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An Infinitary Probability Logic for Type Spaces

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Abstract

Type spaces in the sense of Harsanyi (1967/68) can be considered as the probabilistic analog of Kripke structures. By an infinitary propositional language with additional operators "individual i assigns probability at least α to" and infinitary inference rules, we axiomatize the class of (Harsanyi) type spaces. We show that our axiom system is strongly sound and strongly complete. To the best of our knowledge, this is the very first strong completeness theorem for a probability logic of the present kind. The result is proved by constructing a canonical type space.

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1 Introduction

It is well-known that Kripke structures (and in particular Knowledge spaces introduced by Aumann (1976)) can be axiomatized in terms of modal logic (See for example Kripke (1963), Aumann (1995), Fagin et al. (1995), Heifetz (1997), and Aumann (1999a)). In this paper we aim to do the same for type spaces in the sense of Harsanyi (1967/68), which can be considered as the probabilistic analog of Kripke structures. Type spaces are the predominant structures to describe incomplete information in an interactive context in game theory. (See Aumann and Heifetz (2001) for a nice and well-accessible introduction to the subject.)

We define an infinitary modal language with operators p_i^α, "individual i assigns probability at least α" for rational $\alpha \in [0,1]$, and then a system of infinitary axioms and inference rules, which we prove to be strongly sound and strongly complete (Theorem 1) with respect to the class of (Harsanyi) type spaces. Strongly complete means that, if a formula φ holds whenever a (possibly infinite) set of formulas Γ holds, then Γ proves φ.

Heifetz and Mongin (2001) - and before Fagin, Halpern and Megiddo (1990) for a much richer syntax also expressing valuations for linear combinations of formulas - axiomatized the class of type spaces in terms of a purely finitary logic. They showed that their axiomatization is sound and complete with respect to the class of (Harsanyi) type spaces.

However, a purely finitary axiomatization cannot be used to get strong soundness and strong completeness for this class of models. This was noted by Heifetz and Mongin (2001) and Aumann (1999b). They argue that for the set of formulas $\left\{p_i^{\frac{1}{2}-\frac{1}{n}}(\varphi) : \; n \geq 2, \; n \in \mathbb{N}\right\} \cup \left\{\neg p_i^{\frac{1}{2}}(\varphi)\right\}$, each finite subset has a model (a type space and a state in it, such that each formula in this finite subset is true at that state), while the whole set itself has no model. This implies that, although we will prove a strong completeness theorem, our logic will not be compact.

We construct (Proposition 3) a canonical model whose points consist of the maximal consistent sets of formulas. This construction requires strong soundness and strong completeness. In a very natural way, the maximal consistent sets of formulas determine already the structure of this space. The spirit of this construction follows the constructions of the canonical models for the infinitary versions of the S5-epistemic logics, proposed by Heifetz (1997) (see Aumann (1999a) for the finitary version). Such constructions show that there is a space which contains “all states of the world” in the syntactic sense. A state of the world in the syntactic sense is a maximal consistent set of formulas of the language. The existence of a canonical model guarantees that, without loss of generality, a game of incomplete information can be modeled by a type space.

Type spaces cannot be axiomatized by an infinitary logic in the sense of Heifetz (1997): An example by Karp (1964) in a purely propositional setting shows already that, in the presence of $\aleph_\gamma$ many formulas (where $\aleph_\gamma$ denotes the γth infinite cardinal number) whose truth values can be chosen independently of one another, if one allows for infinite conjunctions of $\aleph_\gamma$ many formulas, then one must also allow for conjunctions of $2^{\aleph_\gamma}$ many formulas and proofs of length of cardinality $\leq 2^{\aleph_\gamma}$ to get strong completeness. This collides with measurability conditions that must be met: When we want to define the validity relation “$\models$” for a type space τ and some point (i.e. state) ω in τ, then, for a formula φ in our language, $(\tau, \omega) \models p_i^\alpha(\varphi)$ can be defined, if $[\varphi]^\tau$, the set of points in τ where φ is true, is a measurable set. Since conjunctions of formulas correspond to intersections of subsets of the structure, uncountable conjunctions cannot be guaranteed to interpret measurable sets unless we do assume that the σ-fields of the type spaces are closed under uncountable intersections, which, of course, would strongly restrict the class of type spaces we could consider.

We resolve this problem by defining a language which takes the advantages and avoids the disadvantages of both the finitary and the infinitary languages. We start with a finitary language $\mathcal{L}_0$ à la Aumann (1995) and Heifetz and Mongin (2001) with operators p_i^α, “individual i assigns probability at least α”. Then, we define an infinitary propositional language $\mathcal{L}$, the primitive propositions of which are the formulas in $\mathcal{L}_0$. So, $\mathcal{L}_0$ is a sublanguage of $\mathcal{L}$.

For the class of type spaces - contrary to the class of knowledge spaces, as was shown by Heifetz and Samet (1998a) - there is also a unique (up to isomorphism) special type space, defined in purely semantic terms, namely the universal type space, a space to which every type space can be mapped by a unique structure preserving map, a so called “type morphism”. The existence of such a space was first proved - under certain topological assumptions - by Mertens and Zamir (1985), followed by many others. Recently Heifetz and Samet (1998b) proved the general measure-theoretic case. In their proof, they - as well as Aumann (1999b) for the proof of the existence of a canonical knowledge-belief system - use a language similar to ours, with one important difference: They do not have a purely syntactic definition of what the maximal consistent sets of formulas are. Instead, such sets are obtained semantically by collecting the sets of formulas that hold true at some point in some type space, thus constructing the universal space (resp. the canonical knowledge-belief system).

It is not too surprising then that the canonical model and the universal space are one and the same. This is stated in our Theorem 2. Hence, we provide here a (up to now missing) characterization of the universal type space (as the space of maximal consistent sets of formulas).

