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ORDERING, EQUITY AND NON-EXPECTED UTILITY

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Abstract

According to Chateauneuf (1996), we compare some inequality criteria. We investigate their properties and characteristics. Then, following the approach that links different levels of deprivation with alternative values of social welfare, we look for a functional that evaluates the well-being of individuals with respect to their relative position in the distribution. After having criticized the approach to evaluate the social welfare using the expected utility theory or the rank dependent one respectively under a descriptive and normative point of view, we introduce an additive subjective utility functional in order to search the behavioral foundations of an equity concerned decision-maker.

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JEL Classification: D31, D63, I31.

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1 Introduction

In this work, we compare some inequality criteria and measure the welfare associated to income distributions by means of an additive (subjective) utility function, pointing out the behavioral foundations of an ethical (i.e. equality concerned) decision-maker. The three orderings we consider are the *Lorenz ordering* (LO), the *absolute differentials ordering* (ADO) and the *relative differentials ordering* (RDO).

The Lorenz ordering is the most utilized criterion for studying inequality. It has both a normative and descriptive appeal. Nevertheless, some scholars, like Moyes (1994), recently noted the existence of some situations in which LO is a too weak criterion for ranking different income distributions. Chateauneuf (1996), for example, analyzed some situations where LO gives rise to a sort of “paradoxes” in comparing distributions. Then, in order to solve what he calls “*pathological situations in which LO is involved*”, he proposes to use as two alternative inequality criteria ADO and RDO, and to apply a *Rank Dependent Expected Value* (RDEV) functional for ranking income profiles otherwise incomparable.

In what follows, we compare LO, ADO and RDO and look for a new approach for evaluating the social welfare associated to a given income distribution. We research a measure of the well-being of individuals with respect to their relative position in the distribution, which rules out pathological situations in ranking alternative income distributions as those described in Chateauneuf (1996). In other words, we measure inequality by using a new class of relative inequality indices, which avoids the problems related to the application of the EU or RDEV that we point out below.

When we measure economic inequality of a population, we use ethical principle, appealing mathematical constructs or simple intuitions in order to give meaning to comparisons of different income distributions. As a principal aspect of social welfare is the economic inequality in the society, thus several social welfare indices are measures of income disparity among individuals. Conversely, the measurement of economic inequality may be based on measures of social welfare as Dalton (1920) pointed out. Since Atkinson (1970) has disclosed the theoretical analogy between ranking income distributions according to a specific criterion (or an index) and an additive social welfare function, several theorists have investigated the formal similarity between the measurement of income inequality and the social preferences concerning its distribution. Later on, in a pioneering paper, Machina (1982) pointed out that the classical measures of welfare and disparity, which are based on utilitarianism, i.e. on the computation of the expected utility of the income

distributions, might be modified and improved according to the generalized theories of choice under risk, i.e. the so-called non-expected utility theories (see Fishburn, (1988)).

Our first aim consists in reviewing the few recent works on the axiomatic foundations of inequality measurement in a non-expected utility (NEU) framework, notably that of Ben-Porath *et alii* (1997), Chew (1983) and Yaari (1987). We therefore show that the expected utility (EU) approach as well as the rank-dependent one cannot satisfactorily deal with the preference pattern that will be described.

The second goal will be to measure inequality of a given income distribution by using a non-expected utility functional. We search behavioral foundations for equity-concerned social preferences in the theory of *disappointment aversion*.¹ The application of this NEU approach to measurement of inequality allows to implement a concern for equity of a hypothetical decision-maker by means of an utility function that takes into account the different positions of each individual (belonging to a distribution), with respect to a reference point (i.e. a threshold, a sort of poverty line). We look for an individualistic notion of equity to be implemented by an ethical decision maker, who is highly sensitive to inequality when the distribution is extremely polarized. To put it shortly, our notion of *concern for equity* is not in relation to “whole” interpersonal distribution, as in the case of an “inequality averse” decision maker that takes her decisions following an evaluation function of expected utility type, but in relation to each position in the distribution that individuals might occupy by chance. The idea, roughly speaking, is the following one. We consider each individual, independently from each other, that competes for the best positions in the distribution. If, *ex post*, she fills positions which fall short a certain threshold, then we can foresee the *sense of loss* (e.g. in her personal dignity) that she might feel when the preferred outcome belongs to a state of affairs that does not occur, but that, *ex ante*, could be. This psychological feeling of disappointment corrects by a negative value the utility evaluation relative to that individualistic consequence. We take into account the decrease in utility which would be felt *ex post* by that individual due to the “complementarity” among possible outcomes of alternative states of affairs.² In this way, an ethical decision-maker, concerned for equity, gets the judgements relative to the equity of each location in the distribution through an indicator, an (equity) evalu-

¹An axiomatization of such an approach is due to Gul (1991).

²Here, *complementarity* is meant in the sense of Allais Paradox. See Fishburn (1988) chapter 2 for a wide discussion on this issue.

ation function, which measures the individual welfare and does not treat inequality as a property of the *whole* interpersonal distribution.

