabilistic demand systems. Both conditional and unconditional (on the individual choice) distributions of compensating variation are considered.

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Key Words: discrete choice models; income effect; compensating variation; equivalent variation.

1 Introduction

Discrete choice models (DCM) have traditionally been used as disaggregate demand models. The first application of DCM was in Transportation, to describe mode choice (choice between public and private mode). Daniel McFadden, a leader in this field, published his earlier and seminal work on DCM in the transportation context (see Domencich and McFadden [6]). At the very beginning of the study of disaggregate demand models, researchers were interested in the computation of welfare measures, since those models were (and are still) mainly used for policy analysis. McFadden [9] provides an exact (deterministic) welfare measure when the marginal utility of money is constant (see also Small and Rosen [17]). In the original formulation, the distribution of the surpluses was not treated nor were the distribution of the compensations, even with no income effect. However, the basic idea of DCM is that individuals are not identical and have probabilistic choice behavior so that the distribution of the welfare measure that is computed in this paper matters.

The focus of the paper is the derivation of the stochastic Hicksian welfare measure in DCM. This is not a new topic. Currently, welfare measures with income effect are studied via numerical simulations. McFadden [11], has developed a sampler for computing compensating variation (CV) caused by a change in the consumer environment. He has also provided an algorithm (the GEV sampler) to estimate welfare effects for the class of generalized extreme value (GEV) models. However, even though the sampler leads to consistent results, it is time consuming since a large number of iterations must be performed in order to obtain numerical approximations of the true welfare impact, with a reasonable level of accuracy.

New econometric techniques have been developed by Berry, Levinsohn and Pakes [3] and applied to estimate a mixed logit demand as well as costs parameters for the U.S. car industry. Berry et al. [4] analyze the welfare effects of the voluntary export restraints placed on Japanese cars in 1981 by using simulation techniques (similar to McFadden's GEV sampler). However, virtually no theoretical work has been performed to analyze policies in the presence of income effects in oligopoly models of imperfect competition.

DCM have became increasingly popular in industrial organization over the last decade to describe consumers making choices of differentiated products (see Anderson, de Palma and Thisse [2] and McFadden [12]). They provide flexible tools for the study of imperfect competition since the existence of a price Nash equilibrium is guaranteed for log-concave taste distributions (see Caplin and Nalebuff [5]). These advances have been used for the study of imperfect competition.[3]

However, these theoretical studies ignore so far income effects. One reason is that there exists no exact formula to perform welfare analysis with income effects. We expect that the derivation of exact welfare formulas for DCM with income effect opens a door to a whole set of new research, both for theoretical and empirical studies in industrial organization.

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[3] See, for example, Anderson, de Palma and Nesterov [1] who analyze the free-entry equilibrium in an oligopoly or Peitz [14] who study price competition with single-product firms.

In this paper, we compute analytically the probability distribution of CV for additive random utility DCM. In Section 2, assumptions on the indirect utility and on the distribution of preferences are made. We also present the definition of the individual CV. In Section 3, Theorem 1 provides the probability distribution of the CV. In Theorem 2, the distribution of the equivalent variation is also derived. In Section 4, we use Theorem 1 to compute the moments of the CV, which are given up to a one-dimensional integral of the choice probabilities. We also introduce a version of the Shephard's Lemma for DCM. We verify that with no income effects, expected CV coincide with the traditional variation of surplus. Section 5 concludes.

2 Assumptions and definitions

In this section we introduce the DCM framework and define the conditional (and unconditional) CV.

2.1 Discrete choice models

We consider a consumer selecting one good in the finite choice set $\mathcal{A} = \{1, ..., n\}$. Her conditional utility u_i from purchasing good i is assumed to be additively separable in two components

$$u_i = v_i + \epsilon_i, \ i \in \mathcal{A}. \tag{1}$$

The first term, v_i, is the observed (by the modeler) component and the second term, ϵ_i, is the unobserved component. We denote hereafter by $v \equiv (v_1, ..., v_n)$

the vector of the observed components of the utility.

It is assumed that v_i depends on the consumer disposable income y_i (after the purchase of good i), on a vector χ_i including other observable attributes (such as price p_i, quality, location, etc.) and parameters to be estimated: $v_i = V(y_i, \chi_i)$, $i \in \mathcal{A}$. We assume that:

A1 (Indirect utility function) $V(.)$ is a continuous indirect utility function.

In particular, it is strictly increasing in income (see, e.g. Mas-Collel, Whinston and Green [8]).

Due to lack of information, the modeler can at best describe the unobserved components of the utility $\epsilon \equiv (\epsilon_1, ..., \epsilon_n)$ as realizations of a random vector denoted by $\widetilde{\epsilon} \equiv (\widetilde{\epsilon}_1, ..., \widetilde{\epsilon}_n)$. We further assume that:

A2 (Random components) $\widetilde{\epsilon}$ is absolutely continuous with respect to the Lebesgue measure over $\mathcal{R}^n$ and has a convex support $\mathcal{B} \subset \mathcal{R}^n$.

In standard DCM, the consumer purchases the good with the highest conditional utility u_i. The event $\mathcal{B}_i(v)$ whereby good i yields maximum utility is the set of unobserved components ϵ which satisfy

$$\mathcal{B}_i(v) = \left\{ \epsilon \in \mathcal{B} \middle| v_i + \epsilon_i = \max_{k \in \mathcal{A}} (v_k + \epsilon_k) \right\}, \; i \in \mathcal{A}. \tag{2}$$

Note that $\mathcal{B}_i(v)$, $i \in \mathcal{A}$, are convex subsets of $\mathcal{B}$ which form almost everywhere (a.e.) a partition[4] of $\mathcal{B}$. Therefore, the probability that the consumer selects

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[4] Since the event that a consumer is indifferent between two (or more) goods occurs with zero probability, according to A2.

good i which is given by $\mathbf{P}_i(v) \equiv \Pr[\mathcal{B}_i(v)]$, can be written as

$$\mathbf{P}_i(v) = \int_{-\infty}^{\infty} F_i(v_i + x - v_1, ..., v_i + x - v_n)\, dx,\ i \in \mathcal{A}, \tag{3}$$

where $F(.)$ is the (multivariate) cumulative density function (c.d.f.) of $\widetilde{\epsilon}$ and $F_i(.)$ is its partial derivative with respect to the ith component (see Anderson et al. [2]).

2.2 Compensating variation in discrete choice models

We wish to compute the compensating variation for a change $(y_i^0, \chi_i^0) \to (y_i^1, \chi_i^1)$ where y_i^0 (resp. y_i^1) is the disposable income and χ_i^0 (resp. χ_i^1) is the vector of observable attributes and parameters of good i at situation 0 (resp. at situation 1). We denote hereafter by $v^0 \equiv (v_1^0, ..., v_n^0)$ the vector of observed components of the utility at situation 0, where $v_i^0 \equiv V(y_i^0, \chi_i^0)$. We study the case where before and after the change the choice set $\mathcal{A}$ and the unobserved components ϵ both remain the same.[5]

The corresponding CV for the consumer who is constrained to select alternative i is defined as the income adjustment, denoted by ψ_i, that equates the utility at situation 0 and 1. It solves the implicit equation[6]

$$v_i^0 = V(y_i^1 - \psi_i, \chi_i^1),\ i \in \mathcal{A}. \tag{4}$$

Note that ϵ_i is dropped from Eq. (4) given the additive structure of the model

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[5] Alternatively, ϵ may be redrawn (from the same distribution) after the change. The invariance of the $\epsilon' s$ is widely accepted and therefore treated in this paper.

[6] In some microeconomic textbooks, an alternative definition for the CV is used: $v_i^0 = V(y_i^1 + \psi_i, \chi_i^1)$. We adopt the same convention here as in McFadden [11].

(see Eq. (1)). We assume regularity condition on $V(.)$ such that:

A3 (Constrained CV adjustment) A solution ψ_i to Eq. (4) exists, $i \in \mathcal{A}$.

