

# The Relation of Time Indexed Formulations of Single Machine Scheduling Problems to the Node Packing Problem

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## Abstract

The relation of time indexed formulations of nonpreemptive single machine scheduling problems to the node packing problem is established and then used to provide simple and intuitive alternate proofs of validity and maximality for previously known results on the facial structure of the scheduling problem. Previous work on the facial structure has focused on describing the convex hull of the set of feasible partial schedules, i.e. schedules in which not all jobs have to be started. The equivalence between the characteristic vectors of this set and those of the set of feasible node packings in a graph whose structure is determined by the parameters of the scheduling problem is established. The main contribution of this paper is to show that the facet inducing inequalities for the convex hull of the set of feasible partial schedules that have integral coefficients and right hand side 1 or 2 are the maximal clique inequalities and the maximally and sequentially lifted 5-hole inequalities of the convex hull of the set of feasible node packings in this graph respectively.

**Key words:** scheduling, node packing, polyhedral methods, facet defining graphs, lifted valid inequalities, facet inducing inequalities.

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# 1 Introduction

In this paper the connection between two seemingly unrelated combinatorial optimization problems, namely, the nonpreemptive single machine scheduling problem (SMSP) and the node packing problem (NPP), is considered. More specifically, the relation of time indexed formulations of SMSP to NPP is established and then used to provide simple and intuitive alternate proofs of validity and maximality for previously known results on the facial structure of time indexed formulations of SMSPs. The dependency of the facial structure on the input data makes the study of scheduling polyhedra difficult. Therefore, the ability to use data independent techniques from combinatorial optimization problems such as NPP to identify strong valid inequalities for SMSP highlights the importance of establishing such relations.

Polyhedral approaches to machine scheduling problems have largely focused on SMSP. Balas (1985) pioneered the study of scheduling polyhedra with his work on the facial structure of the job shop scheduling problem on a single machine. Since then many formulations of different variants of SMSP have been investigated. The reader is referred to Queyranne and Schulz (1994) for a comprehensive survey on polyhedral approaches to machine scheduling.

The time indexed formulation of SMSP considered in this paper was first investigated by Sousa and Wolsey (1992). They introduced three classes of valid inequalities that have integral coefficients. The first class consists of facet inducing inequalities that have right hand side 1. The second and third classes consist of valid inequalities that have right hand side  $b \in \{2, \dots, n-1\}$ . Van den Akker, Van Hoesel and Savelsbergh (1999) completely characterized all of the facet inducing inequalities that have integral coefficients and right hand side 1 or 2. The work of these authors on the facial structure of a time indexed formulation of SMSP has focused on describing the convex hull of the set of feasible partial schedules, i.e. schedules in which not all jobs have to be started. Conditions are given under which these facet inducing inequalities are also facet inducing for the convex hull of the set of feasible complete schedules, i.e. schedules in which all jobs have to be started. Crama and Spieksma (1996) studied the convex hull of the set of feasible complete schedules for problems in which the jobs have equal processing times. They independently characterized all of the facet inducing inequalities that have right hand side 1 and presented two other classes of facet inducing inequalities that have right hand side  $b \in \{2, \dots, n-1\}$ .

The problem of finding a feasible partial schedule to SMSP is a special case of the job interval scheduling (or selection) problem (JISP). JISP is a simple and generally applicable scheduling model that is a natural generalization of NPP on interval graphs. The approximability of such problems has been studied by Spieksma (1999).

NPP is among the most studied combinatorial optimization problems. Much of the polyhedral research related to NPP has focused on identifying facet defining graphs (see, e.g. Padberg, 1977). A fundamental class of facet defining graphs is the class of rank minimal graphs, which includes cliques, odd holes (Padberg, 1973), and odd antiholes (Nemhauser and Trotter, 1974). Typically,

valid inequalities for the convex hull of the set of feasible node packings in general graphs are of low dimension and do not induce facets. The idea of lifting valid inequalities to obtain higher dimensional valid inequalities was introduced by Gomory (1969) in the context of the group problem and has since been generalized by many authors. Padberg (1973) and Nemhauser and Trotter (1974) describe the lifting of valid inequalities in the context of NPP. Computationally, node packing relaxations have been used successfully to derive cutting planes for many integer programming (IP) problems (see, e.g. Hoffman and Padberg, 1993; Bixby and Lee, 1998; Atamtürk, Nemhauser and Savelsbergh, 2000; Borndörfer and Weismantel, 2000). Many commercial IP solvers in use today derive a node packing relaxation of the IP being solved and generate valid clique inequalities (see, e.g. ILOG, Inc., 2001; Dash Associates, 2001).

The remainder of this paper is organized as follows. In Section 2 a time indexed formulation of SMSP is presented and known results pertaining to facet inducing inequalities for the convex hull of the set of feasible partial schedules that have integral coefficients and right hand side 1 or 2 are reviewed. In Section 3 rank inequalities for the convex hull of the set of feasible node packings in a graph with right hand side 1 or 2 are introduced, sequential lifting of low dimensional valid inequalities to higher dimensions in the context of NPP is reviewed, and interval graphs and claw free graphs are discussed. In Section 4 the notion of the intersection graph of a JISP is defined and several important properties are given. Section 5 contains the contributions of this paper. The problem of finding a feasible partial schedule to SMSP is expressed as the problem of finding a feasible node packing in the intersection graph of the corresponding JISP whose structure is determined by the parameters of SMSP. The facet inducing inequalities of the convex hull of the set of feasible partial schedules that have integral coefficients and right hand side 1 or 2 are shown to be the maximal clique inequalities and the maximally and sequentially lifted 5-hole inequalities of the convex hull of the set of feasible node packings in this graph respectively. It is shown that if a 5-hole exists in this graph, then there exists a 5-hole that has one of two minimal structures. The parameters of SMSP for which these minimal structures exist are characterized. These results further demonstrate the value of efficient and effective techniques for deriving and evaluating node packing relaxations in the solution of difficult IPs. In Section 6 concluding remarks are given.

## 2 A time indexed formulation of single machine scheduling problems

An instance of SMSP consists of  $n$  jobs and a processing time  $p_j$  for each job  $j \in \{1, \dots, n\}$ . A schedule is a set of starting times, or equivalently completion times, for the jobs. The problem is to find a schedule such that each job  $j$  receives uninterrupted processing for a period of length  $p_j$  and the machine processes at most one job at a time.