Our paper consists of three sections. In section 2, we briefly discuss the related literature and introduce the inequality criteria we concern with, explaining our basic definitions and crucial concepts analytically as well as intuitively. Moreover, we discuss the issue of using a relative inequality index based on alternative evaluation functions, showing the positive and normative inconsistencies of EU and RDEV respectively. Section 3 is devoted to recall shortly the *disappointment utility model* (DUT). Afterwards, we introduce a class of inequality indices based on DUT, which avoids the problems related to the applications of the evaluation functions mentioned above (namely, EU and RDEU), and rules out what Chateauneuf (1996) calls paradoxes in ranking income distributions. Finally, some remarks on issues relative to the application of different inequality measures conclude the paper.

2 Equity and social evaluation functions: economic literature, motivations and critiques

2.1 Related literature

The last two decades have witnessed some derivations of classes of functions that generalize both expected utility in a context of choice under risk and the indices of inequality in a disparity evaluation context. Chew (1983), in order to solve the so-called Allais paradox, generalizes the class of quasilinear mean of de Finetti (1931). He point out the relation between a family of relative and absolute inequality measures and a related family of weighted utilitarian social welfare functions. In a more general context than that relative to EU, Chew extends the approach due to Atkinson (1970), who disclosed the theoretical analogy between the measures of income inequality and an additive, separable (i.e. utilitarian) and symmetric social welfare function (i.e. a quasilinear mean). Yaari [(1987), (1988)] characterized a class of linear inequality measures. He shows the link between the “rank dependent probabilities” approach to risk and a generalization of the Gini index. A result furthermore extended by Ben-Porath and Gilboa (1994). Ben-Porath, Gilboa and Schmeidler (1997), in a remarkable work, criticize the previous economic literature focused on “comonotonically linear” inequality indices, proposing a richer family of inequality measures in order to take into account notions of ex-ante and ex-post disparity. Chateauneuf (1996) analyzes some

situations where Lorenz dominance criterion fails as a device for measuring inequality, proposing, according to Moyes (1994) and Quiggin (1988), to use as alternatives ADO and RDO³ and to apply a *Rank Dependent Expected Value* (RDEV) functional for ranking incomparable income distributions.

An example of what Chateauneuf calls “paradoxical situations in which LO is involved” could be the following one. Let us suppose that the initial income distribution among four individuals is represented by the vector $x = (10, 15, 20, 25)$. If we compare x with the distribution $y = (11, 14, 21, 24)$, obtained from x through several transfers of shares of income among individuals, we see that the Lorenz curve associated with y lies above that of x . This should mean that the distribution y is *more even* than distribution x . Nevertheless, if we simply consider subgroup $\{2\}$, $\{3\}$, inequality increases as we pass from situation x to situation y . The income distribution we obtain is obviously more polarized as inequality is increased between such groups. LO instead shows that the obtained vector y presents less disparity with respect to x . It seems to Chateauneuf and to some other scholars there is room for introducing alternative inequality orderings as ADO and RDO.

ADO, as Quiggin (1988) has noted in a framework of decision under uncertainty, suggests that a distribution vector y should be considered as unambiguously less risky (hence, in our context, less unequal) than x only if $y_i = x_i + h_i \quad \forall i \in \{1, \dots, n\}$, with $h_i \geq h_{i+1}$ and $\sum_{i=1}^n h_i = 0$. This means that y could be obtained from x through transfers of amounts of money from the richer to the poorer, in an absolutely progressive way. As such a ranking appears to be a suitable inequality criterion, so it seems too restrictive, because it does not allow direct comparison of income profiles with different global income.

On the contrary, RDO permits comparisons between distributions with different total income and, at the same time, it is in accordance with progressive income taxation, usually perceived as inequality reducing. Indeed, if we suppose that x_i is the income of individual i before tax and $y_i = f(x_i)$ the same income after tax, we can interpret RDO as tantamount to a progressive income taxation, i.e. $\frac{f(z)}{z}$ nonincreasing on $(0, \infty)$. Hence, it is a suitable inequality criterion too.

In what follows, we formally define LO, ADO and RDO in order to point out the relation among such partial orderings.

³Notice that these orderings were introduced by Marshall *et alii* (1967) and studied by Marshall and Olkin (1979) in the field of the *Theory of Majorization*.

2.2 Notation and definitions

We consider a given population of $N = \{1, \dots, i, \dots, n\}$ types of individuals. A vector $X = (x_1, p_1; \dots; x_n, p_n)$ represents an income distribution. A pair of elements (x_i, p_i) of X respectively denotes the income of individual i and the percentage of individuals in the populations that receives income x_i . The components x_i of a vector $x \in \mathbb{R}_{++}^n$, are ordered such that $0 < x_1 < \dots < x_n$ and each $p_i \in \mathbb{R}_{++}$ is such that $\sum_i p_i = 1$. The set of all non-decreasing rank-ordered income distributions is denoted by \aleph . We define the cumulative distribution function of X as follows:

$$F(x) = \sum_{\{i|x_i \leq x\}} p_i \quad \forall x \in \mathbb{R},$$

where $F(x)$ is the percentage of people in the distribution X receiving income less than or equal to the income x . The left continuous version of the inverse of F , the so-called quantile function, is a function $F^{-1} : [0, 1] \rightarrow \mathbb{R}_+$, that is denoted as

$$F^{-1}(p) = \text{Inf} \{x \in \mathbb{R}_+, F(x) \geq p : p \in [0, 1]\}.$$

For $X \in \aleph$, $E[X]$ denotes the mean of X .