In the literature, "type spaces" usually are what we will call here "product type spaces". Other authors who consider the more general version are Heifetz and Mongin (2001), who call it also "type spaces" and Mertens and Zamir (1985), who call these spaces "beliefs spaces". As it turns out in their topological setting, the universal type space of Mertens and Zamir is a product type space.

In Theorem 3 we prove that this is still true in our topology-free setting, namely our canonical model is a product type space.

Furthermore, everywhere in the literature except in (Heifetz and Mongin (2001)), only type spaces are considered where the players know their own beliefs (we call these spaces, in this paper, like Heifetz and Mongin (2001), "Harsanyi type spaces").

We construct our canonical model with and without this property and establish the first proof of the existence of a universal type space for the class of type spaces without introspection.

In the topological cases of Mertens and Zamir (1985), Brandenburger and Dekel (1993), Heifetz (1993) and Mertens, Sorin and Zamir (1994) it was shown that the universal type space is beliefs complete (i.e. for each player, every probability measure on the product of the space of states of nature and the other players' component spaces is the marginal of a type of that player). However, the general measure-theoretic case was left open up to now.

We show here in Theorem 4 that this is also still true in the general measure-theoretic setting, in the introspective as well as in the non-introspective case. Moreover the component space of each player is - as a measurable space - isomorphic to the space of probability measures on the whole space in the nonintrospective case, and in the introspective case, the component space of each player is isomorphic to the space of probability measures on the product of the space of states of nature and the other players' component spaces.

2 Preliminaries

For every set M, denote by $|M|$ the cardinality of M. For this paper, fix two nonempty sets X (the set of primitive propositions (to be interpreted as statements about nature, i.e. the primary source of uncertainty for the players))[2] and I (the set of players) and assume without loss of generality that $0 \notin I$ and define $I_0 := I \cup \{0\}$. Let $\aleph_\gamma := \max\{|I|, |X|, \aleph_0\}$.

In this paper α and β denote rational numbers $\in [0,1]$, φ, χ, ψ formulas and ω formulas that are conjunctions of maximal consistent sets of finitary formulas.

Definition 1 The set $\mathcal{L}_0$ of *finitary formulas* is the least set such that:

(1) each $x \in X \cup \{\top\}$ is a finitary formula,

(2) if φ is a finitary formula, then $(\neg\varphi)$ is a finitary formula,

(3) if φ and ψ are finitary formulas, then $(\varphi \wedge \psi)$ is a finitary formula,

(4) if φ is a finitary formula, then for every $i \in I$ and rational $\alpha \in [0,1]$: $(p_i^\alpha(\varphi))$ is a finitary formula.

Definition 2 The set $\mathcal{L}$ of *formulas* is the least set such that:

(1) each $\varphi \in \mathcal{L}_0$ is a formula,

(2) if φ is a formula, then $(\neg\varphi)$ is a formula,

(3) if Φ is a set of formulas of cardinality $\leq 2^{\aleph_\gamma}$, then $\left(\bigwedge_{\varphi\in\Phi}\varphi\right)$ is a formula.[3]

[2]If we define things in this way, the space of states of nature in the literature corresponds to $\mathrm{Pow}(X)$ and the σ-field on the space of states of nature corresponds to the σ-field on $\mathrm{Pow}(X)$ generated by the sets $\{s \subseteq X \mid x \in s\}$, where $x \in X$.

[3]If we write "$\varphi \wedge \psi$", where $\varphi, \psi \in \mathcal{L}$, we mean implicitly the formula $\bigwedge_{\chi\in\{\varphi,\psi\}}\chi$.

As usual, "$\leftrightarrow$", "$\rightarrow$", "$\vee$" and "$\bigvee$" are abbreviations, defined in the usual way: $\left(\bigvee_{\varphi\in\Phi}\varphi\right) := \left(\neg\left(\bigwedge_{\varphi\in\Phi}(\neg\varphi)\right)\right)$, $(\varphi\rightarrow\psi) := ((\neg\varphi)\vee\psi)$, $(\varphi\leftrightarrow\psi) := ((\varphi\rightarrow\psi)\wedge(\psi\rightarrow\varphi))$.

To avoid the use of too many brackets, we take the usual convention of decreasing priority: $\neg$, $\bigwedge$, $\wedge$, $\bigvee$, $\vee$, $\rightarrow$, $\leftrightarrow$. This means, for example, that "$\neg\varphi\wedge\psi$" is an abbreviation for "$((\neg\varphi)\wedge\psi)$".

Definition 3 The set $\mathcal{L}^0$ of 0-*formulas* is the set of (infinitary) propositional formulas in $\mathcal{L}$. More formally, it is the least set of formulas (and obviously: subset of $\mathcal{L}$) such that:

(1) each $x\in X\cup\{\top\}$ is a 0-formula,

(2) if φ is a 0-formula, then $\neg\varphi$ is a 0-formula,

(3) if Φ is a set of 0-formulas of cardinality $\leq 2^{\aleph_\gamma}$, then $\bigwedge_{\varphi\in\Phi}\varphi$ is a 0-formula.

Definition 4 Let $i\in I$. The set $\mathcal{L}^i$ of i-*formulas* is the least set of formulas (and obviously: subset of $\mathcal{L}$) such that:

(1) if $\varphi\in\mathcal{L}_0$, then for every rational $\alpha\in[0,1]: p_i^\alpha(\varphi)$ is an i-formula,

(2) if φ is an i-formula, then $\neg\varphi$ is an i-formula,

(3) if Φ is a set of i-formulas of cardinality $\leq 2^{\aleph_\gamma}$, then $\bigwedge_{\varphi\in\Phi}\varphi$ is an i-formula.

Definition 5 For $i\in I_0$ define $\mathcal{L}_0^i := \mathcal{L}_0\cap\mathcal{L}^i$.