Since $V(.)$ is strictly increasing in income (see A1), ψ_i is unique. Below, we shall use the notations $\psi_- \equiv \min_{k\in\mathcal{A}} \psi_k$ and $\psi_+ \equiv \max_{k\in\mathcal{A}} \psi_k$.

Consider now the case where the consumer is allowed to modify his choice after the change and after compensation. If at situation 1, an (arbitrary) income adjustment of z is operated, the maximum utility of the consumer is: $\max_{k\in\mathcal{A}} \left[V\left(y_k^1 - z, \chi_k^1\right) + \epsilon_k\right]$. Let $cv_i(\epsilon)$ denotes[7] the conditional CV for a consumer who selects good i at situation 0, i.e. $\epsilon \in \mathcal{B}_i\left(v^0\right)$.

Definition 1 (Conditional CV) *The conditional CV for a consumer selecting good i at situation 0 is the income adjustment $cv_i(\epsilon)$ that equates maximum utility at situation 1 to utility of good i chosen at situation 0:*

$$v_i^0 + \epsilon_i = \max_{k\in\mathcal{A}} \left[V\left(y_k^1 - cv_i(\epsilon), \chi_k^1\right) + \epsilon_k\right], \ i \in \mathcal{A}. \tag{5}$$

Let $cv(\epsilon)$ denote the unconditional CV:

Definition 2 (Unconditional CV) *The unconditional CV is the income adjustment $cv(\epsilon)$ that equates maximum utility at situation 1 to maximum utility at situation 0:*

$$\max_{k\in\mathcal{A}} \left(v_k^0 + \epsilon_k\right) = \max_{k\in\mathcal{A}} \left[V\left(y_k^1 - cv(\epsilon), \chi_k^1\right) + \epsilon_k\right]. \tag{6}$$

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[7] The conditional CV also depends on y_i^0, χ_i^0, and on y_k^1, χ_k^1, $k \in \mathcal{A}$. For simplicity, these arguments are dropped.

Note that the unconditional CV is used if the choice of the consumer is not observed by the modeler.

3 Distribution of compensating variation

in this section, we introduce an exact formula for the distribution of the (un)conditional CV. Since the vector ϵ is not observed, $cv_i(\epsilon)$ and $cv(\epsilon)$ are the realizations of random variables which are denoted by $\widetilde{cv}_i$ and $\widetilde{cv}$, respectively. In Theorem 1, we provide the (un)conditional probability distribution of the compensations.

3.1 Preliminary results

We first consider two lemmata and provide a geometric illustration for the main result of the paper for the 3-goods case. In Lemma 1, we show that $cv_i(\epsilon)$ exists and we provide bounds which depends on the initial choice (i) and on the final choice (after the adjustment). For $\epsilon \in \mathcal{B}_i\left(v^0\right)$, let $k_i(\epsilon)$ denote the good selected at situation 1 after an income adjustment of $cv_i(\epsilon)$. When $cv_i(\epsilon)$ exists, $k_i(\epsilon)$ is a.e. uniquely defined by

$$k_i(\epsilon) = \arg\max_{k \in \mathcal{A}} \left[V\left(y_k^1 - cv_i(\epsilon), \chi_k^1\right) + \epsilon_k \right], \ i \in \mathcal{A}. \tag{7}$$

We have (see proof in Appendix):

Lemma 1 (Existence of CV and transitions) *Assume A1-A3 hold. For a consumer selecting good $i \in \mathcal{A}$ at situation 0, $cv_i(\epsilon)$ exists, is unique and*

$cv_i(\epsilon) \in \left[\psi_i, \psi_+\right]$. *Moreover, a.e.*

$$\begin{cases} k_i(\epsilon) = i \Leftrightarrow cv_i(\epsilon) = \psi_i, \\ k_i(\epsilon) = k, \ k \neq i \Rightarrow \psi_i < cv_i(\epsilon) < \psi_k. \end{cases} \tag{8}$$

As a consequence of Lemma 1, for each $\epsilon \in \mathcal{B}$, $cv(\epsilon)$ exists, is unique and $cv(\epsilon) \in \left[\psi_-, \psi_+\right]$.

Lemma 1 can be explained intuitively as follows. Consider a consumer selecting good i at situation 0. First, if she selects again good i at situation 1 (once the income adjustment takes place), she needs necessarily an income adjustment of ψ_i (see Eq. (4)). Conversely, if she receives an income adjustment of ψ_i, she cannot shift to a different good since by doing so she would derive a greater utility than her achieved utility at situation 0. Hence ψ_i is the smallest income adjustment. Second, if she shifts towards a good $k \neq i$, an income adjustment of ψ_k will restore the utility of good k to its level at situation 0. Moreover, she would derive a lower utility than initially since good i was preferred to k. Therefore, the income adjustment must be lower than ψ_k. Lemma 1 implies that with appropriate ranking of the goods, the transition probability matrix is triangular.

Lemma 2 characterizes the set of consumers who initially choose i and require a compensation lower than a given amount z. This lemma provides an expression for $\mathcal{C}_i(z)$, the event that a consumer receives a CV lower or equal to z and selects good i before the change

$$\mathcal{C}_i(z) \equiv \left\{\epsilon \in \mathcal{B}_i\left(v^0\right) | cv_i(\epsilon) \leq z\right\}, \, i \in \mathcal{A}. \tag{9}$$

We show that $\mathcal{C}_i\left(z\right)$ can be written in a form similar to (2), i.e. *as a choice event in a DCM.* Lemma 2 plays a central role in the computation of the c.d.f. of $\widetilde{cv}_i$. We have:

Lemma 2 (Choice event and CV) *Assume A1-A3 hold. The event* $\mathcal{C}_i\left(z\right)$, $i \in \mathcal{A}$, *can be written as:*

$$\mathcal{C}_i\left(z\right) = \begin{cases} \varnothing, & \textit{if } z < \psi_i, \\ \mathcal{B}_i\left(v^*\left(z\right)\right), & \textit{otherwise,} \end{cases} \tag{10}$$

where $v^*\left(z\right)$ *the* n*-vector with* k*th component,* $k \in \mathcal{A}$, *is given by:*

$$v_k^*\left(z\right) \equiv \begin{cases} V\left(y_k^1 - z, \chi_k^1\right), & \textit{if } z < \psi_k, \\ v_k^0, & \textit{otherwise.} \end{cases} \tag{11}$$

Proof. First, since $cv_i\left(\epsilon\right) \in \left[\psi_i, \psi_+\right]$ (see Lemma 1), $\mathcal{C}_i\left(z\right) = \varnothing$ if $z < \psi_i$. Second, let $\epsilon \in \mathcal{B}_i\left(v^0\right)$ and $z \geq \psi_i$. If $cv_i\left(\epsilon\right) \leq z$ then $V\left(y_k^1 - cv_i\left(\epsilon\right), \chi_k^1\right) \geq V\left(y_k^1 - z, \chi_k^1\right)$, $\forall\, k$. Since $v_i^0 + \epsilon_i = \max_{k \in \mathcal{A}}\left[V\left(y_k^1 - cv_i\left(\epsilon\right), \chi_k^1\right) + \epsilon_k\right]$, we get:

$$v_i^0 + \epsilon_i \geq V\left(y_k^1 - z, \chi_k^1\right) + \epsilon_k, \ \forall\, k.$$