An *inequality criterion* \preceq is a *partial ordering* on \aleph , i.e. a asymmetric and transitive relation. When two income distributions X, Y in \aleph satisfy $Y \preceq X$, we shall say that Y is less unequal than X .

The three inequality criteria considered in this work are defined below:

Definition 1 (Inequality Criteria) *Given two income distributions $X, Y \in \aleph$, we say that:*

1. *Y is less unequal than X in the sense of Lorenz, denoted $Y \preceq_{LO} X$, if*

$$\sum_z F_Y^{-1}(z) \geq \sum_z F_X^{-1}(z) \quad \forall z \in [0, 1) \text{ and } F_Y^{-1}(1) = F_X^{-1}(1);$$

2. *Y is less unequal than X for the absolute differentials ordering, denoted $Y \preceq_{ADO} X$, if*

$$F_Y^{-1}(v) - F_Y^{-1}(u) \leq F_X^{-1}(v) - F_X^{-1}(u), \quad \forall 0 < u < v \leq 1;$$

3. Y is less unequal than X for the relative differentials ordering, denoted $Y \preceq_{RDO} X$ if

$$\frac{F_Y^{-1}(v)}{F_Y^{-1}(u)} \leq \frac{F_X^{-1}(v)}{F_X^{-1}(u)} \quad \forall 0 < u < v \leq 1.$$

ADO and RDO are sub-orderings of LO.⁴ Such a relation can be provided by relating all binary relations in *definition 1* to the sign-changed orderings.⁵

Let F and G be the cumulative distributions respectively of $X, Y \in \mathfrak{N}$. A common way to compare the dispersion (inequality) of F and G is via the sign changes of $(F - G)$. If $S(h)$ is the number of sign changes of the function $h(t)$, a natural condition on $(F - G)$ corresponding to F being in some sense more variable than G is:

$$S(F - G) = 1 \quad \text{with sign sequence } +, -.$$

Further, let us define $\varphi(z) = F^{-1}(G(z))$ as *starshaped* if the function $\varphi(z) : [0, 1] \rightarrow [0, 1]$ is such that $\varphi(cz) \leq c\varphi(z)$ for $c \in (0, 1)$ or equivalently $\frac{\varphi(z)}{z}$ is nondecreasing on $[0, 1]$. Then, we show that:

Proposition 1 *RDO (ADO) is a subordering of LO.*

Proof. It is well known that $Y \preceq_{LO} X$ if and only if $S(F - G) \leq 1$.⁶ As $Y \preceq_{RDO} X$ is equivalent to the function $\varphi(x) = F^{-1}(G(x))$ starshaped, a sufficient condition for $S(F - G) \leq 1$ is that the two distribution functions F and G cut one another at most once. That should mean that RDO implies LO, while the contrary does not necessary hold. Indeed, $\varphi(x) = F^{-1}(G(x))$ starshaped is tantamount to $\varphi(cx) \leq c\varphi(x)$ for all $c \in (0, 1)$ and $x \geq 0$. This last expression can be written in the form $\varphi(cx)/cx \leq \varphi(x)/x$ so that $\varphi(x)/x$ is a non-decreasing function, and hence $F^{-1}(G(x))$ crosses any line $X(x) = Kx$ at most once, and from below, and, having crossed it, never touches it again. Taking $K = 1$ and using the non-decreasing nature of F^{-1} , it follows that, if there exist a crossing point ξ ,

$$G(x) \geq F(x) \quad \text{as } x \geq \xi$$

⁴Alternative proofs of this result are in Marshall and Olkin (1979) proposition B.1 on page 129, Chateauneuf (1996) proposition 6 on page 13, and Savaglio (2002).

⁵See Karlin (1968) vol. I, chapters 5 and 6.

⁶See Karlin (1968) vol. I, chapter 5.

that completes the claim.⁷

■

As ADO and RDO imply LO, they compare less distributions than Lorenz ordering does. In order to rule out “paradoxical situations in which LO is involved” and to be able to rank as many distributions as it is possible, scholars introduce (relative) inequality indices (complete ordering). In such a way, they measure the welfare associated to an income distributions and identify the vector which shows less inequality.

In what follows, we provide a critique (due to Chateauneuf (1996)) to using a well-known class of inequality indices and we show how an alternative proposal of disparity measurement is questionable as well.

2.3 Indices of inequality and evaluation functions

In a pioneering work, Atkinson (1970) has investigated the relationship that goes from a social welfare evaluation function (W) to an inequality index (I), (i.e. $W \rightarrow I$). Following the Harsanyi’s approach (1953), he has utilized expected utility theory as a mathematical instrument by which an ethical decision-maker expresses social preferences. The utility differences, resulting from interpersonal comparisons, is indeed deemed as the determinant of judgements of social welfare. In this way, Atkinson has provided natural assumptions for an inequality index.