Definition 6 Let M be a nonempty set and let Σ be a σ-field on M. We denote by $\Delta(M,\Sigma)$ - or short: $\Delta(M)$ - the set of all σ-additive probability measures on (M,Σ). Unless stated differently, we consider $\Delta(M,\Sigma)$ as a measurable space with the σ-field Σ_Δ generated by all the sets of the form $b^\alpha(E) := \{\mu\in\Delta(M,\Sigma)\mid\mu(E)\geq\alpha\}$, for an event $E\in\Sigma$ and $\alpha\in[0,1]\cap\mathbb{Q}$.

Note that if $r \in [0,1]$ and $E \in \Sigma$, then $b^r(E) = \bigcap_{\alpha \in [0,r] \cap \mathbb{Q}} b^\alpha(E) \in \Sigma_\Delta$. Therefore Σ_Δ is also generated by all the sets $b^r(E)$, where $E \in \Sigma$ and $r \in [0,1]$.

Definition 7 A *type space on* X (for player set I) is a 4-tuple

$$\underline{M} = \langle M, \Sigma, (T_i)_{i \in I,} v \rangle,$$

where

- M is a nonempty set,
- Σ is a σ-field on M,
- T_i is a measurable mapping from M to $\Delta(M,\Sigma)$, the space of probability measures on (M,Σ), for $i \in I$,
- v is a mapping from $M \times (X \cup \{\top\})$ to $\{0,1\}$, such that $v(\cdot, x)$ is measurable in M, for every $x \in X$, and such that $v(m,\top) = 1$, for all $m \in M$.

Definition 8 For a type space $\langle M, \Sigma, (T_i)_{i \in I,} v \rangle$ define

$$[T_i(m)] := \{m' \in M \mid T_i(m') = T_i(m)\},$$

for $m \in M$ and $i \in I$.

$$\langle M, \Sigma, (T_i)_{i \in I,} v \rangle$$

is called a *Harsanyi type space* iff for all $A \in \Sigma$, $m \in M$ and $i \in I$: $A \supseteq [T_i(m)]$ implies $T_i(m)(A) = 1$.[4]

The following lemma, which will be needed in the proof of the Completeness Theorem, is a slightly changed version of Lemma 2.1. of Heifetz and Samet (1999):

[4]Note that if $[T_i(m)]$ is measurable, then this condition reduces to: $T_i(m)([T_i(m)]) = 1$.

Lemma 1 *Let M be a nonempty set, let $\mathcal{F}$ be a field on M that generates the σ-field Σ on M and let $\mathcal{F}_\Delta$ be the σ-field on $\Delta(M)$ generated by the sets of the form*

$$b^p(E) := \{\mu \in \Delta(M, \Sigma) \mid \mu(E) \geq p\},$$

where $E \in \mathcal{F}$ and $p \in [0,1] \cap \mathbb{Q}$.
Then $\mathcal{F}_\Delta = \Sigma_\Delta$.

Proof The proof is the same as the proof of Lemma 2.1. of Heifetz and Samet (1999), if we replace there "such that $\beta^p(F) \in \mathcal{F}_\Delta$ for all $0 \leq p \leq 1$" by "such that $\beta^p(F) \in \mathcal{F}_\Delta$ for all $p \in [0,1] \cap \mathbb{Q}$". ■

Definition 9 Let $\underline{M} = \langle M, \Sigma, (T_i)_{i\in I}, v\rangle$ be a type space on X (for player set I). We define:

- $(\underline{M}, m) \vDash \top$ in any case,
- $(\underline{M}, m) \vDash x$ iff $v(m, x) = 1$,
- for all $\varphi, \psi \in \mathcal{L}$:
  $(\underline{M}, m) \vDash \varphi \wedge \psi$ iff $(\underline{M}, m) \vDash \varphi$ and $(\underline{M}, m) \vDash \psi$,
- for every $\varphi \in \mathcal{L}$:
  $(\underline{M}, m) \vDash \neg\varphi$ iff $(\underline{M}, m) \nvDash \varphi$,
- for $\varphi \in \mathcal{L}_0$, such that $[\varphi]^{\underline{M}} := \{m \in M \mid (\underline{M}, m) \vDash \varphi\} \in \Sigma$, and for $i \in I$ and rational $\alpha \in [0,1]$:
  $(\underline{M}, m) \vDash p_i^\alpha(\varphi)$ iff $T_i(m)([\varphi]^{\underline{M}}) \geq \alpha$.

It is easy to show by induction on the formation of the formulas in $\mathcal{L}_0$ that $[\varphi]^{\underline{M}} \in \Sigma$, for every $\varphi \in \mathcal{L}_0$ (in particular, since $T_i : M \to \Delta(M)$ is $\Sigma - \Sigma_\Delta$-measurable, it follows that $[\varphi]^{\underline{M}} \in \Sigma$ implies $[p_i^\alpha\varphi]^{\underline{M}} \in \Sigma$), so the relation "$(\underline{M}, m) \vDash \varphi$?" is well-defined for every $\varphi \in \mathcal{L}_0$.

- If $\Phi \subseteq \mathcal{L}$ and $|\Phi| \leq 2^{\aleph_\gamma}$, then:
  $(\underline{M}, m) \vDash \bigwedge_{\varphi\in\Phi} \varphi$ iff for every $\varphi \in \Phi : (\underline{M}, m) \vDash \varphi$.

It is now easy to show, that the relation "$(\underline{M}, m) \models \varphi$?" is well-defined for every $\varphi \in \mathcal{L}$.

Definition 10 A formula $\varphi \in \mathcal{L}$ is *valid*, iff for every type space (resp. Harsanyi type space) $\underline{M} = \langle M, \Sigma, (T_i)_{i\in I,}\ v\rangle$ and every $m \in M : (\underline{M}, m) \models \varphi$.

Notation 1 Let $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$. We write $\Gamma \models \varphi$, iff for every type space (resp. Harsanyi type space) $\underline{M}$ and every $m \in M : (\underline{M}, m) \models \psi$, for all $\psi \in \Gamma$ implies $(\underline{M}, m) \models \varphi$.