Conversely, assume that $v_i^0 + \epsilon_i \geq V\left(y_k^1 - z, \chi_k^1\right) + \epsilon_k$, $\forall\, k$. Since $v_i^0 + \epsilon_i = V\left(y_{k_i(\epsilon)}^1 - cv_i\left(\epsilon\right), \chi_{k_i(\epsilon)}^1\right) + \epsilon_{k_i(\epsilon)}$, we obtain:

$$V\left(y_{k_i(\epsilon)}^1 - cv_i\left(\epsilon\right), \chi_{k_i(\epsilon)}^1\right) \geq V\left(y_{k_i(\epsilon)}^1 - z, \chi_{k_i(\epsilon)}^1\right),$$

so that $cv_i\left(\epsilon\right) \leq z$. Henceforth:

$$\mathcal{C}_i\left(z\right) = \left\{\epsilon \in \mathcal{B}_i\left(v^0\right) \left| v_i^0 + \epsilon_i \geq V\left(y_k^1 - z, \chi_k^1\right) + \epsilon_k \right., \forall k\right\}.$$

Now, since $\mathcal{B}_i\left(v^0\right) \equiv \left\{\epsilon \in \mathcal{B} \left| v_i^0 + \epsilon_i = \max_{k \in \mathcal{A}}\left(v_k^0 + \epsilon_k\right)\right.\right\}$, we deduce

$$\mathcal{C}_i\left(z\right) = \left\{\epsilon \in \mathcal{B} \left| v_i^0 + \epsilon_i = \max_{k \in \mathcal{A}}\left[v_k^0 + \epsilon_k, V\left(y_k^1 - z, \chi_k^1\right) + \epsilon_k\right], \forall k \right.\right\}.$$

Defining $v_k^{\#}(z) \equiv \max\left[v_k^0, V\left(y_k^1 - z, \chi_k^1\right)\right]$ and using the fact that

$$\max_{k\in\mathcal{A}}\left[v_k^0 + \epsilon_k, V\left(y_k^1 - z, \chi_k^1\right) + \epsilon_k\right] = \max_{k\in\mathcal{A}}\left[v_k^{\#}(z) + \epsilon_k\right],$$

we can rewrite the above expression as follows:

$$\mathcal{C}_i(z) = \left\{\epsilon \in \mathcal{B} \middle| v_i^0 + \epsilon_i = \max_{k\in\mathcal{A}}\left[v_k^{\#}(z) + \epsilon_k\right], \forall k\right\}. \tag{12}$$

Notice that $z < \psi_k$ implies $V\left(y_k^1 - z, \chi_k^1\right) > v_k^0 = V\left(y_k^1 - \psi_k, \chi_k^1\right)$, so that $v_k^{\#}(z) = V\left(y_k^1 - z, \chi_k^1\right)$. On the other hand, $z \geq \psi_k$ implies $V\left(y_k^1 - z, \chi_k^1\right) \leq v_k^0$, and therefore $v_k^{\#}(z) = v_k^0$. Therefore $v_k^{\#}(z)$ and $v_k^*(z)$ defined by Eq. (11) coincide. Consequently, using Eq. (12), Eq. (10) is obtained. ■

The intuition behind Lemma 2 is illustrated by Figures 1-3 for the 3-goods case. Consumers are located in a two-dimensional space and described by the components $(\epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_3)$.

In Figure 1, the consumer located at S^0 is indifferent among the three goods. The half line Δ_{ij}^0 corresponds to the set of consumers who are indifferent between good i and j, $i \neq j$. The area at the East of S^0 with boundaries Δ_{12}^0 and Δ_{13}^0 corresponds to the set $\mathcal{B}_1\left(v^0\right)$, and the area to the West of S^0 with boundaries Δ_{12}^0 and Δ_{23}^0 corresponds to the set $\mathcal{B}_2\left(v^0\right)$. After the change, S^0 shifts to S^1. Transitions occur from good 1 to good 2 or good 3 and from good 2 to good 3. Consumers who hold on to their initial good are represented by the dashed area.[8]

INSERT Fig. 1 HERE

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[8] The set of consumers who hold on to good 1 coincide with the set of consumers who choose good 1 after the change since no shifts occur towards good 1.

In Figure 2, we consider an income adjustment of z_1, with $\psi_1 \leq z_1 < \psi_2$. According to Lemma 1, only consumers choosing good 1 at situation 0 can be compensated with an amount less or equal to z_1. The area to the East of S^{z_1} with boundaries $\Delta_{12}^{z_1}$ and $\Delta_{13}^{z_1}$ corresponds to the set $\mathcal{C}_1(z_1)$ of consumers for whom an income adjustment less or equal to z_1 is required.[9] Since they stick to good 1, they already derive the same utility as before the change, i.e. $v_1^0 + \epsilon_1$. As noted above, this set coincides with $\mathcal{B}_1(v^*(z_1))$ and therefore, $\mathcal{C}_1(z_1) = \mathcal{B}_1(v^*(z_1))$.

In Figure 3, we consider an income adjustment of z_2, with $\psi_2 \leq z_2 \leq \psi_3$. According to Lemma 1, only consumers choosing good 1 or good 2 at situation 0 can be compensated with an amount less or equal to z_2. The area to the East of S^{z_2} with boundaries $\Delta_{12}^{z_2}$ and $\Delta_{13}^{z_2}$ corresponds to the set $\mathcal{C}_1(z_2)$ of consumers selecting good 1 at situation 0 for whom an income adjustment less or equal to z_2 is needed. This set coincides with $\mathcal{B}_1(v^*(z_2))$; therefore $\mathcal{C}_1(z_2) = \mathcal{B}_1(v^*(z_2))$. The area to the West of S^{z_2} with boundaries $\Delta_{12}^{z_2}$ and $\Delta_{23}^{z_2}$ corresponds to the set $\mathcal{C}_2(z_2)$ of consumers selecting good 2 at situation 0 for whom an income adjustment less or equal to z_2 is needed. This set coincides with $\mathcal{B}_2(v^*(z_2))$, and therefore $\mathcal{C}_2(z_2) = \mathcal{B}_2(v^*(z_2))$.

INSERT Fig. 2 and Fig. 3 HERE

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[9] The consumer located at S^{z_1} is exactly compensated with z_1.

3.2 The main theorem

Denote by $\mathbf{\Phi}_i(.)$ the c.d.f. of the conditional CV, $\widetilde{cv}_i$, (i.e. conditional on the choice of good i before the change). Then

$$\mathbf{\Phi}_i(z) \equiv \frac{\Pr\left[\mathcal{C}_i(z)\right]}{\mathbf{P}_i^0},\ i \in \mathcal{A}, \tag{13}$$

where $\mathbf{P}_i^0 \equiv \mathbf{P}_i\left(v^0\right)$. Using Lemma 2, the c.d.f. of the conditional CV, $\mathbf{\Phi}_i(.)$, is computed analytically:

Theorem 1 (Conditional distribution of CV) *Assume A1-A3 hold. The c.d.f. of the conditional compensating variation* $\widetilde{cv}_i$ *on its support* $\left[\psi_i, \psi_+\right]$ *is:*

$$\mathbf{\Phi}_i(z) = \frac{\mathbf{P}_i\left(v^*(z)\right)}{\mathbf{P}_i^0},\ i \in \mathcal{A}. \tag{14}$$

Proof. According to Lemma 2, for $z \in \left[\psi_i, \psi_+\right]$, $\mathcal{C}_i(z) = \mathcal{B}_i\left(v^*(z)\right)$. Henceforth, computing the probability of $\mathcal{C}_i(z)$ amounts to compute the probability of choosing good i with a vector of observed components of the utility given by $v^*(z)$. This leads to Eq. (14). ■

Note that the unconditional CV, $\widetilde{cv}$, has a c.d.f. denoted by $\mathbf{\Phi}(.)$, with $\mathbf{\Phi}(z) = \sum_{i\in\mathcal{A}} \mathbf{P}_i^0 \mathbf{\Phi}_i(z)$, where $\mathbf{\Phi}_i(.)$ on its support $\left[\psi_i, \psi_+\right]$ is given by Eq. (14).