Denote with $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ a strictly increasing and continuous evaluation function. Given an income distribution X , Atkinson (1970) defines a relative inequality index as:

$$A(X) = 1 - \frac{\tilde{x}}{\bar{X}} \quad (1)$$

where $\bar{X} = \sum_{i=1}^n p_i x_i$ represents the average income and \tilde{x} is the *equally distributed equivalent level of income* (for short, *edei*), defined as:

$$W(\tilde{x}, \tilde{x}, \dots, \tilde{x}) = \sum_{i=1}^n p_i u(x_i) = W(X),$$

i.e. as the income that if it were imputed to every income-receiver in the population, would yield the same level of social welfare W as the actual income distribution X .

⁷A similar argument shows that ADO implies LO.

Formula 1 represents a general approach to the measurement of social welfare, that we interpret as an aversion to inequality, as expression 1 implicitly shows the rate of trade-off of a society of equality against the mean income.

Nevertheless, Chateauneuf (1996) noted that the so-called class of Atkinson-Kolm-Sen inequality indices (in short, AKS indices), does not distinguish among LO, ADO and RDO. This means that an EU based inequality index cannot be consistent with ADO and RDO while it rules out some paradoxical situations as those investigated by Chateauneuf (1996). In words, if we use EU in order to evaluate the welfare associated with a given income distribution, we cannot discriminate between situations where we need stronger inequality criteria for capturing the differences in disparity among people and situations where weaker (i.e. less demanding under an informational point of view), criteria apply. It is then impossible to draw welfaristic conclusions on the so-called “paradoxical situations” by using the EU evaluation approach. AKS inequality indices, being based on the *expected utility theory*, lead to a concave (utility) function in every case:

Proposition 2 (Chateauneuf 1996) *Let $X, Y \in \mathfrak{N}$, then*

1. *If $Y \preceq_{LO} X$, with $E(X) = E(Y)$, implies $A(Y) \leq A(X)$, then u is concave;*
2. *If $Y \preceq_{ADO} X$, with $E(X) = E(Y)$, implies $A(Y) \leq A(X)$, then u is concave;*
3. *If $Y \preceq_{RDO} X$, implies $A(Y) \leq A(X)$, then u is concave.*

A justification to the introduction of alternative (to EU) evaluation functions is then to distinguish among LO, ADO and RDO and avoid paradoxical situations in which Lorenz is involved. In order to evaluate the welfare relative to an income distribution, an alternative motivation for not using EU functional is that some economic experiments and a lot of empirical evidence have shown violations of the axioms and underlying principles of expected utility by intelligent and well-informed people. Systematic failures of the independence axiom, continuity or expectation principle have received special attention by many theorists, who have tried to accommodate violations of the theory of von Neumann and Morgenstern through more general approaches of greater normative and descriptive interest.

For these reasons, several scholars (as Chateauneuf (1996)), have proposed to adopt the more flexible (than EU) approach of the *Rank Dependent*

Expected Value theory, building relative inequality indices where the *edei* is computed by using such a model. RDEV is a non-expected utility model, that appears to be of great interest. Its functional form relies on an independence axiom for indifference applied to distributions called *co-monotonicity axiom* and defined as follows.⁸

Suppose x and y such that $x_1 < x_2 < \dots < x_n$, $y_1 < y_2 < \dots < y_n$; $x_i \leq y_i$ for all i , $x_i < y_i$ for some i , $p_i > 0$ for $i = 1, \dots, n$ and $\sum_i p_i = 1$. Define simple distribution functions F and G by $F(x_i) = G(x_i) = p_i$ for $i = 1, \dots, n$. Let x° denote the distribution with probability 1 for x , and let $e(x)$ denote the certainty equivalent so that $e(x) \sim x$. Then:

Axiom 1 $\lambda e(F)^\circ + (1 - \lambda) e(G)^\circ \sim \sum p_i e(\lambda x_i^\circ + (1 - \lambda) y_i^\circ)^\circ$ for all $\lambda \in (0, 1)$.

Axiom 1 guaranties, given a function $f : [0, 1] \rightarrow [0, 1]$, the existence of a functional J on \aleph defined as:

$$J(X) = x_1 + \sum_{i=1}^{n-1} (x_{i+1} - x_i) f(p_{i+1} + \dots + p_n)$$

such that, given the binary (preference) relation \succeq on $\aleph \times \aleph$:

$$X \succeq Y \Leftrightarrow J(X) \geq J(Y) \quad \text{for all } X, Y \in \aleph$$

The proposal of building a relative index of inequality computing the *edei* by using the RDEV functional $J(X)$ in the expression 1, consists in a weighted estimation of poor and rich through the function f . This should represent a more realistic formulation than EU one, because it allows to draw welfare conclusions on ranking of vector distributions also in the case where the EU model is silent (in the sense of proposition 2), on the value of the welfare associated to alternative distributions and on the differences among inequality criteria we apply.