Time indexed formulations of SMSP consider a planning horizon of length  $T$  that is discretized into the periods  $1, \dots, T$ . Period  $t$  starts at time  $t-1$  and ends at time  $t$ . Let  $J = \{1, \dots, n\}$  denote the index set of jobs. Let the interval  $[a, b]$  denote the set of periods  $\{a+1, \dots, b\} \cap \{1, \dots, T\}$ . Note that  $[a, b] = \emptyset$  if  $a \geq b$ . Let  $T_j = [0, T - p_j + 1]$  for each  $j \in J$ . It is assumed that the problem data is integer,  $p_j \geq 1$  for all  $j \in J$ , and

$$T \geq \sum_{j \in J} p_j.$$

Let

$$n^* = \sum_{j \in J} (T - p_j + 1).$$

The time indexed formulation of SMSP is a 0-1 IP with variables indexed by a job-period pair  $(j, t)$  where  $j \in J$  and  $t \in T_j$ . The variable  $x_{jt} = 1$  indicates that job  $j$  is started in period  $t$  and  $x_{jt} = 0$  otherwise. The 0-1 IP is the following.

$$\min \sum_{j \in J} \sum_{t \in T_j} c_{jt} x_{jt} \quad (1a)$$

$$\text{s.t.} \quad \sum_{j \in J} \sum_{s \in [t-p_j, t]} x_{js} \leq 1, \quad t \in [0, T] \quad (1b)$$

$$\sum_{t \in T_j} x_{jt} = 1, \quad j \in J \quad (1c)$$

$$x_{jt} \geq 0, \quad j \in J \quad (1d)$$

$$x_{jt} \text{ integer}, \quad j \in J. \quad (1e)$$

The formulation (1) can be used to model different variants of SMSP. All standard minsum optimality criteria are linear in the time indexed variables and can be formulated using an appropriate choice of objective coefficients in the objective function (1a). The machine constraints (1b) ensure that at most one job can be processed on the machine at a time and that each job  $j$  receives uninterrupted processing for a period of length  $p_j$ . The job constraints (1c) ensure that each job  $j$  is processed exactly once. Release dates and deadlines can be modeled by restricting the set of variables. Note that (1) has a pseudo polynomial number of variables as the planning horizon length  $T$  is not necessarily bounded by a polynomial function in  $n$ .

Many SMSP that can be modeled by (1) are strongly  $\mathcal{NP}$ -hard (see, e.g. Lawler, Lenstra, Rinnooy Kan and Shmoys, 1993). Therefore, the time indexed formulations of these SMSP are also strongly  $\mathcal{NP}$ -hard, even if the planning horizon length  $T$  is polynomially bounded. In fact, Crama and Spieksma (1996) have shown that a time indexed formulation of SMSP is strongly  $\mathcal{NP}$ -hard, even if the processing times  $p_j = 2$  for all  $j \in J$ , the objective coefficients  $c_{jt} \in \{0, 1\}$  for all  $j \in J$  and  $t \in T_j$ , and the planning horizon length  $T$  is part of the input.

A characteristic vector  $x$  of (1) indicates a feasible complete schedule, i.e. a schedule in which each job has to be processed exactly once. Let the set

$$X_{\text{CS}} = P_{\text{CS}} \cap \mathbb{Z}^{n*}$$

denote the set of feasible complete schedules where

$$P_{\text{CS}} = \{x \in \mathbb{R}_+^{n*} : (1b) \text{ and } (1c)\}.$$

The convex hull  $\text{conv}(X_{\text{CS}})$  of  $X_{\text{CS}}$  is not full dimensional. If

$$T \geq \sum_{j \in J} p_j + p_{\max}$$

then the dimension

$$\dim(\text{conv}(X_{\text{CS}})) = nT - \sum_{j \in J} p_j$$

where  $p_{\max} = \max\{p_j : j \in J\}$  (Sousa and Wolsey, 1992). To study the facial structure of (1) the convex hull  $\text{conv}(X_{\text{PS}})$  of the set  $X_{\text{PS}}$  of all feasible partial schedules, i.e. schedules in which each job  $j$  can be processed at most once, is considered. Feasible partial schedules arise when the constraint (1c), that each job  $j$  is processed exactly once, is relaxed to

$$\sum_{t \in T_j} x_{jt} \leq 1, \quad j \in J, \tag{1c'}$$

that each job is processed at most once. Let the set

$$X_{\text{PS}} = P_{\text{PS}} \cap \mathbb{Z}^{n*}$$

denote the set of all feasible partial schedules where

$$P_{\text{PS}} = \{x \in \mathbb{R}_+^{n*} : (1b) \text{ and } (1c')\}.$$

This relaxation has the advantage that  $\text{conv}(X_{\text{PS}})$  is full dimensional as it contains all of the unit vectors and the origin which are affinely independent. Since  $\text{conv}(X_{\text{CS}})$  is a face of  $\text{conv}(X_{\text{PS}})$ , a valid inequality for  $\text{conv}(X_{\text{PS}})$  is also valid for  $\text{conv}(X_{\text{CS}})$ .

Before describing any facet inducing inequalities for  $\text{conv}(X_{\text{PS}})$  some notation is introduced. Let the set  $S$  denote the index set of variables with nonzero coefficients in an inequality. The set of variables with nonzero coefficients in an inequality associated with job  $j \in J$  defines a set  $S_j = \{t : (j, t) \in S\}$  of time periods. Let  $l_j = \min\{t : t - p_j + 1 \in S_j\}$  and  $u_j = \max\{t : t \in S_j\}$ . If  $S_j = \emptyset$  let  $l_j = \infty$  and  $u_j = -\infty$ . If  $S_j \neq \emptyset$  then  $l_j$  is the first period in which job  $j$  can be finished if it is started in  $S$  and  $u_j$  is the last period in which job  $j$  can be started in  $S$ . Let  $l = \min\{l_j : j \in J\}$  and  $u = \max\{u_j : j \in J\}$ . Finally, let  $x(S)$  denote

$$\sum_{j \in J} \sum_{t \in S_j} x_{jt}.$$

Unless otherwise indicated the remainder of this section is a summary of the results presented in Van den Akker et al. (1999).

A result of Hammer, Johnson and Peled (1975) for down monotone 0-1 polytopes implies that all facet inducing inequalities with right hand side 0 for  $\text{conv}(X_{\text{PS}})$  have the form  $x_{jt} \geq 0$  where  $j \in J$  and  $t \in T_j$ . These inequalities are facet inducing for  $\text{conv}(X_{\text{CS}})$  if

$$T \geq \sum_{j \in J} p_j + p_{\max}.$$

A further consequence of the result of Hammer et al. is that valid inequalities that have integral coefficients and right hand side 1 are of the form  $x(S) \leq 1$  and valid inequalities that have integral coefficients and right hand side 2 are of the form  $x(S^1) + 2x(S^2) \leq 2$  where  $S = S^1 \cup S^2$  and  $S^1 \cap S^2 = \emptyset$ .

## 2.1 Facet inducing inequalities $x(S) \leq 1$ for $\text{conv}(X_{\text{PS}})$

A facet inducing inequality  $x(S) \leq 1$  for  $\text{conv}(X_{\text{PS}})$  is determined by one job, which without loss of generality is called job 1, and two time periods  $l$  and  $u$ . The inequality has the structure

$$\begin{aligned} S_1 &= [l - p_1, u], \\ S_j &= [u - p_j, l], \quad j \in J \setminus \{1\} \end{aligned} \tag{2}$$

where  $l = l_1 \leq u_1 = u$ . Note that if  $l = u$  the inequalities with structure (2) coincide with the inequalities (1b). If  $l = p_1$ ,  $u = T - p_1 + 1$  and  $S_j = \emptyset$  for all  $j \in J \setminus \{1\}$  the inequalities with structure (2) coincide with the inequalities (1c').