3.3 Diversion on equivalent variation

Another usual welfare measure is the equivalent variation (EV). We use the notations $v^1 \equiv \left(v_1^1, ..., v_n^1\right)$ where $v_i^1 \equiv V\left(y_i^1, \chi_i^1\right)$, and $\mathbf{P}_i^1 \equiv \mathbf{P}_i\left(v^1\right)$, $i \in \mathcal{A}$. Assume the consumer selects i at situation 1; the conditional EV given the final choice of i, denoted by $ev_i(.)$, solves

$$\max_{k\in\mathcal{A}}\left[V\left(y_k^0+ev_i\left(\epsilon\right),\chi_k^0\right)+\epsilon_k\right]=v_i^1+\epsilon_i,\; i\in\mathcal{A}, \tag{15}$$

while the unconditional EV, denoted by $ev(.)$, verifies

$$\max_{k\in\mathcal{A}}\left[V\left(y_k^0+ev\left(\epsilon\right),\chi_k^0\right)+\epsilon_k\right]=\max_{k\in\mathcal{A}}\left(v_k^1+\epsilon_k\right). \tag{16}$$

Within this framework, we define ψ_i (with slight abuse of notation) as the amount of income, which solves the equation

$$V\left(y_i^0+\psi_i,\chi_i^0\right)=v_i^1,\; i\in\mathcal{A}. \tag{17}$$

Assumption A3 is now replaced by:

A3' (Constrained EV adjustment) A solution ψ_i to Eq. (17) exists, $i\in\mathcal{A}$.

Theorem 1 can be adapted directly to compute the distribution of the conditional EV, denoted by $\widetilde{ev}_i$. We obtain:

Theorem 2 (Conditional distribution of EV) *Assume A1, A2, A3' hold. The survival function of the conditional equivalent variation* $\widetilde{ev}_i$ *on its support* $\left[\psi_-,\psi_i\right]$ *is:*

$$\mathbf{S}_i\left(z\right)=\frac{\mathbf{P}_i\left(\omega^*\left(z\right)\right)}{\mathbf{P}_i^1},\; i\in\mathcal{A}, \tag{18}$$

where $w^*\left(z\right)$ *the* n*-vector with* k*th component,* $k\in\mathcal{A}$*, is given by:*

$$w_k^*\left(z\right)\equiv\begin{cases}V\left(y_k^0+z,\chi_k^0\right), & \text{if } z>\psi_k,\\ v_k^1, & \text{otherwise.}\end{cases} \tag{19}$$

Proof. Rewrite Eq. (15) as

$$v_i^1 + \epsilon_i = \max_{k \in \mathcal{A}} \left[V\left(y_k^0 - \left(-ev_i\left(\epsilon\right)\right), \chi_k^0\right) + \epsilon_k \right].$$

This equation is analogous to Eq. (5), where the state indices are permuted, where $-ev_i(\epsilon)$ plays the role of $cv_i(\epsilon)$ and where $-\psi_i$ plays the role of ψ_i. Therefore, according to Lemma 1, $ev_i(\epsilon)$ exists, is unique and has support $\left[\psi_-, \psi_i\right]$. Now, since $\widetilde{ev}_i$ is continuous for $z < \psi_i$ and according to Theorem 1, the expression for $\mathbf{S}_i(z) \equiv \Pr\left[\widetilde{ev}_i > z\right]$ is

$$\mathbf{S}_i(z) = \frac{\mathbf{P}_i\left(w^*(z)\right)}{\mathbf{P}_i^1},\ z < \psi_i,$$

where $w^*(z)$ has components defined by (19). Finally, since $\mathbf{S}_i(z)$ is continuous to the right and $\mathbf{P}_i\left(w^*(z)\right)/P_i^1$ is continuous, we conclude that necessarily: $\mathbf{S}_i\left(\psi_i\right) = \mathbf{P}_i\left(w^*\left(\psi_i\right)\right)/\mathbf{P}_i^1$. ■

Note that the unconditional EV, denoted by $\widetilde{ev}$, has a survival function, denoted by $\mathbf{S}(.)$, with $\mathbf{S}(z) = \sum_{i \in \mathcal{A}} \mathbf{P}_i^0 \mathbf{S}_i(z)$, where $\mathbf{S}_i(.)$ on its support $\left[\psi_-, \psi_i\right]$ is given by Eq. (18).

4 Moments of compensating variation

In this section, we compute the (un)conditional moments of the distribution of CV. Then, using expected CV, we extend Shephard's Lemma to the probabilistic demand systems and verify that with no income effects, see coincide with the traditional variation of surplus.

4.1 (Un)Conditional moments of CV

As a corollary of Theorem 1, we show that the conditional moments can be obtained up to a one dimensional integral of the choice probabilities:

Corollary 1 (Conditional moments of CV) *Assume A1-A3 hold. The mth conditional moment of the compensating variation verifies:*

$$\mathbb{E}\left[\widetilde{cv}_i^m\right] = \psi_+^m - \frac{m}{\mathbf{P}_i^0} \int_{\psi_i}^{\psi_+} z^{m-1} \mathbf{P}_i \left(v^* \left(z\right)\right) dz \,, \ i \in \mathcal{A}. \tag{20}$$

Proof. For $0 \leq \pi \leq 1$, define the conditional quantile function $\mathbf{\Phi}_i^{-1}\left(\pi\right) \equiv \sup \left\{ z \in \left[\psi_i, \psi_+\right] \left| \mathbf{\Phi}_i \left(z\right) \leq \pi \right. \right\}$, $i \in \mathcal{A}$, which is the inverse of conditional c.d.f. of CV. By definition, the mth conditional moment of CV is such that $\mathbb{E}\left[\widetilde{cv}_i^m\right] \equiv \int_0^1 \left[\mathbf{\Phi}_i^{-1}\left(\pi\right)\right]^m d\pi$. Recall that (see Eq. (14)) $\mathbf{\Phi}_i\left(z\right) = \mathbf{P}_i\left(v^*\left(z\right)\right) / \mathbf{P}_i^0$, for $z \in \left[\psi_i, \psi_+\right]$. The choice probabilities $\mathbf{P}_i\left(.\right)$ given by Eq. (3) are differentiable since the conditions for differentiation under the integral sign are satisfied (see Schwartz [16, Theorem 115]). Moreover, since $v^*\left(.\right)$ is continuous, $\mathbf{\Phi}_i\left(.\right)$ is continuous and monotonic and therefore is differentiable a.e. according to the Lebesgue theorem (see Rudin [15, Theorem 8.19]). Hence $\mathbf{\Phi}_i\left(.\right)$ is a.e. differentiable, a p.d.f. $\varphi_i\left(.\right)$ can be defined. Using the change of variable: $\pi = \mathbf{\Phi}_i\left(z\right)$, with $z \in \left[\psi_i, \psi_+\right]$, we get $\mathbb{E}\left[\widetilde{cv}_i^m\right] = \psi_i^m \boldsymbol{\pi}_i + \int_{\psi_i}^{\psi_+} z^m \varphi_i\left(z\right) dz$, where $\boldsymbol{\pi}_i = \mathbf{\Phi}_i\left(\psi_i\right)$ represents the (conditional) probability that the consumer selecting good i at situation 0 sticks to good i. The expected result (20) is obtained using an integration by parts. ■

Therefore, we obtain the following result for the unconditional moments:

Corollary 2 (Unconditional moments of CV) *Assume A1-A3 hold. The mth unconditional moment of the compensating variation verifies:*

$$\mathbb{E}\left[\widetilde{cv}^m\right] = \psi_+^m - m \sum_{i \in \mathcal{A}} \int_{\psi_i}^{\psi_+} z^{m-1} \mathbf{P}_i \left(v^* \left(z\right)\right) dz. \tag{21}$$

Proof. Using the fact that $\mathbb{E}\left[\widetilde{cv}^m\right] = \sum_{i \in \mathcal{A}} \mathbf{P}_i^0 \mathbb{E}\left[\widetilde{cv}_i^m\right]$, where $\mathbb{E}\left[\widetilde{cv}_i^m\right]$ is given by Eq. (20), the expected result follows immediately. ■

Generally, the integrals which appear in Eq. (20) and (21) cannot be written in closed form, even for logit or GEV models. These integrals can be approximated analytically through simulation by generating (uniform) random draws[10] of z, calculating the probability $\mathbf{P}_i \left(v^* \left(z\right)\right)$ and then averaging the values $z^{m-1} \mathbf{P}_i \left(v^* \left(z\right)\right)$.