A question but arises: ‘‘Has the RDEV’s approach a greater normative content than EU’s one, in analyzing inequality of alternative income distributions?’’

The next subsection is dedicated to such an answer.

⁸See Quiggin (1988), Yaari (1987) and Segal (1989) for more details on RD functional.

2.4 A critique to the use of the RDEV

The RDEV is a flexible approach for evaluating the welfare associated to a given individual income distribution. Nevertheless, it is unattractive under a positive and normative point of view. Let us see why.

RDEV requires that if $X^{(\alpha)}$ is the risky prospect obtained multiplying each outcome of distribution X by $\alpha > 0$, then $X^{(\alpha)} \sim Y^{(\alpha)}$ for all $\alpha > 0$ (within limits) when $X \sim Y$, which is, without any doubt, an unattractive description of how rational agents take decisions.⁹

Secondarily, the co-monotonicity axiom C, that delineates the functional of RDEV, appears disputable under a normative point of view.

For the special case $n = 2$, with $x_1 < x_2$, $y_1 < y_2$, $x_1 \leq y_1$, $x_2 \leq y_2$, with $x_i < y_i$ for some i , $p \in (0, 1)$ such an axiom requires that:

$$\frac{1}{2} \left[e\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) \right] + \frac{1}{2} \left[e\left(\frac{1}{2}y_1 + \frac{1}{2}y_2\right) \right] \sim \frac{1}{2} \left[e\left(\frac{1}{2}x_1 + \frac{1}{2}y_1\right) \right] + \frac{1}{2} \left[e\left(\frac{1}{2}x_2 + \frac{1}{2}y_2\right) \right]$$

We show that, in some situations, an *ethical* (i.e. concerned for equity) *decision-maker* will likely violate it.

Let us imagine, indeed, the following two situations A and B, in which individuals 1 and 2 receive arbitrary possible outcomes relative to the cases s and t , which have the same probability to happen:

		s	t		s	t
A	1	-100	200		1	-100 0
	2	0	202	B	2	200 202

An ethical decision-maker, concerned for equity, who evaluates situation A and B, is oriented to avoid a too much unequal distribution of outcomes among individuals, giving an higher importance to the less favorable positions in the determination of social welfare. In other words, she cannot prefer B to A because, in both cases of B, individual 1 receives a too low outcome x_B or y_B . At the same time, the normative prescription that the decision-maker can draw from the main axiom of RDEV is that situations A and B are perfectly indifferent. It is straightforward to conclude that RDEV does not provide a sufficient normative basis to the choices of an ethical decision-maker. We need a different approach to equity-concerned social preferences. In a few words, we look for a utility functional that is

⁹For a wider analysis of this kind of critique see Fishburn (1988, chapter 3) and Kolm (1977).

consistent (i.e. distinguish among) with LO, ADO and RDO and evaluate the well-being of individuals with respect to their relative positions in the distribution. An alternative way to show the relation between different level of deprivations and alternative values of social welfare is then proposed.

3 Measuring inequality: an approach through non-expected (subjective) utility model

Prior to about 1980, the main alternative to Savage’s additive expected utility theory was Allais’s additive nonexpected intensive utility theory for decision under uncertainty.¹⁰ Since that time, several new alternative models have been developed to accommodate violations of independence, substitution, reduction and transitivity. *Disappointment Utility Theory* (DUT) is a theory presented by Loomes and Sugden (1986) and axiomatized by Gul (1991). DUT is a different line of research in NEU, characterized by the absence of “separability” among possible outcomes of the states of the world.¹¹ It besides is a model of decision making under uncertainty that includes expected utility theory as a special case and solves the so-called *Allais Paradox*. The theory of Loomes and Sugden is represented by an expectation functional that involves a concept of disappointment/elation. In a few words, the expected utility is modified by a factor that takes into account of the differences between outcome utilities and their mean. If this difference is positive, then one is elated by doing better than the mean, but if it is negative, then there is disappointment.

Gul (1991) develops his axiomatic model, considering a binary (preference) relation complete and transitive, a continuity axiom and a symmetry one and replacing the independence axiom (IA) of expected utility theory with an alternative axiom, that he calls *weak independence* (WI). WI is a particular version of IA that avoids the Allais Paradox, enabling the independence axiom to fail when disappointment effects are present.

Given these four axioms, Gul characterizes preferences, that are represented by the following functional:

$$\Gamma_X(a, x^*) = \phi(a) \sum_{x > x^*} x_i \frac{p_i}{a} + (1 - \phi(a)) \sum_{x < x^*} x_i \frac{p_i}{1 - a} \quad (2)$$

where $a = \sum p_i$ for $x > x^*$, with x^* a given (exogenous) threshold

¹⁰See Fishburn (1988) for more details on different approaches to choice under uncertainty.

¹¹The respect of separability across individual is instead warranted.

or reference point (e.g. the average), and $\phi(a) = \frac{a}{[1+(1-a)\delta]}$, which is a function from $[0, 1]$ to $[0, 1]$ with $\phi(0) = 0$ and $\phi(1) = 1$, representing the decision-maker's evaluation. We suppose $\delta > 0$ in order to describe strictly disappointment averse individual preferences (see Gul, (1991)).