A valid inequality  $x(S) \leq 1$  is maximal if there does not exist a valid inequality  $x(W) \leq 1$  with  $S \subset W$  where  $S$  is a proper subset of  $W$ . If

$$T \geq \sum_{j \in J} p_j + 3p_{\max}$$

then a valid inequality  $x(S) \leq 1$  with structure (2) that is maximal is a facet inducing inequality for  $\text{conv}(X_{\text{CS}})$  (Sousa and Wolsey, 1992). The number of facet inducing inequalities with structure (2) is polynomial in the size of the formulation.

## 2.2 Facet inducing inequalities $x(S^1) + 2x(S^2) \leq 2$ for $\text{conv}(X_{\text{PS}})$

For a valid inequality  $x(S^1) + 2x(S^2) \leq 2$ , at most two jobs can be started in  $S = S^1 \cup S^2$ . If a job  $j \in J$  is started in period  $t \in S_j$  either

- (i) it is impossible to start any job in  $S$  before job  $j$  and at most one job can be started in  $S$  after job  $j$ ; or
- (ii) there exists a job  $i \in J \setminus \{j\}$  such that job  $i$  can be started in  $S$  before or after job  $j$  and any job  $k \in J \setminus \{i, j\}$  cannot be started in  $S$ ; or

- (iii) at most one job can be started in  $S$  before job  $j$  and it is impossible to start any job in  $S$  after job  $j$ .

Thus, the set  $S$  can be written as  $S = L \cup M \cup U$  where  $L \subseteq S$  is the set of variables for which case (i) holds,  $M \subseteq S$  is the set of variables for which case (ii) holds and  $U \subseteq S$  is the set of variables for which case (iii) holds. Similarly, the sets  $S_j$  can be written as  $S_j = L_j \cup M_j \cup U_j$  although each of the intervals  $L_j$ ,  $M_j$  and  $U_j$  may be empty. If job  $j$  is started in a period in  $S_j^2$  then it is impossible to start any job in  $S$  before or after job  $j$ . Thus  $S_j^2 \subseteq L_j \cap U_j$  for all  $j \in J$  and so  $S^2 \subseteq L \cap U$ . By definition  $L_j \cap M_j = \emptyset$  and  $M_j \cap U_j = \emptyset$ . If  $L_j \cap U_j \neq \emptyset$  then  $M_j = \emptyset$ . A valid inequality  $x(S^1) + 2x(S^2) \leq 2$  is nondecomposable if it cannot be written as the sum of two valid inequalities  $x(W) \leq 1$  and  $x(W') \leq 1$ . A valid inequality  $x(S^1) + 2x(S^2) \leq 2$  is maximal if there does not exist a valid inequality  $x(W^1) + 2x(W^2) \leq 2$  with  $S^1 \subseteq (W^1 \cup W^2)$ ,  $S^2 \subseteq W^2$  and  $S^1 \neq W^1$  or  $S^2 \neq W^2$ .

A facet inducing inequality  $x(S^1) + 2x(S^2) \leq 2$  for  $\text{conv}(X_{\text{PS}})$  is determined by at most three jobs and six time periods. The inequality can be represented by a collection of sets  $L_j$ ,  $M_j$  and  $U_j$  which are given in Table 1. Three cases have to be considered.

- (1a) Let  $l^* = \min\{l_j : j \in J \setminus \{1, 2\}\}$  and  $u^* = \max\{u_j : j \in J \setminus \{1, 2\}\}$ . A facet inducing inequality  $x(S^1) + 2x(S^2) \leq 2$  with  $l = l_1 < l_2 \leq l^*$  and  $u = u_1 > u_2 \geq u^*$  having the structure given in Table 1, where  $[u_2 - p_j, l] \subseteq L_j$  and  $[u - p_j, l_2] \subseteq U_j$  for all  $j \in J \setminus \{1, 2\}$ , is nondecomposable if and only if  $M_j \neq \emptyset$  for some  $j \in J$  and  $l^* < u_2$  or  $l_2 < u^*$ .
- (1b) Let  $l^* = \min\{l_j : j \in J \setminus \{1, 2\}\}$  and  $u^* = \max\{u_j : j \in J \setminus \{1, 3\}\}$ . A facet inducing inequality  $x(S^1) + 2x(S^2) \leq 2$  for  $\text{conv}(X_{\text{PS}})$  with  $l = l_1 < l_2 \leq l^*$ ,  $u = u_1 > u_3 \geq u^*$  and  $l_2 < l^*$  or  $u_3 > u^*$  having the structure given in Table 1 is nondecomposable if and only if  $M_j \neq \emptyset$  for some  $j \in J \setminus \{1\}$ .
- (2) Let  $l' = \min\{l_j : j \in J \setminus \{1, 2\}\}$  and  $u' = \max\{u_j : j \in J \setminus \{1, 2\}\}$ . These parameters differ from  $l^*$  and  $u^*$  as it is possible that  $l' < l_2$  or  $u' > u_1$ . A facet inducing inequality  $x(S^1) + 2x(S^2) \leq 2$  for  $\text{conv}(X_{\text{PS}})$  with  $l = l_1$  and  $u = u_2$  having the structure given in Table 1, where  $[l' - p_2, l] \subseteq L_2$  and  $[u - p_1, u'] \subseteq U_1$ , is nondecomposable if and only if  $M_1 \neq \emptyset$  or  $M_2 \neq \emptyset$  and  $l' < u_1$  or  $l_2 < u'$ .

For conditions of maximality the reader is referred to Van den Akker (1994).  
If

$$T \geq \sum_{j \in J} p_j + 5p_{\max}$$

then a valid inequality  $x(S^1) + 2x(S^2) \leq 2$  with one of the structures given in Table 1 that is nondecomposable and maximal is facet inducing for  $\text{conv}(X_{\text{CS}})$ . The number of facet inducing inequalities for  $\text{conv}(X_{\text{PS}})$  with one of these structures is polynomial in the size of the formulation.