3 Strong Completeness

In this section we shall define our axioms and inference rules, our notion of "proof" (in the sense of our logic) and prove strong soundness and - by constructing the canonical model - strong completeness.

The List of Axioms

In all what follows (in this paper), α and β are rational numbers in $[0, 1]$.

(A0) $\top$

(A1) $\varphi \to (\psi \to \varphi)$, for $\varphi, \psi \in \mathcal{L}$,

(A2) $(\varphi \to (\psi \to \varrho)) \to ((\varphi \to \psi) \to (\varphi \to \varrho))$, for $\varphi, \psi, \varrho \in \mathcal{L}$,

(A3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$, for $\varphi, \psi \in \mathcal{L}$,

(A4) $\bigwedge_{\varphi\in\Phi}(\psi \to \varphi) \to (\psi \to \bigwedge_{\varphi\in\Phi} \varphi)$, for $\psi \in \mathcal{L}$ and $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \leq 2^{\aleph_\gamma}$,

(A5) $\bigwedge_{\varphi\in\Phi} \varphi \to \psi$, for $\psi \in \Phi$, where $\Phi \subseteq \mathcal{L}$ such that $|\Phi| \leq 2^{\aleph_\gamma}$,

(A6) $\bigwedge_{a\in A}(\bigvee_{b\in A} \varphi_{a,b}) \to \bigvee_{g\in A^A}(\bigwedge_{a\in A} \varphi_{a,g(a)})$, whenever $|A| \leq \aleph_\gamma$,[5]

[5] A^A denotes here the set of all functions from A to A.

(P1) $p_i^0(\varphi)$, for $\varphi \in \mathcal{L}_0$,

(P2) $p_i^1(\top)$,

(P3) $\bigwedge_{\alpha<\beta} p_i^\alpha(\varphi) \to p_i^\beta(\varphi)$, for $\varphi \in \mathcal{L}_0$,

(P4) $\left(p_i^\alpha(\varphi \wedge \psi) \wedge p_i^\beta(\varphi \wedge \neg\psi)\right) \to p_i^{\alpha+\beta}(\varphi)$, for α, β with $\alpha + \beta \leq 1$ and $\varphi, \psi \in \mathcal{L}_0$,

(P5) $\left(\neg p_i^\alpha(\varphi \wedge \psi) \wedge \neg p_i^\beta(\varphi \wedge \neg\psi)\right) \to \neg p_i^{\alpha+\beta}(\varphi)$, for α, β with $\alpha + \beta \leq 1$ and $\varphi, \psi \in \mathcal{L}_0$,

(P6) $p_i^\alpha(\varphi) \to \neg p_i^\beta(\neg\varphi)$, for α, β with $\alpha + \beta > 1$ and $\varphi \in \mathcal{L}_0$,

(P7) $p_i^\alpha(\varphi) \to p_i^\beta(\varphi)$, for α, β with $\beta < \alpha$ and $\varphi \in \mathcal{L}_0$,

(P8) $p_i^1(\varphi \to \psi) \to (p_i^\alpha(\varphi) \to p_i^\alpha(\psi))$, for $\varphi, \psi \in \mathcal{L}_0$,

(I1) $p_i^\alpha(\varphi) \to p_i^1\left(p_i^\alpha(\varphi)\right)$, for $\varphi \in \mathcal{L}_0$,

(I2) $\neg\, p_i^\alpha(\varphi) \to p_i^1(\neg\, p_i^\alpha(\varphi))$, for $\varphi \in \mathcal{L}_0$.

Except (A0) and (P2), all the above axioms are in fact axiom schemes, i.e. lists of axioms.

We adopt the following *inference rules*:

- *Modus Ponens*: From φ and $\varphi \to \psi$ infer ψ.
- *Conjunction*: From Φ infer $\bigwedge_{\varphi\in\Phi} \varphi$, if $\Phi \subseteq \mathcal{L}$ such that $\mid \Phi \mid \leq 2^{\aleph_\gamma}$.
- *Necessitation*: From φ infer $p_i^1(\varphi)$, if $\varphi \in \mathcal{L}_0$.
- *Continuity at $\varnothing$* : From $\bigwedge_{n\in\mathbb{N}} \varphi_n \to \neg\top$, where $\varphi_n \in \mathcal{L}_0$, for all $n \in \mathbb{N}$,

  infer $\bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}}(\bigwedge_{n\leq l} \varphi_n)$.
- *Uncountable Introspection*: From $\varphi \to \bigvee_{n\in\mathbb{N}} \varphi_n$, where $\varphi \in \mathcal{L}^i$ and $\varphi_n \in \mathcal{L}_0$ for all $n \in \mathbb{N}$,

  infer $\varphi \to \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}(\bigvee_{n\leq l} \varphi_n)$.

(A0) – (A6) are the axioms and "Modus Ponens" and "Conjunction" are the inference rules for infinitary propositional logic, where the language is the propositional part, $\mathcal{L}^0$, of our infinitary language $\mathcal{L}$. Karp (1964), has proved strong soundness and strong completeness (Karp (1964, Theorem 5.5.4)) for this logic. We will use this result, sometimes without referring to it explicitly.

Most of the axioms (P1) - (P8), (I1), (I2) above can be found in (Aumann (1995)) and (Heifetz and Mongin (2001)).

Note that the above set of axioms is not minimal:

- (P1) follows from (P3), (A0) and Modus Ponens, if we adopt the usual convention that $\bigwedge_{\varphi \in \emptyset} \varphi := \top$.
- (P2) follows from (A0) and Necessitation.
- Heifetz and Mongin (2001) proved that (P7) follows from (A0) - (A6), (P3) - (P6) and (P8).
- The proof of the Completeness Theorem will also show that (A0) - (A6), (P3) - (P6), (P8) and (I1) imply (I2).
- It is easy to see that the inference rule "Uncountable Introspection" implies (together with (A0) - (A6), (P3) - (P6) and (P8) and the other inference rules) the axioms (I1) and (I2).
- The proof of the Completeness Theorem shows that, in the case of $\aleph_\gamma = \aleph_0$, (I1) (together with (A0) - (A6), (P3) - (P6) and (P8) and the other inference rules) implies the inference rule "Uncountable Introspection".