In our model, DUT implies that the concern for equity of an hypothetical decision-maker, who decides given social preferences, is relative to every possible position, which an individual can occupy by chance. The concern for equity is expressed with respect to the point of view of every individual, taking into account her relative position in the income distribution. Precisely, DUT pays special attention to the particularly lower values of the distribution, avoiding the critique concerning RDEV. An ethical decision-maker, evaluating social preferences through DUT, takes into account the deprivation of individuals belonging to the quantiles under a given threshold (e.g. a poverty line). The disappointment evaluation function ϕ captures the negative psychological feeling of an individual that knows to have not the same *opportunity* of others *of being or doing something*. As Farina (1996) noticed, the utility function, on which we will therefore dwell, is modified by a concern for equity, which is not considered in relation to *whole* interpersonal distributions, as in the case of the inequality averse decision-maker. In this last case (i.e. the EU's one), the evaluation of any different income profiles is given by computing their alternative expected values. Here, we have an utility function where the concern for equity is expressed in relation to *each* location which individuals might occupy by chance. We consider the distributive issue under an individualistic perspective. Because an individual compares her *capacities to be or to do something* with respect to the capacities of others, evaluating her personal value of equality.

Then, following the standard approach that links different levels of deprivation with alternative values of social welfare, we apply the DUT functional in order to evaluate the well-being of individuals, with respect to their relative position in the distribution. We show to evaluate inequality in a more normatively consistent way than that of EU or of RDEV. We distinguish among LO, ADO and RDO inequality criteria and we express social preferences on alternative distributional prospects, taking into account the different level of disparity of each location in the distribution.

Formally, given the disappointment inequality aversion functional 2, we denote with:

$$G(X) = 1 - \frac{\Gamma(X)}{\bar{X}} \quad (3)$$

the relative index of inequality, defined by using Gul's functional. We then identify a class of *ordering-preserving functions* for ADO and RDO, following

the approach due to Marshall and Olkin (1979, chapter 1). In general:

Definition 2 *If \succsim is an ordering defined on some set $\aleph \subset \mathbb{R}^n$, a functions $\varphi : \aleph \rightarrow \mathbb{R}$ is said order-preserving or isotonic if*

$$X \succsim Y \Rightarrow \varphi(X) \geq \varphi(Y) \quad X, Y \in \aleph.$$

Then, we check the conditions under which G respects (is consistent with) ADO:

Proposition 3 $\forall X, Y \in \aleph$,

$$Y \preceq_{ADO} X \Rightarrow G(Y) \leq G(X)$$

if and only if

$$\phi(a) \leq a$$

is satisfied $\forall a \in [0, 1]$.

Proof. \Rightarrow Given $a \in [0, 1]$ and fixed $l, n \in \mathbb{R}_+$ with $l < n$, we can get $Y \preceq_{ADO} X$ if

$$Y = [(1-a)l + an, 1-a; (1-a)l + an, a] \text{ and } X = [l, 1-a; n, a],$$

hence $G(Y) \leq G(X)$ entails:

$$\phi(a)(n-l) + l \leq a(n-l) + l,$$

i.e. $\phi(a) \leq a$.

\Leftarrow Let $X, Y \in \aleph$ such that $Y \preceq_{ADO} X$. We have seen that ADO is equivalent to set $y_i = x_i + h_i, \forall i \in [1, \dots, n]$, with $h_i \geq h_{i+1} \forall i$ and $\sum_{i=1}^n h_i = 0$. Moreover, by assumption, $\phi(a) \leq a \forall a \in [0, 1]$. Then, we have to prove that $G(Y) \leq G(X)$ or

$$\Delta[h] = G(Y) - G(X) \leq 0.$$

This means that:

$$\Delta[h] = \phi(a) \sum_{x > x^*} h_i \frac{p_i}{a} + (1 - \phi(a)) \sum_{x < x^*} h_i \frac{p_i}{i - a}$$

is majorized (given that $\phi(a) \leq a$) by $\frac{\sum_{i=1}^n h_i}{a}$. The remark that $\sum_{i=1}^n h_i = 0$ entails the claim. ■

Let us characterize a class of functions that are consistent with RDO, by showing that G respects relative differentials ordering if and only if the function φ is starshaped above 0:

Proposition 4 $\forall X, Y \in \aleph$,

$$Y \preceq_{RDO} X \Rightarrow G(Y) \leq G(X)$$

if and only if

ϕ is starshaped above 0.