Case	Structure		
(1a)	$L_1 = [l - p_1, l_2],$ $L_2 = [l_2 - p_2, l],$ $L_j = [l^* - p_j, l],$	$M_1 = [u^* - p_1, l^*] \setminus (L_1 \cup U_1),$ $M_2 = [\max\{u^*, l_2\} - p_2, \min\{l^*, u_2\}] \setminus (L_2 \cup U_2),$ $M_j = [u_2 - p_j, l_2] \setminus (L_j \cup U_j),$	$U_1 = [u_2 - p_1, u],$ $U_2 = [u - p_2, u_2],$ $U_j = [u - p_j, u^*],$
(1b)	$L_1 = [l - p_1, l_2],$ $L_2 = [l_2 - p_2, l],$ $L_3 = [l^* - p_3, l],$ $L_j = [l^* - p_j, l],$	$M_1 = \emptyset,$ $M_2 = [u_3 - p_2, l^*] \setminus (L_2 \cup U_2),$ $M_3 = [u^* - p_3, l_2] \setminus (L_3 \cup U_3),$ $M_j = [u_3 - p_j, l_2] \setminus (L_j \cup U_j),$	$U_1 = [u_3 - p_1, u],$ $U_2 = [u - p_2, u^*],$ $U_3 = [u - p_3, u_3],$ $U_j = [u - p_j, u^*],$
(2)	$L_1 = [l - p_1, \min\{l_2, l'\}],$ $L_2 = [l_2 - p_2, l],$ $L_j = [l' - p_j, l],$	$M_1 = [u' - p_1, \min\{l', u_1\}] \setminus (L_1 \cup U_1),$ $M_2 = [\max\{u', l_2\} - p_2, l'] \setminus (L_2 \cup U_2),$ $M_j = \emptyset,$	$U_1 = [u - p_1, u_1],$ $U_2 = [\max\{u_1, u'\} - p_2, u],$ $U_j = [u - p_j, u'],$

$\infty$

Table 1: Structure of facet inducing inequalities  $x(S^1) + 2x(S^2) \leq 2$  for  $\text{conv}(X_{\text{Ps}})$ .

### 3 Node packing problems

An instance of NPP consists of a simple undirected graph  $G = (V, E)$  and a weight function

$$w : V \rightarrow \mathbb{R}^{|V|}.$$

A node packing in the graph  $G$  is a set  $U \subseteq V$  of nodes that are not pairwise adjacent. The problem is to find a node packing in  $G$  that has maximum weight. The convex hull  $\text{conv}(X_{\text{NP}})$  of the set  $X_{\text{NP}}$  of feasible node packings in  $G$  is full dimensional as it contains the unit vectors and the origin which are affinely independent. Since there is a one to one correspondence between nodes of  $G$  and the characteristic vectors of  $X_{\text{NP}}$ , a distinction is not made.

Let  $G(S)$  denote the subgraph of  $G$  induced by the set  $S \subseteq V$  of nodes. If an inequality has nonzero support  $S$  then the graph  $G(S)$  is said to produce the inequality. If an inequality is facet inducing for  $\text{conv}(X_{\text{NP}})$ , then the inequality is said to be facet inducing for  $G$ . A subgraph  $G(S)$  is facet defining if it produces a facet inducing inequality for  $G(S)$ . Although there are many facet defining graphs for NPP, for our purposes it is sufficient to consider cliques and odd antiholes.

A complete graph is a graph in which all of the nodes are pairwise adjacent. A clique is a maximal complete graph. If  $G(S)$  induces a clique in  $G$ , then the clique inequality  $x(S) \leq 1$  is a facet inducing inequality for  $G(S)$ . Furthermore,  $x(S) \leq 1$  is facet inducing for  $G$  when  $S$  is maximal with respect to set inclusion.

A  $k$ -hole is a chordless cycle induced by a set of nodes of cardinality  $k$ , where  $k \geq 4$ . A hole is odd for odd values of  $k$ . A  $k$ -antihole is the complement of a  $k$ -hole. Note that a 5-antihole is isomorphic to a 5-hole, so 5-holes are both odd holes and odd antiholes. If  $G(S)$  induces an odd  $k$ -antihole in  $G$ , then the odd antihole inequality  $x(S) \leq 2$  is a facet inducing inequality for  $G(S)$ . However,  $x(S) \leq 2$  is unlikely to induce a facet of  $G$ . Using a technique called lifting a higher dimensional lifted odd antihole inequality  $x(S^1) + 2x(S^2) \leq 2$  can be obtained where  $S \subseteq S^1$  and  $S^1 \cap S^2 = \emptyset$ .

The stability number  $\alpha(G)$  is the cardinality of a maximum node packing in  $G$ . Let  $\alpha(S)$  denote the stability number of  $G(S)$ . A graph  $G$  is rank minimal if and only if  $G$  is a clique or the rank inequality  $x(V) \leq \alpha(G)$  is facet inducing for  $G$  and, for each  $S \subset V$  the rank inequality  $x(S) \leq \alpha(S)$  does not induce a facet of  $G(S)$ . A graph  $G(S)$  with  $\alpha(S) = 1$  is rank minimal if and only if it is a maximal clique (Padberg, 1973). A graph  $G(S)$  with  $\alpha(S) = 2$  is rank minimal if and only if it is an odd antihole (Nemhauser and Trotter, 1974). Every rank facet producing graph contains an induced rank minimal graph with the same stability number.

Typically, valid inequalities for the convex hull of the set of feasible node packings in general graphs are of low dimension and do not induce facets. Sequential lifting can often be used to obtain higher dimensional valid inequalities. For the NPP, an instance of the sequential lifting problem for a valid inequality  $x(S^1) + 2x(S^2) + \dots + bx(S^b) \leq b$  for  $G$ , consists of the sets  $S^0, \dots, S^b$  and a

node  $v_r \in V \setminus (V(S^0) \cup \dots \cup V(S^b))$  where  $S^i \cap S^j = \emptyset$  for all  $0 \leq i < j \leq b$ . Note that the sets  $S^0, \dots, S^b$  may be empty. The lifting problem is to determine  $\gamma$  where

$$\gamma = \max\{x(S^1) + 2x(S^2) + \dots + bx(S^b) : x_r = 1, x \in X_{\text{NP}}\}.$$

The inequality  $a_r x_r + x(S^1) + 2x(S^2) + \dots + bx(S^b) \leq b$  is valid for  $G$  for any  $a_r \leq b - \gamma$ . If  $a_r = b - \gamma$  then the lifting is said to be maximum. The set  $S^{a_r}$  is updated to include the index  $r$ . A lifted inequality  $x(S^1) + 2x(S^2) + \dots + bx(S^b) \leq b$  where  $V(S^0) \cup \dots \cup V(S^b) = V$  induces a facet of  $G$  if the lifting is maximum. The coefficients  $a_r$  are dependent on the order in which the variables are lifted. Thus, a family of lifted inequalities can be obtained by considering different orderings of the nodes in  $V \setminus (V(S^0) \cup \dots \cup V(S^b))$ . For more on lifting see, e.g. Nemhauser and Wolsey (1988).

A graph  $G_I$  is an interval graph if there exists a family  $I$  of intervals in  $\mathbb{R}$  and a one to one correspondence between the nodes of  $G_I$  and the intervals in  $I$  such that two nodes of  $G_I$  are adjacent if and only if the corresponding intervals have a non empty intersection. Two important properties of interval graphs are that every induced subgraph of an interval graph is an interval graph and that every interval graph contains a simplicial node whose neighborhood, i.e. the set of adjacent nodes, induces a clique. For more on graph theory see, e.g. Golumbic (1980).