Definition 11 (1) The *system P* consists of the axioms (A0) - (A6), (P3) - (P6), (P8), and the inference rules "Modus Ponens", "Conjunction", "Necessitation", and "Continuity at $\varnothing$".

(2) The *system H* is the system P together with the additional axiom (I1) if $\aleph_\gamma = \aleph_0$ and the system H is the system P together with the inference rule "Uncountable Introspection" otherwise.

Definition 12 (1) The *set of theorems of the system P* is the minimal set of formulas that contains the axioms (A0) - (A6), (P3) - (P6), (P8), and that is closed under "Modus Ponens", "Conjunction", "Necessitation" and "Continuity at $\varnothing$".

(2) The *set of theorems of the system H* is the minimal set of formulas that contains the axioms (A0) - (A6), (P3) - (P6), (P8), (I1), and that is closed under "Modus Ponens", "Conjunction", "Necessitation" and "Continuity at $\varnothing$", in the case $\aleph_\gamma = \aleph_0$.
And if $\aleph_\gamma > \aleph_0$: The set of theorems of the system H is the minimal set of formulas that contains the axioms (A0) - (A6), (P3) - (P6), (P8), and that is closed under "Modus Ponens", "Conjunction", "Necessitation", "Continuity at $\varnothing$" and "Uncountable Introspection".

In fact we have here two papers in one: Given a nonempty set of players I and a nonempty set of primitive propositions X, if nothing else is said, we do all what follows for the system P on the syntactic side and for the class of type spaces on the semantic side. And we also do all what follows for the system H on the syntactic side and for the class of Harsanyi type spaces on the semantic side. We only specify the system if there is a difference between the two cases in the proofs or the statements in the Lemmas, Propositions or Theorems.

Definition 13 Let Γ be a set of formulas in $\mathcal{L}$. A *proof of* φ *from* Γ is a sequence whose length is smaller than $(2^{\aleph_\gamma})^+$ and whose last formula is φ, such that each formula in the proof is in Γ, a theorem of the system, or inferred from the previous formulas by "Modus Ponens" or "Conjunction".[6]

If there is a proof of φ from Γ, we write $\Gamma \vdash \varphi$. In particular, "$\vdash \varphi$" means that φ is a theorem.

Definition 14 • The system P (resp. H) is strongly sound iff for every $\Gamma \subseteq \mathcal{L}$ and every $\varphi \in \mathcal{L} : \Gamma \vdash \varphi$ implies $\Gamma \vDash \varphi$.

[6]Of course, whether φ is a theorem of the system, resp., whether there is a proof of φ from Γ, depends on the system under consideration, i.e. there might be a proof of φ from Γ in the system H, but not in the system P. It follows also that the notion of consistency depends on the system.

- The system P (resp. H) is strongly complete iff for every $\Gamma \subseteq \mathcal{L}$ and every $\varphi \in \mathcal{L} : \Gamma \vDash \varphi$ implies $\Gamma \vdash \varphi$.

Definition 15 Γ is *consistent* if there is no formula $\varphi \in \mathcal{L}$ such that there are proofs of φ and $\neg\varphi$ from Γ.

Lemma 2 *(1) If $\Phi \subseteq \mathcal{L}$ and $|\Phi| \leq 2^{\aleph_\gamma}$, then $\Phi \vdash \varphi$ iff $\left\{\bigwedge_{\psi\in\Phi} \psi\right\} \vdash \varphi$.*

(2) $\{\psi\} \vdash \varphi$ iff $\vdash \psi \to \varphi$.

(3) If $\Gamma \vdash \varphi \to \psi$ and $\Gamma \vdash \psi \to \chi$, then $\Gamma \vdash \varphi \to \chi$.

(4) $\vdash \varphi \to \neg(\neg\varphi)$.

(5) If $\psi \in \Phi$ and $|\Phi| \leq 2^{\aleph_\gamma}$, then $\vdash \neg\psi \to \neg \bigwedge_{\varphi\in\Phi} \varphi$.

Proof

(1) "If" follows by applying the inference rule "Conjunction" to Φ. "Only if" follows by replacing in the proof of φ from Φ every occurrence of a $\psi \in \Phi$ by the sequence $\bigwedge_{\psi'\in\Phi} \psi'$, $\bigwedge_{\psi'\in\Phi} \psi' \to \psi$, ψ. This yields then a proof of φ from $\left\{\bigwedge_{\psi\in\Phi} \psi\right\}$.

(2) "If" follows immediately by Modus Ponens. "Only if" follows by induction on the length of the proof of φ from $\{\psi\}$. There are four cases:

  (a) $\varphi = \psi$: By (A5) applied to $\{\psi\}$, it follows that $\vdash \psi \to \psi$.

  (b) φ is a theorem: By (A1), $\vdash \varphi \to (\psi \to \varphi)$, and by Modus Ponens it follows that $\vdash \psi \to \varphi$.

  (c) φ follows by Modus Ponens: Then there is a χ such that χ and $\chi \to \varphi$ occur in the proof of φ. The sequences up to (and including) χ and $\chi \to \varphi$ are proofs of χ and $\chi \to \varphi$ from $\{\psi\}$ of shorter length. Hence, by the induction hypothesis, $\vdash \psi \to \chi$ and $\vdash \psi \to (\chi \to \varphi)$. $(\psi \to (\chi \to \varphi)) \to ((\psi \to \chi) \to (\psi \to \varphi))$ is a theorem (A2), so by applying Modus Ponens two times we get $\vdash \psi \to \varphi$.