Proof. \Rightarrow Let p_1, p_2, p_3 belong to $(0, 1)$, and suppose $p_1 = 1 - (p_2 + p_3)$. We have to prove that:

$$\frac{\phi(p_1 + p_2)}{p_1 + p_2} \geq \frac{\phi(p_3)}{p_3}.$$

Given a vector of transfers h , we consider:

$$Y = (x_1 + h_1, p_1; x_2 + h_2, p_2; x_3 + h_3, p_3)$$

and

$$X = (x_1, p_1; x_2, p_2; x_3, p_3)$$

such that $y_i = x_i + h_i$, with $i \in [1, 2, 3]$. Let us suppose:

$$\begin{cases} h_1 = h \\ h_2 = h + \frac{1}{p_2 + p_3} \\ h_3 = \frac{1}{p_3} \left[-\frac{p_2}{p_2 + p_3} + (p_3 - 1)h \right] \end{cases}$$

and

$$\begin{cases} x_1 = h_1 \\ x_2 = h_2 \\ x_3 = h_2 + \frac{1+h}{p_3}. \end{cases}$$

An easy computation shows that $X, Y \in \aleph$ and that $Y \preceq_{RDO} X$ because $\sum_{i=1}^3 h_i = 0$. But, we know, by hypothesis, that $G(Y) \leq G(X)$ i.e.

$$\Delta[h] = G(Y) - G(X) \leq 0,$$

that implies:

$$\frac{\phi(p_3)}{p_3} \left[\frac{-p_2 + (p_2 + p_3)(p_3 - 1)h}{p_2 + p_3} \right] \leq \frac{\phi(p_1 + p_2)}{p_2 + p_1} \left[\frac{hp_2(p_2 + p_3) + p_2 + hp_1(p_2 + p_3)}{p_2 + p_3} \right]$$

i.e. the desired result.

\Leftarrow (i) Let $X, Y \in \aleph$ be such that $Y \preceq_{RDO} X$.

Note that $X' = \left(\frac{x_1}{E(X)}, p_1; \dots; \frac{x_n}{E(X)}, p_n\right)$ and $Y' = \left(\frac{y_1}{E(Y)}, p_1; \dots; \frac{y_n}{E(Y)}, p_n\right)$ belong to \aleph and are such that $Y' \preceq_{RDO} X'$ with $E(Y') = E(X') = 1$. As we are looking for an inequality index $G : \aleph \rightarrow [0, 1]$ that be invariant with respect to equiproportional increase of all incomes (a relative index), and $G(Y') = G(Y)$, $G(X') = G(X)$, it is enough to prove that $Y \preceq_{RDO} X$ with $\sum_{i=1}^n y_i = \sum_{i=1}^n x_i$ entails $G(Y) \leq G(X)$.

(ii) Let φ be the policy maker's evaluation function of inequality, and assume φ star-shaped above 0. We intend to prove that $\Delta = G(Y) - G(X) \leq 0$, i.e. $\Gamma(Y) - \Gamma(X) \geq 0$.

Let assume $Y \neq X$. Set $y_i = x_i + h_i$, hence $\frac{y_{i+1}}{y_i} \leq \frac{x_{i+1}}{x_i}$ entails $\frac{h_{i+1}}{h_i} \leq \frac{x_{i+1}}{x_i}$, therefore, since $\frac{x_{i+1}}{x_i} \geq 1$, it follows that $h_i \leq 0$ implies $h_{i+1} \leq 0$ with $h_{i+1} \leq h_i$. Moreover $\sum_{i=1}^n y_i = \sum_{i=1}^n x_i$ entails $\sum_{i=0}^n h_i = 0$, hence since $Y \neq X$, $\exists k_0 \in [1, \dots, n-1]$ such that $i \leq k_0 \Rightarrow h_i \geq 0$ and $i > k_0 \Rightarrow h_i < 0$.

As

$$\Delta[h] = \varphi(a) \sum_{t>t^*} h_i \frac{p_i}{a} + [1 - \varphi(a)] \sum_{t<t^*} h_i \frac{p_i}{1-a} \geq 0$$

we notice that t^* is a relative threshold, obtained dividing the distributions for their means in order to respect the requirement that G is a relative index.

(iii) We set $k \in [1, \dots, n-1]$ and we notice that $\sum_{t>t^*} h_i p_i$ must be negative for two reasons:

a) we can imagine that, given a threshold t^* , a sort of poverty line, each transfer must occur around t^* , that means that people belonging to quantiles of distribution on the right of threshold give money to poorer people belonging to quantiles under the poverty line;

b) $\sum_{t>t^*} h_i p_i = \sum_{i=k+1}^n h_i p_i \leq 0$, with $k+1 > t^*(k_0)$, then $h_i \leq 0 \forall i \in [k+1, \dots, n]$, then $\sum_{t>t^*} h_i p_i \leq 0$ as we know from (ii).