4.2 Shephard's lemma

As a consequence of Eq. (21), expected CV is given by

$$\mathbb{E}\left[\widetilde{cv}\right] = \psi_+ - \sum_{i \in \mathcal{A}} \int_{\psi_i}^{\psi_+} \mathbf{P}_i \left(v^* \left(z\right)\right) dz.. \tag{22}$$

This expression is reminiscent of the standard treatment of surplus, which involves the computation of areas under the compensated probability curves (The computation of expected consumer surplus was also envisaged by Small and Rosen [17] and by Karlstrom[11] [7]).

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[10] Alternatively, the GEV sampler consists in draws of ϵ using Monte Carlo markov chain methods. The generated values $cv^m \left(\epsilon\right)$ are then averaged. As noted by McFadden [11], this procedure is computationally burden.

[11] Karlstom [7] research is focused on $\mathbb{E}\left[\widetilde{cv}\right]$ while the distribution of $\widetilde{cv}$ is derived in this paper. Moreover, he derives an integral expression for the first moment, where the integrand

Let $\mathbf{X}_i$ denote the aggregate (expected) demand for good i. The mass of consumers which are assumed to be statistically identical, is normalized to one. The aggregate demand is: $\mathbf{X}_i = \mathbf{P}_i(v)\, x_i$, where x_i is the conditional (individual) demand for good i. For a price variation Δp_i of good i, let $\psi_i(\Delta p_i)$ be the (constrained) CV associated to this price variation Δp_i (see Eq. (4)). In this case, we can use Shephard's Lemma which allows to recover the conditional (individual) demand: $\lim_{\Delta p_i \to 0} \psi_i(\Delta p_i) / \Delta p_i = x_i$, $i \in \mathcal{A}$.

In Proposition 1, we show that the (aggregate) demand for good i can also be recovered from expected CV denoted by $\mathbb{E}[\widetilde{cv}(\Delta p_i)]$. This result provides a version of Shephard's Lemma for an aggregate population of consumers buying according to a DCM. We have:

Proposition 1 (Shephard's Lemma revisited) *Assume A1-A3 hold. If the indirect utility function $V(.)$ is twice continuously differentiable in income and price, then:*

$$\lim_{\Delta p_i \to 0} \frac{\mathbb{E}[\widetilde{cv}(\Delta p_i)]}{\Delta p_i} = \mathbf{X}_i, \ i \in \mathcal{A}.$$

Proof. Consider w.l.o.g. a change in price of good 1. To reduce notations, we omit below the argument Δv_1 from ψ_1 and $\mathbb{E}[\widetilde{cv}]$. Moreover, since only variations in price are considered, we ignore the remaining arguments of χ_i, and replace χ_i by p_i in our notations.

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involves the partial derivative of the choice probabilities. By contrast, our expression given by Eq. (22) involves only choice probabilities.

First, for $\Delta p_1 > 0$, Eq. (22) reduces to

$$\mathbb{E}\left[\widetilde{cv}\right] = -\int_{\psi_1}^{0} I\left(z\right) dz,$$

where $I\left(z\right) \equiv \mathbf{P}_1\left(v_1, V\left(y_2 - z, p_2\right), ..., V\left(y_n - z, p_n\right)\right)$. Therefore

$$\mathbb{E}\left[\widetilde{cv}\right] - \mathbf{P}_1\left(v\right)\psi_1 = \int_{\psi_1}^{0} \left[\mathbf{P}_1\left(v\right) - I\left(z\right)\right] dz.$$

Since $I\left(z\right)$ is differentiable at $z = 0$, there exists $K > 0$ such that for $z \in \left[\psi_1, 0\right]$ small enough, $0 \leq \mathbf{P}_1\left(v\right) - I\left(z\right) \leq Kz$. Therefore $0 \leq \mathbb{E}\left[\widetilde{cv}\right] - \mathbf{P}_1\left(v\right)\psi_1 \leq K\psi_1^2/2$. It follows that

$$\lim_{\Delta p_1 \to 0^+} \frac{\mathbb{E}\left[\widetilde{cv}\right]}{\Delta p_1} = \mathbf{P}_1\left(v\right) \lim_{\Delta p_1 \to 0^+} \frac{\psi_1}{\Delta p_1},$$

and therefore: $\lim_{\Delta p_1 \to 0^+} \mathbb{E}\left[\widetilde{cv}\right]/\Delta p_1 = \mathbf{X}_1$.

Second, for $\Delta p_1 < 0$, we get $\mathbb{E}\left[\widetilde{cv}\right] = \int_0^{\psi_1} J\left(z, \Delta p_1\right) dz$, where $J\left(z, \Delta p_1\right) \equiv \mathbf{P}_1\left(V\left(y_1 - z, p_1 + \Delta p_1\right), v_2, ..., v_n\right)$. Therefore

$$\mathbb{E}\left[\widetilde{cv}\right] - \mathbf{P}_1\left(v\right)\psi_1 = \int_0^{\psi_1} \left[J\left(z, \Delta p_1\right) - \mathbf{P}_1\left(v\right)\right] dz.$$

From the following inequalities: $J\left(z, 0\right) \leq J\left(z, \Delta p_1\right) \leq J\left(0, \Delta p_1\right)$, $z \in \left[0, \psi_1\right]$, we get

$$\int_0^{\psi_1} \left[J\left(z, 0\right) - \mathbf{P}_1\left(v\right)\right] dz \leq \mathbb{E}\left[\widetilde{cv}\right] - \mathbf{P}_1\left(v\right)\psi_1 \leq \left[J\left(0, \Delta p_1\right) - \mathbf{P}_1\left(v\right)\right]\psi_1.$$

Differentiability of $J\left(z, \Delta p_1\right)$ at $z = 0$ implies that there exists $K > 0$ such that for $z \in \left[0, \psi_1\right]$ small enough, $-Kz \leq J\left(z, 0\right) - \mathbf{P}_1\left(v\right)$, which yields $-K\psi_1^2/2 \leq \mathbb{E}\left[\widetilde{cv}\right] - \mathbf{P}_1\left(v\right)\psi_1$. By continuity, $\lim_{\Delta p_1 \to 0^+} \left[J\left(0, \Delta p_1\right) - \mathbf{P}_1\left(v\right)\right] = 0$. Therefore

$$\lim_{\Delta p_1 \to 0^-} \frac{\mathbb{E}\left[\widetilde{cv}\right]}{\Delta p_1} = \mathbf{P}_1\left(v\right) \lim_{\Delta p_1 \to 0^-} \frac{\psi_1}{\Delta p_1}.$$

and therefore: $\lim_{\Delta p_1 \to 0^-} \mathbb{E}\left[\widetilde{cv}\right]/\Delta p_1 = \mathbf{X}_1$. ■

According to Eq. (22), $\mathbb{E}\left[\widetilde{cv}\right]$ is the sum of the integrals of parametrized choice probabilities $\mathbf{P}_i\left(v^*\left(z\right)\right)$. Conversely, Shephard's Lemma shows that the expected demand function is the derivative of expected CV, with respect to price.