(d) φ follows by Conjunction: Then $\varphi = \bigwedge_{\chi\in\Phi} \chi$ with $|\Phi| \leq 2^{\aleph_\gamma}$. By the induction hypothesis (since each χ must occur before φ in the proof), we have $\vdash \psi \to \chi$, for every $\chi \in \Phi$. By conjunction, we get $\vdash \bigwedge_{\chi\in\Phi} (\psi \to \chi)$ and by applying Modus Ponens to (A4), $\vdash \psi \to \bigwedge_{\chi\in\Phi} \chi$.

(3) We have $\Gamma \vdash \psi \to \chi$. By (A1), $(\psi \to \chi) \to (\varphi \to (\psi \to \chi))$ is an axiom. Modus Ponens yields $\Gamma \vdash \varphi \to (\psi \to \chi)$. By (A2), $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$ is an axiom. Modus Ponens yields $\Gamma \vdash (\varphi \to \psi) \to (\varphi \to \chi)$. Together with $\Gamma \vdash \varphi \to \psi$, Modus Ponens yields now $\Gamma \vdash \varphi \to \chi$.

(4) and 5. are well-known tautologies of Propositional Calculus, so, according to the Completeness Theorem of Karp (1964, Theorem 5.5.4), theorems of our system.

■

Proposition 1 • *The system P is strongly sound with respect to type spaces.*

• *The system H is strongly sound with respect to Harsanyi type spaces.*

Proof

(1) We have to show that if $\vdash \varphi$ (i.e. φ is a theorem) in the system P (resp. in the system H) and if $\underline{M}$ is a type space (resp. Harsanyi type space) and $m \in M$, then $(\underline{M}, m) \vDash \varphi$.

(2) And we have to show that if $\underline{M}$ is a type space (resp. Harsanyi type space), $m \in M$, $(\underline{M}, m) \vDash \Gamma$ and if $\Gamma \vdash \varphi$ in the system P (resp. in the system H), then $(\underline{M}, m) \vDash \varphi$.

To:

(1) It suffices to show:

(a) If φ is an axiom of the system P (resp. of the system H) and if $\underline{M}$ is a type space (resp. Harsanyi type space) and $m \in M$, then $(\underline{M}, m) \vDash \varphi$.

(b) If φ is valid and $\varphi \to \psi$ is valid, then ψ is valid.

(c) If φ is valid, then $p_i^1(\varphi)$ is valid.

(d) If $|\Phi| \le 2^{\aleph_\gamma}$ and each $\varphi \in \Phi$ is valid, then $\bigwedge_{\varphi\in\Phi} \varphi$ is valid.

(e) If $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$, and $\bigwedge \varphi_n \to \neg\top$ is valid, then $\bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}} (\bigwedge_{n\le l} \varphi_n)$ is valid.

(f) If $\varphi \in \mathcal{L}^i$ and $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$, then the validity of $\varphi \to \bigvee_{n\in\mathbb{N}} \varphi_n$ in the class of Harsanyi type spaces implies that $\varphi \to \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}} (\bigvee_{n\le l} \varphi_n)$ is valid in the class of Harsanyi type spaces.

To:

(a) That the axioms are valid is an easy check, (A6) is valid, provided we include the axiom of choice (like always) in our underlying set theory. (P1) – (P8) correspond to well-known properties of probability measures. (I1): Let $\underline{M}$ be a Harsanyi type space, $\varphi \in \mathcal{L}_0$ and $m \in M$. Then, $(\underline{M}, m) \vDash \neg p_i^\alpha(\varphi) \vee p_i^1(p_i^\alpha(\varphi))$ iff $(\underline{M}, m) \vDash \neg p_i^\alpha(\varphi)$ or $(\underline{M}, m) \vDash p_i^1(p_i^\alpha(\varphi))$. Let $(\underline{M}, m) \vDash p_i^\alpha(\varphi)$. This means that $T_i(m)\left([\varphi]^{\underline{M}}\right) \ge \alpha$. But then $[T_i(m)]^{\underline{M}} \subseteq [p_i^\alpha(\varphi)]^{\underline{M}}$, hence $T_i(m)\left([p_i^\alpha(\varphi)]^{\underline{M}}\right) = 1$ and (I1) is valid. (I2) follows in the same manner.

(b) - (d) above are clear,

(e) corresponds to the continuity at $\varnothing$, a well-known property of σ-additive probability measures: Let $\underline{M}$ be a type space (resp. Harsanyi type space) and $m \in M$. By the definition of "$\vDash$", we have $\left[\bigwedge_{n\in\mathbb{N}} \varphi_n \to \neg\top\right]^{\underline{M}} = \left(M \setminus \bigcap_{n\in\mathbb{N}} [\varphi_n]^{\underline{M}}\right) \cup \left(M \setminus [\top]^{\underline{M}}\right) = \left(M \setminus \bigcap_{n\in\mathbb{N}} [\varphi_n]^{\underline{M}}\right)$. If $\bigwedge_{n\in\mathbb{N}} \varphi_n \to \neg\top$ is valid, then $\bigcap_{n\in\mathbb{N}} [\varphi_n]^{\underline{M}} = \varnothing$. In this case, we have for $E_l := \bigcap_{n\le l} [\varphi_n]^{\underline{M}} = \left[\bigwedge_{n\le l} \varphi_n\right]^{\underline{M}}$, that $E_l \downarrow \varnothing$. So, for every $m \in M$ and $k \in \mathbb{N}$ there is a $l(k,m) \in \mathbb{N}$ such that $T_i(m)\left(E_{l(k,m)}\right) < \frac{1}{k}$. By definition of "$\vDash$", we have $(\underline{M}, m) \vDash \neg p_i^{\frac{1}{k}}\left(\bigwedge_{n\le l(k,m)} \varphi_n\right)$. Again by definition of "$\vDash$", it follows that $(\underline{M}, m) \vDash \bigwedge_{k\in\mathbb{N}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}} (\bigwedge_{n\le l} \varphi_n)$. Hence $\bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}} (\bigwedge_{n\le l} \varphi_n)$ is valid.