(iv) Now, let us call $\sum_{t>t^*} h_i p_i = A$ and $\sum_{t<t^*} h_i p_i = B$. We have:

$$\Delta[h] = \frac{\varphi(a)}{a} A + \frac{[1 - \varphi(a)]}{1-a} B \geq 0$$

and we know that $A + B = 0$ (with $A \leq 0$, that means that $B \geq 0$). Then,

$$\Delta[h] = \frac{\varphi(a)}{a} (-B) + \frac{[1 - \varphi(a)]}{1-a} B \geq 0,$$

i.e. $\Delta[h] = B \left[\frac{[1-\varphi(a)]}{1-a} - \frac{\varphi(a)}{a} \right]$. As $B \geq 0$, we have to prove

$$\frac{[1-\varphi(a)]}{1-a} \geq \frac{\varphi(a)}{a}$$

that is true as φ is star-shaped. In fact, we know $\frac{\varphi(a)}{a} \leq 1$ and from this that $\frac{[1-\varphi(a)]}{1-a} \geq 1$. ■

As star-shaped functions are inequality-reducing transformations, we can interpret such a class of functions as if the equally distributed equivalent level of income $\Gamma(X)$ was the average level of the income after a progressive income taxation of the distribution X (for a more extensive interpretation of this point see Moyes (1994) and Savaglio (2001)). We have shown that, by using DUT for evaluating disparity among individuals, we apply a more realistic normative formulation than EUT's and RDU's ones. We distinguish among LO, ADO and RDO, but providing a different type of information in ranking alternative income distributions. Finally we provide an individualistic notion of equity, implemented by an ethical decision maker, who is highly sensitive to disparity.

3.1 Ruling out paradoxes in ranking income distributions

We show, as a result, that by using the index $G(\cdot)$, it is possible to respect ADO or RDO and ruling out some paradoxes as those analyzed by Chateauneuf (1996).

Let us recall the vectors $x = (10, 15, 20, 25)$ and $y = (11, 14, 21, 24)$. We know that the Lorenz curve associated with y lies above that x . Nevertheless, if we simply consider subgroup $\{2\}$, $\{3\}$ inequality increases as we pass from situation x to situation y (i.e., $x_3 - x_2 = 20 - 15 = 5 < 7 = 21 - 14 = y_3 - y_2$). Let h_i represent the share of income that the i th individual receives (or transfers) from (to) another individual j . Then, $y = (14, 14, 21, 21)$ could be *also* obtained from x through the transfers $h_2 = -1$ and $h_3 = 1$, even if $y \preceq_{LO} x$. Finally, $y = (10, 16, 19, 25)$, which still Lorenz dominates x , could be obtained through the following transfers $h_1 = 0$, $h_2 = 1$, $h_3 = -1$ and $h_4 = 0$: a richer person has obtained an increase of income while the poorest person has not, furthermore a relatively poorer person has suffered a reduction of income while the richest one has not. LO then appears to be too weak as a criterion, because it treats a distribution as a *whole* and does not take into account the differences of income among individuals belonging to different quantiles. Chateauneuf (1996) has called these examples “disputable situations”, proposing ADO and RDO as suitable alternatives to LO. We

have shown in another work (see Savaglio, (2001)), that if we strength the Pigou-Dalton principle of transfer we can avoid all “paradoxical situations” mentioned above. Here, we show how, using the relative inequality index $G(\cdot)$ and evaluating the welfare associated to the different distributions, it is possible to rank all these situations, solving what Chateauneuf (1996) calls “*paradoxes in which LO is involved*”.

Suppose $x^* = 17, 5$, $\delta = 2$, $\varphi(a) = 1/8$, and $(1 - \varphi(a)) = 7/8$. Let us take $x = (10, 15, 20, 25)$ and $y = (11, 14, 21, 24)$ obtained from x through two Pigou-Dalton transfers $\{h_1 = +1, h_2 = -1\}$ and $\{h_3 = +1, h_4 = -1\}$ such that transfers are not around the fixed threshold, our relative inequality index 3 entails $\Gamma(Y) - \Gamma(X) = 0$.¹² It means that the welfare in y does not change with respect to x . With $y = (10, 16, 19, 25)$, $\Gamma(Y) - \Gamma(X) = 1/16$, i.e. distribution y shows more welfare and less inequality. Analogously, for the only ADO or RDO rankable distribution $y = (11, 15, 20, 24)$, $\Gamma(Y) - \Gamma(X) = 3/8$. While, for $y = (14, 14, 21, 21)$, the welfare is given by $\Gamma(Y) - \Gamma(X) = 9/8$. Finally, if we consider the evident unfair distribution $y = (0, 0, 0, 70)$, which cannot be ranked by using ADO or RDO, index G shows that it represents a clear loss of social welfare with respect to distribution x : $\Gamma(Y) - \Gamma(X) = -\frac{55}{8}$.

4 Conclusion

It has noted that, in some situations, Lorenz order involves paradoxes in ranking different income distributions. The proposal of using two different inequality criteria, namely ADO and RDO, in order to solve such disputable situations led us to deepen if such alternative orderings capture, in some sense, more information about income distribution than LO does.

The welfaristic part of our work has concerned the relation between different levels of individual deprivation and alternative values of well-being. We had used an alternative non-linear welfare functional, that 1) evaluates disparity among individuals, in a more robust way than the approach via EU or RDEV both under a descriptive and normative point of view; and 2) is coherent with the inequality criteria analyzed above. By using such an alternative utility functional, namely DUT, an hypothetical ethical decision-maker shows her concerning for equity taking into account the value of inequality of each individual in the distribution.

¹²Of course, X and Y are the income distributions associated with the income profiles x and y .

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