4.3 Consumer's surplus

Consider the case where the marginal utility of income is constant and the same for all goods[12] (equal to 1, w.l.o.g). The Williams-Daly-Zachary Theorem states that with no income effects, the expected maximum utility is a representative or social utility function for a population of statistically identical consumers making discrete choices among the n goods (see McFadden [10]). More precisely,

$$CS\left(v\right) \equiv \mathbb{E}\left[\max_{k\in\mathcal{A}}\left(v_k + \widetilde{\epsilon}_k\right)\right], \tag{23}$$

coincides with the conventional Marshallian consumer surplus. Applying Roy's Identity to $CS\left(.\right)$ allows to recover the choice probabilities

$$\mathbf{P}_i\left(v\right) = \frac{\partial CS\left(v\right)}{\partial v_i},\ i \in \mathcal{A}. \tag{24}$$

For a change where v^0 and v^1 are the vectors of observed components of the utility before and after the change, respectively, the variation in consumer

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[12] Since utilities are defined up to a positive multiplicative constant.

surplus coincides with the area to the left of the choice probability curves[13]

$$CS\left(v^1\right)-CS\left(v^0\right)=\sum_{i\in\mathcal{A}}\int_{v^0}^{v^1}\mathbf{P}_i\left(v\right)dv_i. \tag{25}$$

With unitary marginal utility of income, the unconditional CV solves (see Eq. (6)) $cv\left(\epsilon\right)=\max_{k\in\mathcal{A}}\left(v_k^1+\epsilon_k\right)-\max_{k\in\mathcal{A}}\left(v_k^0+\epsilon_k\right)$. Therefore, from Eq. (23), expected CV satisfies $\mathbb{E}\left[\widetilde{cv}\right]=CS(v^1)-CS(cv^0)$. In Proposition 2, we verify that the expression derived for expected CV (see Eq. (22)) coincides with the variation in consumer surplus:

Proposition 2 (Consumer's surplus) *Assume A1-A3 hold. If the marginal utility of income is constant and the same for all goods, then:*

$$\mathbb{E}\left[\widetilde{cv}\right]=CS(v^1)-CS(cv^0), \tag{26}$$

where $CS\left(.\right)$ is the aggregate consumer surplus.

Proof. Eq. (22) can be rewritten as $\mathbb{E}\left[\widetilde{cv}\right]=\psi_-+\sum_{k\in\mathcal{A}}\int_{\psi_-}^{\psi_k}\mathbf{P}_k\left(v^*\left(z\right)\right)dz$. The Roy's identity (24) yields

$$\mathbb{E}\left[\widetilde{cv}\right]=\psi_-+\sum_{k\in\mathcal{A}}\int_{\psi_-}^{\psi_k}\frac{\partial CS\left(v^*\left(z\right)\right)}{\partial v_k}dz.$$

Now, with no income effects, for $k\in\mathcal{A}$,

$$\frac{dv_k^*\left(z\right)}{dz}\equiv\begin{cases}-1, & \text{if } z<\psi_k,\\ 0, & \text{if } z>\psi_k.\end{cases}$$

Hence

$$\mathbb{E}\left[\widetilde{cv}\right]=\psi_--\sum_{k\in\mathcal{A}}\int_{\psi_-}^{\psi_+}\frac{\partial CS\left(v^*\left(z\right)\right)}{\partial v_k}\frac{dv_k^*\left(z\right)}{dz}dz.$$

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[13] This integral is *path independent* since the Jacobian matrix $\left(\partial\mathbf{P}_i\,/\partial v_j\,\right)$ is symmetric (cf. Small and Rosen [17]).

which can be rewritten as

$$\mathbb{E}\left[\widetilde{cv}\right] = \psi_- - \int_{\psi_-}^{\psi_+} \frac{dCS\left(v^*\left(z\right)\right)}{dz} dz.$$

Now, continuity of $CS\left(v^*\left(z\right)\right)$ on $\left[\psi_-, \psi_+\right]$ implies

$$\mathbb{E}\left[\widetilde{cv}\right] = \psi_- + CS\left(v^*\left(\psi_-\right)\right) - CS\left(v^*\left(\psi_+\right)\right).$$

Since $v_k^*\left(\psi_+\right) = v_k^0$, and $v_k^*\left(\psi_-\right) = v_k^1 - \psi_-$, Eq. (26) is obtained. ■

Note however that with income effect the expression $CS\left(v^1\right) - CS\left(v^0\right)$ does not represent an intuitive welfare measure (although it is sometimes used as such in empirical studies with income effects). With income effect, expected CV does provide a sound basis for welfare evaluations.[14]

5 Concluding remarks

We have proposed a formula which allow to compute the distribution of compensating variation for random utility models. The surprise is that an analytical formula exists and involves choice probabilities only. The formula can be written very simply for the GEV and the logit. The formula for the logit model is of particular interest since any system of choice probabilities can be written(under mild regularity conditions) as an integral of a logit model, (see McFadden and Train [13]) called a mixed logit model. Therefore, the computation of the compensation of a mixed logit can be greatly simplified since the individual com-

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[14] It can be shown that for concave in income indirect utility functions which are further assumed to be additively separable in income and the other arguments, we have: $CS\left(v^1\right) - CS\left(v^0\right) \geq \mathbb{E}\left[\widetilde{cv}\right]$.

pensation is straightforward. The issue will be explored in further empirical research.

Appendix

Proof. [Lemma 1] Given that $\epsilon \in \mathcal{B}_i\left(v^0\right)$, define by

$$\mathcal{K}_i(\epsilon) \equiv \left\{k \left| \exists c,\, v_i^0 + \epsilon_i = V\left(y_k^1 - c, \chi_k^1\right) + \epsilon_k \right.\right\},$$

as the set of goods for which a compensation is possible. According to A3, $\mathcal{K}_i(\epsilon) \neq \emptyset$ since $i \in \mathcal{K}_i(\epsilon)$. Notice that

$$v_i^0 + \epsilon_i > V\left(y_k^1 - c, \chi_k^1\right) + \epsilon_k,\ \forall c, \text{ if } k \notin \mathcal{K}_i(\epsilon), \tag{27}$$

since $V\left(y_k^1 - \psi_k, \chi_k^1\right) + \epsilon_k = v_k^0 + \epsilon_k < v_i^0 + \epsilon_i$, and by continuity of $V(.)$ (see A1). Now, for $k \in \mathcal{K}_i(\epsilon)$, define $cv_{ik}(\epsilon)$ as the unique solution of equation

$$v_i^0 + \epsilon_i = V\left(y_k^1 - cv_{ik}(\epsilon), \chi_k^1\right) + \epsilon_k. \tag{28}$$

Monotonicity of $V(.)$ implies

$$\begin{aligned} v_i^0 + \epsilon_i \geq V\left(y_k^1 - c, \chi_k^1\right) + \epsilon_k \Leftrightarrow cv_{ik}(\epsilon) \leq c, \\ v_i^0 + \epsilon_i \leq V\left(y_k^1 - c, \chi_k^1\right) + \epsilon_k \Leftrightarrow cv_{ik}(\epsilon) \geq c. \end{aligned} \tag{29}$$

Notice also that $v_i^0 + \epsilon_i \geq v_k^0 + \epsilon_k$ since i is selected before the change. By definition of ψ_k: $v_k^0 + \epsilon_k = V\left(y_k^1 - \psi_k, \chi_k^1\right) + \epsilon_k$. As a consequence: $v_i^0 + \epsilon_i \geq V\left(y_k^1 - \psi_k, \chi_k^1\right) + \epsilon_k$ and by (29)

$$cv_{ik}(\epsilon) \leq \psi_k,\ k \in \mathcal{K}_i(\epsilon). \tag{30}$$

Now, using Eq. (27) and Eq. (29), $cv_i(.)$ is the largest compensation among the goods for which a compensation exists. Assume this is not the case. Then there exists $\widehat{k}$ such that $cv_{i\widehat{k}}(\epsilon) > cv_i(\epsilon)$ and by (29) we have

$$v_i^0 + \epsilon_i < V\left(y_{\widehat{k}}^1 - cv_i(\epsilon), \chi_{\widehat{k}}^1\right) + \epsilon_{\widehat{k}},$$

a contradiction since for the optimal choice we have

$$v_i^0 + \epsilon_i = \max_{k=1..n} \left[V\left(y_k^1 - cv_i\left(\epsilon\right), \chi_k^1\right) + \epsilon_k \right].$$

Therefore

$$cv_i\left(\epsilon\right) = \max_{k \in \mathcal{K}_i(\epsilon)} \left[cv_{ik}\left(\epsilon\right) \right]. \tag{31}$$

Since $cv_{ii}\left(\epsilon\right) = \psi_i$ and using (31), then $cv_i\left(\epsilon\right) \geq \psi_i$. Moreover, from (30) we deduce $cv_i\left(\epsilon\right) \leq \psi_+$. Therefore $\psi_i \leq cv_i\left(\epsilon\right) \leq \psi_+$.