A claw is a graph with the set  $\{v_1, v_2, v_3, v_4\}$  of nodes and the set  $\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}\}$  of edges. A graph is said to be claw free if it does not contain a claw as an induced subgraph. A claw free graph is rank facet producing if and only if it can be built up from rank minimal graphs by iterating sequential lifting and complete join operations (Galluccio and Sassano, 1997). A facet inducing graph  $G$  is said to be obtained from one of its induced subgraphs  $G(S)$  by sequential lifting if the rank inequality  $x(V) \leq \alpha(G)$  is obtained by sequential lifting from the rank inequality  $x(S) \leq \alpha(S)$  where  $\alpha(S) = \alpha(G)$ . The complete join of two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  is the graph  $G = (V, E)$  with  $V = V_1 \cup V_2$  and  $E = E_1 \cup E_2 \cup \{\{v_1, v_2\} : v_1 \in V_1, v_2 \in V_2\}$ . To characterize all of the rank facet inducing inequalities of a claw free graph it is sufficient to characterize all of the rank minimal claw free graphs. Thus, a graph  $G(S)$  with  $\alpha(S) = 2$  is rank facet inducing if and only if it can be built up from odd antiholes by iterating sequential lifting and complete join operations.

## 4 Job interval scheduling problems

An instance of JISP consists of  $n$  jobs and a set  $I_j$  of intervals in  $\mathbb{R}$  for each job  $j \in \{1, \dots, n\}$ . A schedule is a set of intervals. The problem is to find a schedule such that at most one interval is selected for each job  $j$  and no two selected intervals intersect. This problem can be formulated as a NPP on a graph  $G$ . The graph  $G$  is constructed by associating a node with each interval. Two nodes in  $G$  are connected if the corresponding intervals are associated with

the same job or if the corresponding intervals intersect. A schedule is a node packing in  $G$ . Thus, a feasible node packing in  $G$  is a feasible schedule to JISP.

The exact structure of  $G$  is determined by the parameters of JISP. Nevertheless,  $G$  has an important basic property, namely, that the set  $E$  of edges can be decomposed as follows. Let  $E_K \subseteq E$  denote those edges of  $G$  that correspond to two intervals of the same job in the JISP. Let  $E_I \subseteq E$  denote those edges of  $G$  that correspond to the intersection of two intervals of the JISP. Note that  $E = E_K \cup E_I$  but that  $E_K \cap E_I \neq \emptyset$ . The graph  $G_K = (V, E_K)$  is composed of disjoint cliques. The graph  $G_I = (V, E_I)$  is an interval graph. Any graph with the above structure is the intersection graph of a JISP. Let such a graph be called a JISP graph.

The following claim identifies some important properties of a JISP graph.

**Lemma 1.** *A JISP graph has the following properties.*

1. *Every induced subgraph of a JISP graph is a JISP graph.*
2. *Every JISP graph contains a node whose neighborhood induces the union of at most two cliques.*
3. *A 5-hole is the only odd antihole that is a JISP graph.*
4. *The complete join of two 5-holes is not a JISP graph.*

*Proof.* Properties 1 and 2 follow from the definition of a JISP graph and the properties of interval graphs. Properties 3 and 4 are a consequence of property 2.  $\square$

## 5 A node packing relaxation of single machine scheduling problems

The problem of finding a feasible partial schedule to SMSP is a special case of JISP. With each potential starting time  $t \in T_j$  of each job  $j \in J$  we simply associate the interval  $[t, t + p_j - 1]$ . The problem can then be formulated as a NPP on the corresponding JISP graph  $G$ . Recall that  $G$  is constructed by associating a node with each interval. Two nodes in  $G$  are connected if the corresponding intervals are associated with the same job or if the corresponding intervals intersect. The exact structure of  $G$  is determined by the parameters of SMSP. A schedule to JISP is a node packing in  $G$ . Similarly, a feasible node packing in  $G$  is a feasible schedule to JISP and, therefore, a feasible partial schedule to SMSP.

The remainder of this text contains the contributions of this paper. Let the graph  $G = (V, E)$  denote the JISP graph arising from the problem of finding a feasible partial schedule to SMSP. Note that  $G$  is simply the column intersection graph of the  $(0, 1)$  clique-node incidence matrix arising from the constraints (1b) and (1c') of SMSP (see, e.g. Nemhauser and Wolsey, 1988). The interval graph  $G_I$  is the column intersection graph of the  $(0, 1)$  clique-node matrix arising from

the constraints (1b). The graph  $G_K$  of disjoint cliques is the column intersection graph of the  $(0, 1)$  clique-node matrix arising from the constraints (1c'). Since there is a one to one correspondence between job-period pairs, the variables of (1) and the nodes of  $G$ , a distinction is not made. For convenience, let  $V_j$  denote the set  $V(\{(j, t) : t \in T_j\})$  of nodes associated with the job  $j \in J$ .

In Section 5.1 it is shown that the facet inducing inequalities for  $G$  that have integral coefficients and right hand side 1 or 2 are the maximal clique inequalities and the maximally and sequentially lifted 5-hole inequalities respectively. The characterization of these facet inducing inequalities in terms of rank minimal claw free subgraphs of  $G$  is not as explicit a characterization as that of Van den Akker et al. (1999) but provides simpler and intuitive alternate proofs of validity and maximality. Furthermore, this characterization holds for JISP. In Section 5.2 it is shown that if a 5-hole exists in  $G$ , then there exists a 5-hole which has one of two minimal structures. The parameters of SMSP for which these structures exist are characterized. The characterization of the 5-holes is specific to SMSP.

Before proceeding, it is appropriate to comment on an important computational aspect of polyhedral approaches to solving IPs, namely, the problem of separation. The problem of separation is the problem of identifying a violated inequality in a class of valid inequalities for an IP, that separates the current solution from the set of feasible solutions, or proving that no such inequality exists. Van den Akker (1994) describes heuristic separation algorithms based on clever enumeration of potentially violated facet inducing inequalities for  $G$  that have integer coefficients and right hand side 1 or 2. The algorithms run in time polynomial in the number of fractional variables in the solution of the current linear program and guarantee the identification of a violated inequality should one exist. Separation algorithms that exploit the relation of time indexed formulations of SMSP to NPP have not been investigated.

## 5.1 Facet inducing inequalities for $G$

In this section it is shown that the facet inducing inequalities for the graph  $G$  with integral coefficients and right hand side 1 or 2 are the maximal clique inequalities and the maximally and sequentially lifted 5-hole inequalities respectively.

Consider a facet inducing inequality  $x(S) \leq 1$  for  $G$ . Recall that a graph  $G(S)$  produces a rank facet inducing inequality of  $G$  with right hand side 1 if and only if it is a clique. This proves the following claim.

**Theorem 2.** *A facet inducing inequality  $x(S) \leq 1$  for the graph  $G$  is a maximal clique inequality.  $\square$*

Consider a facet inducing inequality  $x(S^1) + 2x(S^2) \leq 2$  for  $G$ . Let  $S = S^1 \cup S^2$ . Since  $x(S^1) + 2x(S^2) \leq 2$  is valid,  $G(S^2)$  must be a clique. Since  $x(S^1) + 2x(S^2) \leq 2$  induces a facet,  $\alpha(S) = 2$  and  $G(S)$  is, necessarily, claw free. Thus, it must be the case that the graph  $G(S^1)$  is rank facet producing. Therefore, the facet inducing inequality  $x(S^1) + 2x(S^2) \leq 2$  for the graph  $G$

can be obtained from the rank facet inducing inequality  $x(S^1) \leq 2$  for  $G(S^1)$  by sequential lifting in any order.