(f) It is easy to see by induction on the formation of formulas $\varphi \in \mathcal{L}^i$: Either $[T_i(m)]^{\underline{M}} \subseteq [\varphi]^{\underline{M}}$ or $[T_i(m)]^{\underline{M}} \cap [\varphi]^{\underline{M}} = \emptyset$. Observe that $(\underline{M}, m) \vDash \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}(\bigvee_{n\leq l} \varphi_n)$ iff $\lim_{l\to\infty} T_i(m)([\bigvee_{n\leq l} \varphi_n])$ $= 1$, which is by σ-additivity the case iff $T_i(m)([\bigvee_{n\in\mathbb{N}} \varphi_n]) = 1$. Assume that $\varphi \to \bigvee_{n\in\mathbb{N}} \varphi_n$ is valid in the class of Harsanyi type spaces and assume for a Harsanyi type space $\underline{M}$ that $(\underline{M}, m) \vDash \varphi$. By the above, $[T_i(m)]^{\underline{M}} \subseteq [\varphi]^{\underline{M}}$, since φ is an i-formula. This implies that $[T_i(m)]^{\underline{M}} \subseteq [\bigvee_{n\in\mathbb{N}} \varphi_n]^{\underline{M}}$ and by the introspection property of the Harsanyi type spaces, $T_i(m)([\bigvee_{n\in\mathbb{N}} \varphi_n]) = 1$. The above observation implies now that
$(\underline{M}, m) \vDash \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}(\bigvee_{n\leq l} \varphi_n)$, hence
$\varphi \to \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}(\bigvee_{n\leq l} \varphi_n)$ is valid in the class of Harsanyi type spaces.

(2) Given 1., we have to show :

(a) If $(\underline{M}, m) \vDash \varphi$ and $(\underline{M}, m) \vDash \varphi \to \psi$, then $(\underline{M}, m) \vDash \psi$. But $(\underline{M}, m) \vDash \varphi \to \psi$ iff $m \in [\neg\varphi \vee \psi]^{\underline{M}} = \left(M \setminus [\varphi]^{\underline{M}}\right) \cup [\psi]^{\underline{M}}$, so $m \in [\varphi]^{\underline{M}}$ implies $m \in [\psi]^{\underline{M}}$.

(b) If $|\Phi| \leq 2^{\aleph_\gamma}$ and for all $\varphi \in \Phi : (\underline{M}, m) \vDash \varphi$, then $(\underline{M}, m) \vDash \bigwedge_{\varphi\in\Phi} \varphi$, but this is clear by the definition of "$\vDash$".

■

Remark 1 $\mid \mathcal{L}_0 \mid = \max\{\mid I \mid, \mid X \mid, \aleph_0\} = \aleph_\gamma$.

Definition 16

- $\Omega :=$ $\left\{\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\psi\in\mathcal{L}_0\setminus\Phi_0} \neg\psi \mid \Phi_0 \subseteq \mathcal{L}_0, \text{ s.t. } \Phi_0 \cup \{\neg\psi | \psi \in \mathcal{L}_0\setminus\Phi_0\} \text{ is consistent}\right\}$,
- $[\psi] := \{\omega \in \Omega \mid\vdash \omega \to \psi\}$,
- for $\Gamma \subseteq \mathcal{L}$ define $[\Gamma] := \bigcap_{\psi\in\Gamma} [\psi]$,
- for $\omega \in \Omega$, such that $\omega = \bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\psi\in\mathcal{L}_0\setminus\Phi_0} \neg\psi$, define $\Psi_\omega := \Phi_0 \cup \{\neg\psi \mid \psi \in \mathcal{L}_0\setminus\Phi_0\}$.

Note that although we write "Ω", we define in fact two Ω's, one corresponding the to system P, and one corresponding to the system H.

Proposition 2 *(1)* $\vdash \bigvee_{\omega\in\Omega} \omega$.

(2) For every formula $\psi \in \mathcal{L}$ *and for every* $\omega \in \Omega$ *: Either* $\vdash \omega \to \psi$ *or* $\vdash \omega \to \neg\psi$ *.*

(3) $\vdash \psi \leftrightarrow \bigvee_{\omega\in[\psi]} \omega$.

(4) If $\Phi \subseteq \mathcal{L}$ *such that* $|\Phi| \leq 2^{\aleph_\gamma}$*, then* $\vdash \bigwedge_{\varphi\in\Phi} \varphi \leftrightarrow \bigvee_{\omega\in[\Phi]} \omega$.

(5) $\vdash \neg\varphi \leftrightarrow \bigvee_{\omega\in\Omega\setminus[\varphi]} \omega$.

(6) $[\neg\varphi] = \Omega \setminus [\varphi]$ *.*

(7) If $\Phi \subseteq \mathcal{L}$ *such that* $|\Phi| \leq 2^{\aleph_\gamma}$*, then* $[\Phi] = \left[\bigwedge_{\varphi\in\Phi} \varphi\right]$ *.*