Now, let $k_i\left(\epsilon\right)$ be the good chosen after the change (and after compensation), which is defined by (7) and is also such that: $k_i\left(\epsilon\right) = \arg\max_{k \in \mathcal{K}_i(\epsilon)} \left[cv_{ik}\left(\epsilon\right) \right]$. According to A2, it is a.e. uniquely defined. Clearly $k_i\left(\epsilon\right) = i$ if $cv_i\left(\epsilon\right) = \psi_i$, since $cv_{ii}\left(\epsilon\right) = \psi_i$. Conversely, if the individual receives a compensation $cv_i\left(\epsilon\right) = \psi_i$ and chooses $k \neq i$, then $V\left(y_i^1 - \psi_i, \chi_i^1\right) + \epsilon_i = v_i^0 + \epsilon_i$ by definition of ψ_i; moreover $V\left(y_k^1 - \psi_i, \chi_k^1\right) + \epsilon_k = v_i^0 + \epsilon_i$ since the individual chooses k after the change and after the compensation of ψ_i. Then $V\left(y_i^1 - \psi_i, \chi_i^1\right) + \epsilon_i = V\left(y_k^1 - \psi_i, \chi_k^1\right) + \epsilon_k$, which occurs with zero probability. Therefore we have proved that a.e. $k_i\left(\epsilon\right) = i \Leftrightarrow cv_i\left(\epsilon\right) = \psi_i$.

If $k_i\left(\epsilon\right) = k \neq i$, then $cv_i\left(\epsilon\right) > \psi_i$ by the above result. Moreover, since $\psi_i \leq cv_i\left(\epsilon\right) \leq \max_{k \in \mathcal{A}} \psi_k$, $cv_i\left(\epsilon\right)$ is such that $V\left(y_k^1 - cv_i\left(\epsilon\right), \chi_k^1\right) + \epsilon_k = v_i^0 + \epsilon_i$. By definition of ψ_k: $V\left(y_k^1 - \psi_k, \chi_k^1\right) + \epsilon_k = v_k^0 + \epsilon_k$. Assume $cv_i\left(\epsilon\right) \geq \psi_k$; in this case, $V\left(y_k^1 - \psi_k, \chi_k^1\right) + \epsilon_k \geq V\left(y_k^1 - cv_i\left(\epsilon\right), \chi_k^1\right) + \epsilon_k$. Then $v_k^0 + \epsilon_k \geq v_i^0 + \epsilon_i$, a contradiction. Therefore, we have proved that a.e.: $k_i\left(\epsilon\right) = k$, $k \neq i \Rightarrow \psi_i < cv_i\left(\epsilon\right) < \psi_k$. ■

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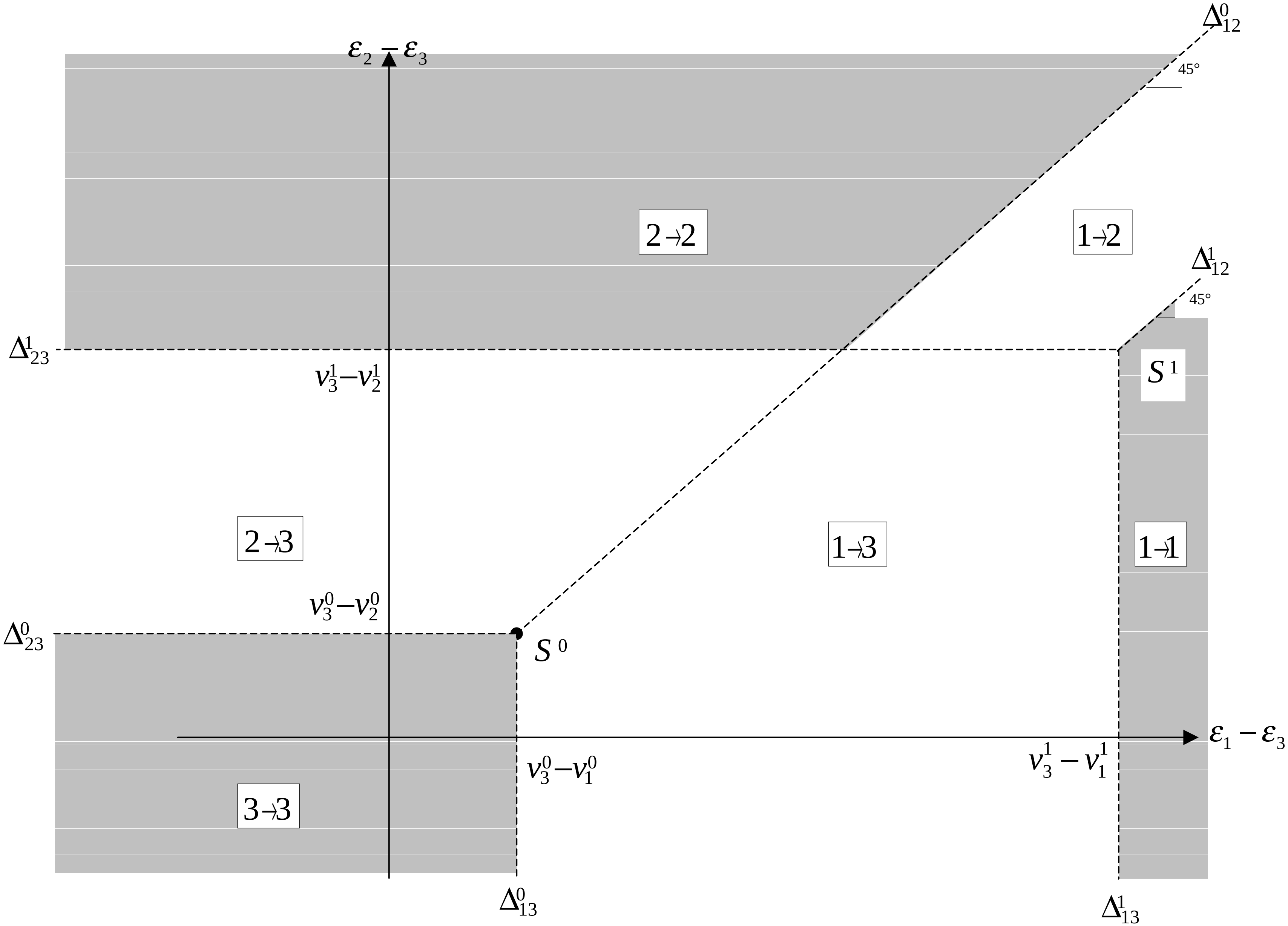