Recall that a claw free graph is rank facet producing if and only if it can be built up from rank minimal graphs by iterating sequential lifting and complete join operations. However, a graph  $G(W)$  with  $\alpha(W) = 2$  is rank minimal if and only if it is an odd antihole. Thus,  $G(S^1)$  is rank facet producing if and only if it can be built up from an odd antihole. Therefore, by properties 3 and 4 of Lemma 1,  $G(S^1)$  must be built up from a 5-hole by sequential lifting. This proves the main result of this section.

**Theorem 3.** *A facet inducing inequality  $x(S^1) + 2x(S^2) \leq 2$  for the graph  $G$  is a maximally and sequentially lifted 5-hole inequality.*  $\square$

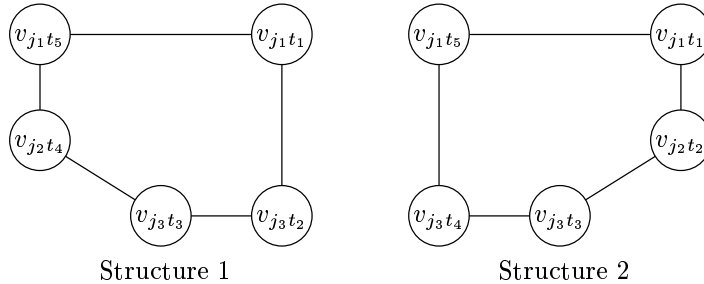
To conclude, the following claim proves that, for a fixed  $S^1$ , the set  $S^2$  is the unique set of nodes that are adjacent to all of the nodes in  $S^1$ .

**Proposition 4.** *There do not exist two facet inducing inequalities  $x(S^1) + 2x(S^2) \leq 2$  and  $x(S^1) + 2x(W^2) \leq 2$  for the graph  $G$  where  $W^2 \neq S^2$ .*

*Proof.* It suffices to show that, for a fixed  $S^1$ , there is a unique set of nodes  $S^2$  such that the inequality  $x(S^1) + 2x(S^2) \leq 2$  is facet inducing for  $G$ . Let  $H^2$  denote those nodes that are adjacent to all of the nodes in  $S^1$ . Thus,  $S^2 \subseteq H^2$ ,  $W^2 \subseteq H^2$  and  $H^2$  is unique. Suppose that there exist two nodes  $v_1, v_2 \in H^2$  that are non adjacent. Consider the graph  $G(W)$  where the set  $W = S^1 \cup \{v_1, v_2\}$ . Property 1 of Lemma 1 implies that  $G(W)$  is a JISP graph. However, no node in  $W$  satisfies Property 2 of Lemma 1 which is a contradiction. Thus, it must be the case that  $G(H^2)$  is a clique. Therefore, it must be the case that  $S^2 = W^2 = H^2$  and so, for a fixed  $S^1$ ,  $S^2$  is the unique set of nodes that are adjacent to all of the nodes in  $S^1$ .  $\square$

## 5.2 Characterization of 5-holes in $G$

In this section it is shown that if a 5-hole exists in the graph  $G$ , then there exists a 5-hole which has one of the following two minimal structures. The parameters of SMSP for which these minimal structures exist are characterized.



A 5-hole  $G(H)$  where  $H = \{(j_1, t_1), \dots, (j_5, t_5)\}$  is determined by jobs  $j_i \in J$  and time periods  $t_i \in T_{j_i}$  for  $i \in \{1, \dots, 5\}$ . The following claim shows that  $H$  is determined by at least three, but no more than four, unique jobs.

**Lemma 5.** *If the graph  $G(H)$  is a 5-hole, then  $3 \leq |\{j \in J : V(H) \cap V_j \neq \emptyset\}| \leq 4$ .*

*Proof.* It is shown that unless  $3 \leq |\{j \in J : V(H) \cap V_j \neq \emptyset\}| \leq 4$  the cycle underlying  $G(H)$  has a chord and therefore cannot be a 5-hole. Let  $G(H)$  be a 5-hole were  $H = \{(j_1, t_1), \dots, (j_5, t_5)\}$  and  $v_{j_1 t_1} - \dots - v_{j_5 t_5} - v_{j_1 t_1}$  is the underlying cycle. Suppose that  $|\{j \in J : V(H) \cap V_j \neq \emptyset\}| \leq 2$ . Then,  $|V(H) \cap V_j| \geq 3$  for some  $j \in J$ . Without loss of generality, assume that  $j_1 = j_3 = j$ . Since  $G_j$  is a clique,  $G(H)$  has the chord  $\{v_{j t_1}, v_{j t_3}\}$ .

Suppose that  $|\{j \in J : V(H) \cap V_j \neq \emptyset\}| = 5$ . Then  $|V(H) \cap V_{j_i}| = 1$  for each  $i \in \{1, \dots, 5\}$ . Without loss of generality, assume that  $t_1 = \min\{t : (j, t) \in H\}$ . Suppose that  $t_2 = \max\{t : (j, t) \in H\}$ . Since  $t_1 \leq t_3 \leq t_2 \leq t_1 + p_{j_1} - 1$ ,  $G(H)$  has the chord  $\{v_{j_1 t_1}, v_{j_3 t_3}\}$ . Similarly, if  $t_5 = \max\{t : (j, t) \in H\}$ , then  $G(H)$  has the chord  $\{v_{j_1 t_1}, v_{j_4 t_4}\}$ . Suppose that  $t_3 = \max\{t : (j, t) \in H\}$ . Then  $t_2 \leq t_3 \leq t_2 + p_{j_2} - 1$  and  $t_4 \leq t_3 \leq t_4 + p_{j_4} - 1$ . Thus,  $1 - p_{j_4} \leq t_4 - t_2 \leq p_{j_2} - 1$  and  $G(H)$  has the chord  $\{v_{j_2 t_2}, v_{j_4 t_4}\}$ . Similarly, if  $t_4 = \max\{t : v_{j t} \in H\}$ , then  $G(H)$  has the chord  $\{v_{j_3 t_3}, v_{j_5 t_5}\}$ .  $\square$

It follows from Lemma 5 that for some permutation of  $J$ ,  $H$  is determined by jobs  $1, \dots, 4$ . Without loss of generality, the following claim characterizes  $H$ .

**Proposition 6.** *If the graph  $G(H)$  is a 5-hole, then the set  $H = \{(1, t_1), (2, t_2), (j, t_3), (3, t_4), (1, t_5)\}$  where  $j \in \{2, 3, 4\}$  and  $v_{1 t_1} - v_{2 t_2} - v_{j t_3} - v_{3 t_4} - v_{1 t_5} - v_{1 t_1}$  is the underlying cycle. Furthermore,  $t_5 < t_1$  and  $\max\{t_4 + p_3 - 1, t_5 + p_1 - 1\} < t_2$ .*

*Proof.* Without loss of generality, assume that  $j_5 = j_1 = 1$  and that  $t_5 < t_1$ . Furthermore,  $j_2, j_4 \in J \setminus \{1\}$  such that  $j_2 \neq j_4$  else  $G(H)$  contains one or more of the chords  $\{v_{1 t_1}, v_{1 t_4}\}$ ,  $\{v_{1 t_2}, v_{1 t_5}\}$  and  $\{v_{j_2 t_2}, v_{j_4 t_4}\}$ . Thus, assume that  $j_2 = 2$  and  $j_4 = 3$ .

Suppose that  $t_2 \leq t_4$ . Then  $t_2 + p_2 - 1 < \min\{t_4, t_5\}$  else  $G(H)$  contains one or more of the chords  $\{v_{2 t_2}, v_{3 t_4}\}$  and  $\{v_{2 t_2}, v_{1 t_5}\}$ . But  $t_5 < t_1 \leq t_2 + p_2 - 1$  which is a contradiction. Thus,  $\max\{t_4 + p_3 - 1, t_5 + p_1 - 1\} < t_2$ .  $\square$

The following claim shows that if  $H$  is determined by four jobs, then there also exist 5-holes in  $G$  with structure 1 and 2.

**Lemma 7.** *If there exists a 5-hole  $G(H)$ , where the set  $H$  is determined by four jobs, then there also exist 5-holes  $G(H_k)$  with structure  $k$  where  $k \in \{1, 2\}$ .*

*Proof.* If  $G(H)$  is a 5-hole, then it follows from Proposition 6 that  $H = \{(1, t_1), (2, t_2), (4, t_3), (3, t_4), (1, t_5)\}$  and  $v_{1 t_1} - v_{2 t_2} - v_{4 t_3} - v_{3 t_4} - v_{1 t_5} - v_{1 t_1}$  is the underlying cycle. The graph  $G(H_1)$  where  $H_1 = \{(1, t_1), (4, t), (4, t_3), (3, t_4), (1, t_5)\}$  is a 5-hole with structure 1 if  $\max\{t_1 - p_4 + 1, t_4 + p_3, t_5 + p_1\} \leq t \leq t_1 + p_1 - 1$ . Suppose not. Since  $p_1, p_4 \geq 1$ ,  $t_1 - p_4 + 1 \leq t_1 + p_1 - 1$ . If  $t_1 + p_1 - 1 < t_5 + p_1$ , then  $t_1 \leq t_5$  which is a contradiction. If  $t_1 + p_1 - 1 < t_4 + p_3$ , then  $t_4 + p_3 < t_2 \leq t_1 + p_1 - 1$  which is a contradiction. Thus,  $\max\{t_1 - p_4 + 1, t_4 + p_3, t_5 + p_1\} \leq t_1 + p_1 - 1$  and  $G(H_1)$  is a 5-hole with structure 1.

Similarly,  $G(H_2)$  where  $H_2 = \{(1, t_1), (2, t_2), (4, t_3), (4, t), (1, t_5)\}$  is a 5-hole with structure 2 if  $t_5 - p_4 + 1 \leq t \leq \min\{t_5 + p_1 - 1, t_1 - p_4, t_2 - p_4\}$ . Suppose not. Since  $p_1, p_4 \geq 1$ ,  $t_5 - p_4 + 1 \leq t_5 + p_1 - 1$ . If  $t_1 - p_4 < t_5 - p_4 + 1$ , then  $t_1 \leq t_5$  which is a contradiction. If  $t_2 - p_4 < t_5 - p_4 + 1$ , then  $t_2 \leq t_5 \leq t_5 + p_1 - 1$  which is a contradiction. Thus,  $t_5 - p_4 + 1 \leq \min\{t_5 + p_1 - 1, t_1 - p_4, t_2 - p_4\}$  and  $G(H_2)$  is a 5-hole with structure 2 in  $G$ .  $\square$

The following claim shows that if  $H$  is determined by three jobs, then  $G(H)$  has one of the structures 1 or 2.

**Lemma 8.** *A 5-hole  $G(H)$  where the set  $H$  is determined by three jobs has one of the structures 1 or 2.*

*Proof.* It follows from Proposition 6 that  $H = \{(1, t_1), (2, t_2), (j, t_3), (3, t_4), (1, t_5)\}$  where  $j \in \{2, 3\}$  and  $v_{1t_1} - v_{2t_2} - v_{jt_3} - v_{3t_4} - v_{1t_5} - v_{1t_1}$  is the underlying cycle of  $G(H)$ . Suppose that  $t_3 \leq t_4$ . Then  $t_3 \leq t_4 \leq t_3 + p_j - 1 < t_5$  or else  $G(H)$  contains the chord  $\{v_{jt_3}, v_{1t_5}\}$ . If  $j = 3$ , then  $t_5 < t_3 + p_j - 1$  since  $t_5 \leq t_5 + p_1 - 1 < t_2 \leq t_3 + p_j - 1$  which is a contradiction. Thus, if  $t_3 \leq t_4$ , then  $j = 2$  and  $t_3 < t_5 - p_2 + 1$ . Therefore,  $G(H)$  is a 5-hole with structure 1.

Suppose that  $t_2 < t_3$ . Then  $t_2 \leq t_1 + p_1 - 1 < t_3 \leq t_2 + p_2 - 1$  or else  $G(H)$  contains the chord  $\{v_{1t_1}, v_{jt_3}\}$ . If  $j = 2$ , then  $t_2 \leq t_4 + p_3 - 1$  since  $t_3 \leq t_4 + p_3 - 1$  which is a contradiction. Thus, if  $t_2 < t_3$ , then  $j = 3$  and  $t_1 + p_1 - 1 < t_3$ . Therefore,  $G(H)$  is a 5-hole with structure 2.

Suppose that  $t_4 < t_3 \leq t_2$ . Then  $\max\{t_2 - p_j, t_5 + p_1 - 1\} < t_3 \leq \min\{t_1 - p_j, t_4 + p_3 - 1\}$ . If  $j = 2$ , then  $G(H)$  is a 5-hole with structure 1. If  $j = 3$ , then  $G(H)$  is a 5-hole with structure 2.  $\square$

The following claim follows from Proposition 6 and Lemmas 7 and 8.