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$\varepsilon_2 - \varepsilon_3$
2→2
1→2
Δ_{12}^{1}
45°
Δ_{23}^{1}
S^1
$v_3^1 - v_2^1$
2→3
1→3
1→1
$v_3^0 - v_2^0$
Δ_{23}^{0}
S^0
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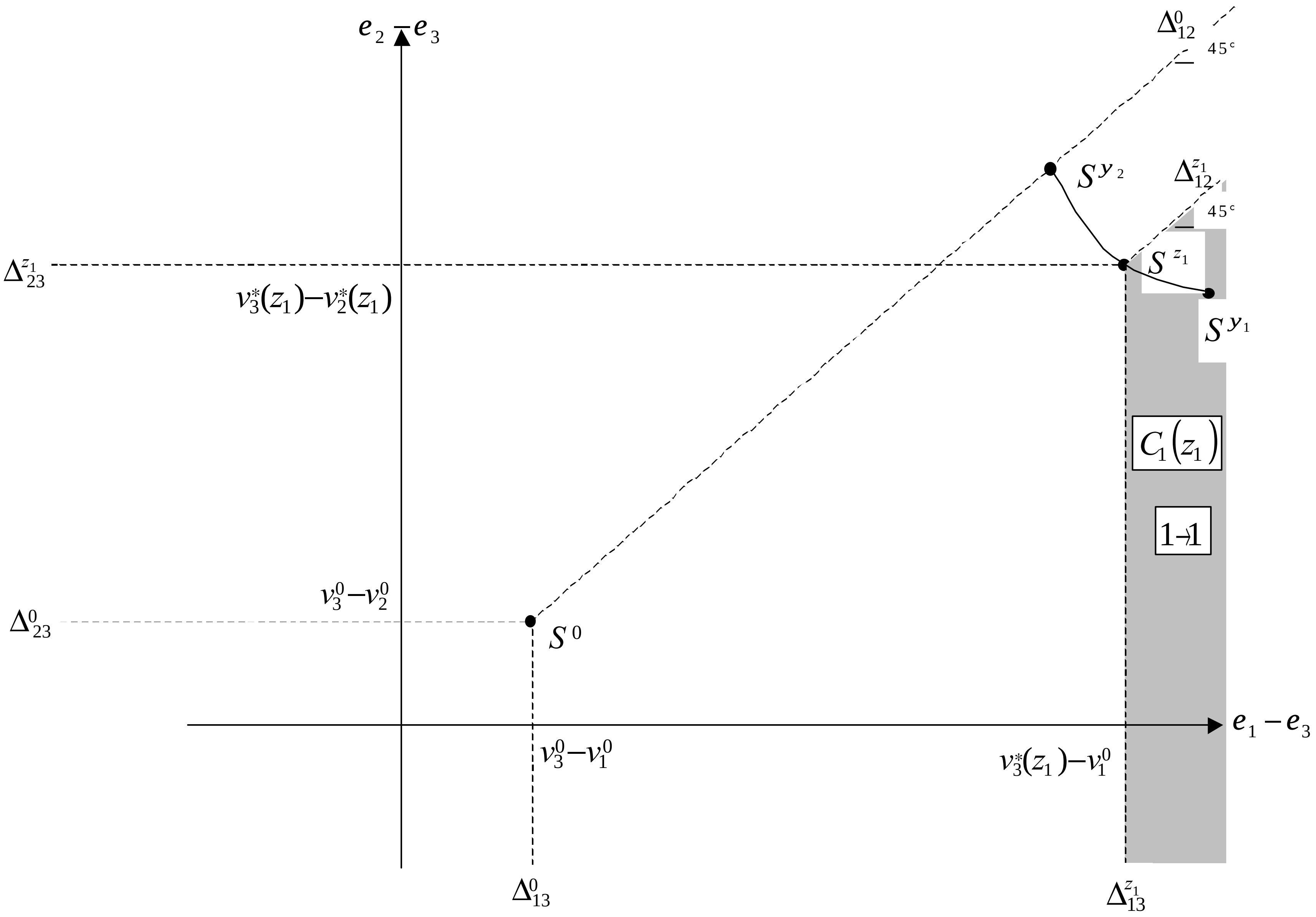
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$v_3^*(z_1) - v_2^*(z_1)$
S^{y_1}
$C_1(z_1)$
$1 \to 1$
$v_3^0 - v_2^0$
Δ_{23}^{0}
S^0
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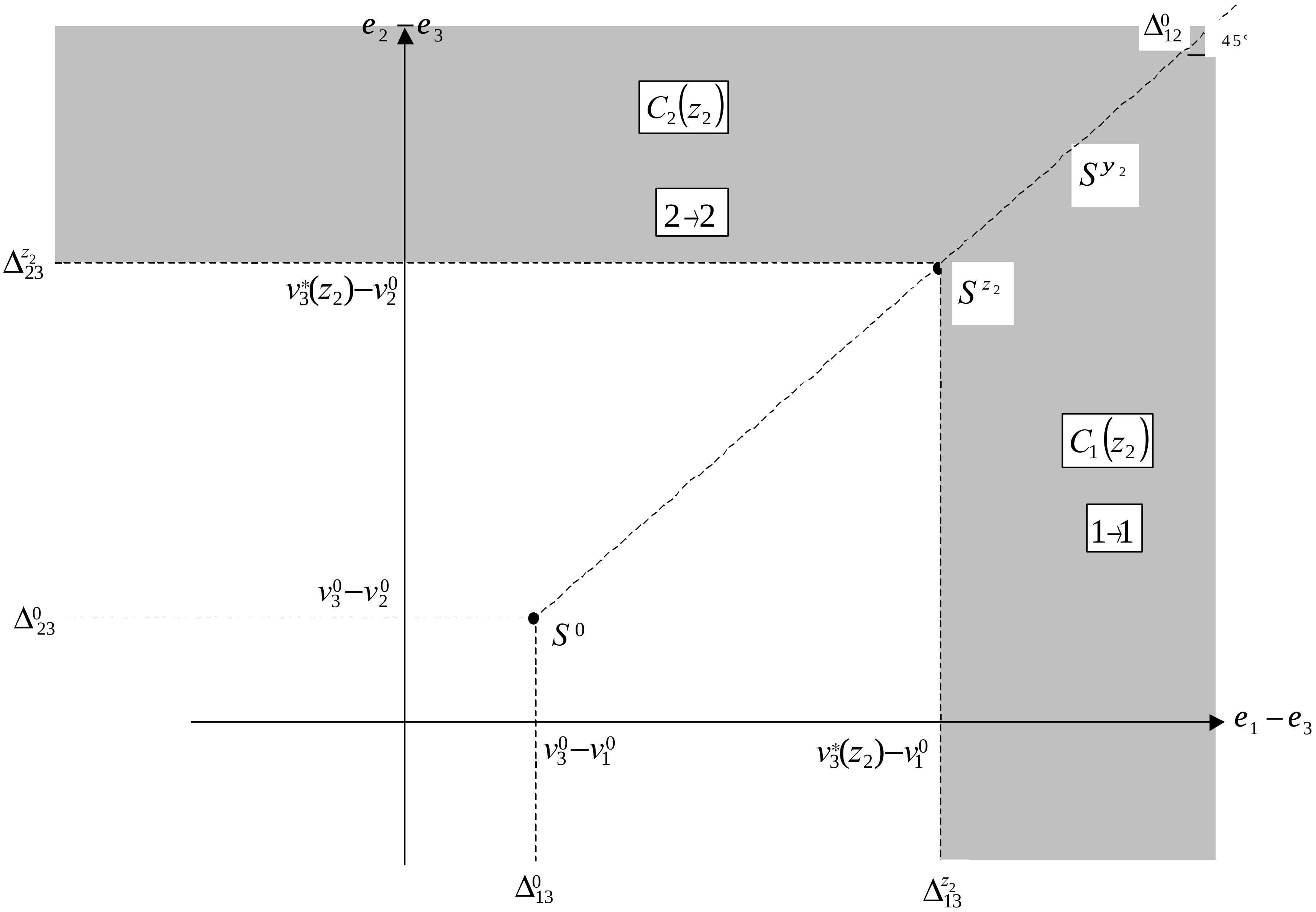
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S^{z_2}
$C_1(z_2)$
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$v_3^0 - v_2^0$
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$e_1 - e_3$
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$v_3^*(z_2) - v_1^0$
Δ_{13}^{0}
$\Delta_{13}^{z_2}$