**Theorem 9.** *If the graph  $G$  contains a 5-hole, then for some permutation of the index set  $J$  there exists a 5-hole with either structure 1 or 2.*  $\square$

In the remainder of this section the parameters of SMSP for which 5-holes with structure 1 or 2 exist are characterized. Consider the assumptions

$$T - p_{j_1} - p_{j_2} - p_{j_3} \geq 0, \quad (3a)$$

$$p_j \geq 1, \quad j \in \{j_1, j_2, j_3\} \quad (3b)$$

on a reduced set  $p = (T, p_{j_1}, p_{j_2}, p_{j_3})$  of parameters of SMSP where  $\{j_1, j_2, j_3\} \subseteq J$ . Let the set

$$\mathcal{X} = \mathcal{P} \cap \mathbb{Z}^4$$

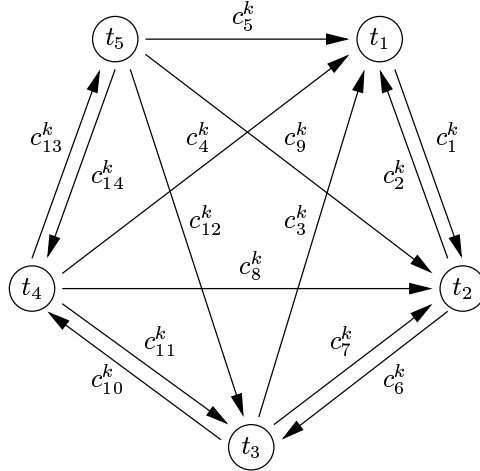
denote the set of parameters satisfying the assumptions where

$$\mathcal{P} = \{p \in \mathbb{R}^4 : (3)\}.$$

Let the vector  $t = (t_1, t_2, t_3, t_4, t_5)$  denote the time periods determining a 5-hole in  $G$ . The necessary conditions on the parameters  $p$  and time periods  $t$  for a 5-hole with structure  $k$  to exist in  $G$  are described by the system of inequalities

$$A^k p + Nt \leq b^k$$

where the matrices  $A^k$ ,  $N$  and  $b^k$  are given in Table 2 for  $k \in \{1, 2\}$ . These necessary conditions are a system of difference constraints in the variables  $t$  and can be represented as a digraph  $D$  with parametric arc costs  $c^k(p) = b^k - A^k p$  for  $k \in \{1, 2\}$  (see, e.g. Ahuja, Magnanti and Orlin, 1993). The digraph  $D$  has a source node  $s$  and five additional nodes  $1, \dots, 5$  corresponding to the variables  $t_1, \dots, t_5$  respectively. For clarity the source node  $s$  has been omitted from the following diagram of  $D$ .



The following claim shows that a 5-hole with structure  $k$  can exist in  $G$  if and only if there are no negative cost directed cycles in  $D$  with the costs  $c^k(p)$  where  $k \in \{1, 2\}$ . The proof is adapted from Ahuja et al. (1993).

**Proposition 10.** *There exists a 5-hole  $G(H)$  with structure  $k$  if and only if there are no negative cost directed cycles in the digraph  $D$  with parametric arc costs  $c^k(p)$  where  $k \in \{1, 2\}$ .*

*Proof.* The necessary conditions for the existence of a 5-hole with structure  $k$  are identical to the optimality conditions for the shortest path problem on  $D$  with costs  $c^k(p)$  where  $k \in \{1, 2\}$ . For each  $j \in \{1, \dots, 5\}$  the variable  $t_j$  denotes the length of some directed path from the source node  $s$  to node  $j$ . Necessary and sufficient conditions for the variables to represent shortest path distances are that  $t_i - t_j \leq c_a^k(p)$  for each arc  $a = (i, j)$  in  $D$ . These conditions are satisfied if and only if  $D$  does not contain a negative cost directed cycle.  $\square$

The necessary conditions on the parameters of SMSP for a 5-hole with structure  $k \in \{1, 2\}$  to exist in  $G$  correspond to inequalities that constrain the sum of the costs on each simple directed cycle in  $D$  with costs  $c^k(p)$  to be nonnegative. Let the set  $\mathcal{X}^H$  denote the set of parameters  $p \in \mathcal{X}$  for which there exists a 5-hole  $G(H)$  with structure  $k \in \{1, 2\}$ . A complete description of the set of parameters  $p$  for which a 5-hole with structure 1 or 2 exists in  $G$  is given in the following remark. These results were determined using PORTA v1.3.2 (Christof and Löbel, 1997). PORTA is a collection of routines for analyzing polytopes and polyhedra.



**Remark 11.** The convex hull  $\text{conv}(\mathcal{X}^H)$  of  $\mathcal{X}^H$  is given by

$$\mathcal{P}^H = \{p \in \mathcal{P} : (4)\}$$

where

$$T - 2p_{j_1} - p_{j_3} \geq 0, \quad (4a)$$

$$p_{j_2} \geq 2. \quad (4b)$$

The polyhedron  $\mathcal{P}^H$  is unbounded and has integer vertices  $(4, 1, 2, 1)$  and  $(5, 2, 2, 1)$ .

There exists a 5-hole in  $G$  if  $p \in \mathcal{X}^H$  for some  $\{j_1, j_2, j_3\} \subseteq J$ . Therefore, to establish that there are no 5-holes in  $G$  it is necessary to verify that  $p \notin \mathcal{X}^H$  for every  $\{j_1, j_2, j_3\} \subseteq J$ .

## 6 Concluding Remarks

Even under mild assumptions on the parameters of SMSP, the graph  $G$  contains a claw. Consequently, the approach taken in Section 5.1, using the characterization of rank minimal claw free graphs of Galluccio and Sassano (1997), cannot be generalized for characterizing facet inducing inequalities for  $G$  with integral coefficients and right hand side greater than 2. However, it seems likely that the results of Section 5.2 can be extended to  $k$ -holes for all odd  $k \geq 5$ . The claim of Lemma 5 can be generalized.

**Lemma 12.** *If the graph  $G(H)$  is a  $k$ -hole, then  $\frac{k+1}{2} \leq |\{j \in J : V(H) \cap V_j \neq \emptyset\}| \leq k - 1$ .*

*Proof.* The proof is similar to that of Lemma 5. □

The claim of Theorem 9 can be extended, assuming that necessary intermediary claims, similar to Proposition 6 and Lemmas 7 and 8, can be proven. The parameters of SMSP for which these structures exist could also be characterized.

It is evident from Lemma 12 that the number of possible  $k$ -hole structures in  $G$  is a function of  $k$ . Without loss of generality, it can be assumed that a  $k$ -hole  $G(H)$  where  $H$  is determined by  $l \in \{\frac{k+1}{2}, \dots, k-1\}$  unique jobs is such that  $|V(H) \cap V_{j_1}| = 2$  and  $|V(H) \cap V_{j_i}| \geq 1$  for  $i \in \{2, \dots, l\}$ . Since  $|V(H) \cap V_{j_i}| \leq 2$  for  $i \in \{2, \dots, l\}$ , the remaining  $k-l-1$  nodes of  $V(H)$  are distributed amongst the sets  $V_{j_i}$  in one of  $\binom{l-1}{k-l-1}$  different ways. Consequently, there are

$$\sum_{l \in \{\frac{k+1}{2}, \dots, k-1\}} \binom{l-1}{k-l-1}$$

different  $k$ -hole structures in  $G$  for each odd  $k \geq 5$ .

Other facet defining graphs discussed in the NPP literature could be investigated. The valid inequalities of Sousa and Wolsey (1992) and Crama and Spieksma (1996) for  $G$  that have integral coefficients and right hand side greater

than 2 must be obtainable, or are otherwise dominated, by lifting valid inequalities for facet defining subgraphs of  $G$ . Many of the facet defining graphs discussed in the NPP literature do not exist due to the structure of  $G$ . This approach seems a promising direction for further research into the facial structure of SMSP.

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