Proof

(1) By (A5), $\vdash \varphi \vee \neg\varphi$. Since $|\mathcal{L}_0| \leq \aleph_\gamma$, it follows by Conjunction that $\vdash \bigwedge_{\varphi\in\mathcal{L}_0} (\varphi \vee \neg\varphi)$. By (A6) and Modus Ponens, it follows that $\vdash \bigvee_{\Phi_0\subseteq\mathcal{L}_0} \left(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\setminus\Phi_0} \neg\varphi\right)$. If $\Phi_0 \cup \{\neg\varphi | \varphi \in \mathcal{L}_0\setminus\Phi_0\}$ is inconsistent (i.e. not consistent), then it follows by 1 and 2 of the Lemma 2 that $\vdash \left(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\setminus\Phi_0} \neg\varphi\right) \to \psi$ and $\vdash \left(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\setminus\Phi_0} \neg\varphi\right) \to \neg\psi$, for a $\psi \in \mathcal{L}$. By Conjunction, (A4) and Modus Ponens, we get $\vdash \left(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\setminus\Phi_0} \neg\varphi\right) \to (\psi \wedge \neg\psi)$. Since $(\chi \to \rho) \to (\neg\rho \to \neg\chi)$ is a tautology of the Propositional Calculus, we get, by Modus Ponens, $\vdash \neg(\psi \wedge \neg\psi) \to \neg\left(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\setminus\Phi_0} \neg\varphi\right)$. $\neg(\psi \wedge \neg\psi)$ is a tautology of the Propositional Calculus, hence Modus Ponens yields $\vdash \neg\left(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\setminus\Phi_0} \neg\varphi\right)$. Let $\mathbf{C}_0$ be the set of all $\Phi_0 \subseteq \mathcal{L}_0$ such that $\Phi_0 \cup \{\neg\varphi | \varphi \in \mathcal{L}_0\setminus\Phi_0\}$ is inconsistent. By Conjunction, $\bigwedge_{\Phi_0\in\mathbf{C}_0} \neg\left(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\setminus\Phi_0} \neg\varphi\right)$ is a theorem. Since $\bigvee_{\Phi_0\subseteq\mathcal{L}_0} \left(\bigwedge_{\varphi\in\Phi_0} \varphi \wedge \bigwedge_{\varphi\in\mathcal{L}_0\setminus\Phi_0} \neg\varphi\right) =$

$\left(\bigwedge_{\Phi_0\in\mathbf{C}_0}\neg\left(\bigwedge_{\varphi\in\Phi_0}\varphi\wedge\bigwedge_{\varphi\in\mathcal{L}_0\setminus\Phi_0}\neg\varphi\right)\right)\rightarrow\left(\bigvee_{\omega\in\Omega}\omega\right)$, by the definition of "$\bigvee$", "$\vee$" and "$\rightarrow$", Modus Ponens yields now $\vdash\bigvee_{\omega\in\Omega}\omega$.

(2) Follows by induction (on the formation of the formulas). If $\vdash\omega\rightarrow\varphi$ and $\vdash\omega\rightarrow\neg\varphi$, then by 1 and 2 of Lemma 2, Ψ_ω is not consistent, a contradiction.
For every $\varphi\in\mathcal{L}_0$ we have $\varphi\in\Psi_\omega$ or $\neg\varphi\in\Psi_\omega$. Again by 1 and 2 of Lemma 2 it follows that $\vdash\omega\rightarrow\varphi$ or $\vdash\omega\rightarrow\neg\varphi$.
If $\varphi=\neg\psi$, then either $\vdash\omega\rightarrow\neg\psi$, or $\vdash\omega\rightarrow\psi$. In the second case, since $\psi\rightarrow\neg(\neg\psi)$ is a tautology of the Propositional Calculus, we have $\vdash\psi\rightarrow\neg\varphi$ and by 3 of Lemma 2 it follows that $\vdash\omega\rightarrow\neg\varphi$.
If $|\Phi|\leq 2^{\aleph_\gamma}$ and $\varphi=\bigwedge_{\psi\in\Phi}\psi$, then by induction hypothesis, either $\vdash\omega\rightarrow\psi$ for every $\psi\in\Phi$ or there is a $\chi\in\Phi$ such that $\vdash\omega\rightarrow\neg\chi$. In the first case, by Conjunction, it follows that $\vdash\bigwedge_{\psi\in\Phi}\omega\rightarrow\psi$ and by (A4) and Modus Ponens, $\vdash\omega\rightarrow\bigwedge_{\psi\in\Phi}\psi$. In the second case, since $\neg\chi\rightarrow\neg\bigwedge_{\psi\in\Phi}\psi$ is a tautology of the Propositional Calculus, hence a theorem, we conclude by 3 of Lemma 2 that $\vdash\omega\rightarrow\neg\bigwedge_{\psi\in\Phi}\psi$.

(3) If $\omega\notin[\psi]$, then $\vdash\omega\rightarrow\neg\psi$. But then, since $(\omega\rightarrow\neg\psi)\rightarrow(\psi\rightarrow\neg\omega)$ is a tautology of the Propositional Calculus, we have $\vdash\psi\rightarrow\neg\omega$. By Conjunction, (A4) and Modus Ponens, we conclude $\vdash\psi\rightarrow\bigwedge_{\omega\in\Omega\setminus[\psi]}\neg\omega$. $\bigwedge_{\omega\in\Omega\setminus[\psi]}\neg\omega\rightarrow\neg\bigvee_{\omega\in\Omega\setminus[\psi]}\omega$ is a tautology of the Propositional Calculus, so we conclude $\vdash\psi\rightarrow\neg\bigvee_{\omega\in\Omega\setminus[\psi]}\omega$.
$\bigvee_{\omega\in\Omega}\omega\rightarrow\left(\neg\bigvee_{\omega\in\Omega\setminus[\psi]}\omega\rightarrow\bigvee_{\omega\in[\psi]}\omega\right)$ is a tautology of the Propositional Calculus, hence a theorem, so we infer by 1. and Modus Ponens that $\vdash\neg\bigvee_{\omega\in\Omega\setminus[\psi]}\omega\rightarrow\bigvee_{\omega\in[\psi]}\omega$. By 3 of Lemma 2, it follows that $\vdash\psi\rightarrow\bigvee_{\omega\in[\psi]}\omega$.
If $\omega\in[\psi]$, then $\vdash\omega\rightarrow\psi$. Since $(\omega\rightarrow\psi)\rightarrow(\neg\psi\rightarrow\neg\omega)$ is a tautology of the Propositional Calculus, hence a theorem, it follows by Modus Ponens that $\vdash\neg\psi\rightarrow\neg\omega$. By Conjunction, (A4) and Modus Ponens, we get $\vdash\neg\psi\rightarrow\bigwedge_{\omega\in[\psi]}\neg\omega$. Then, since $\left(\neg\psi\rightarrow\bigwedge_{\omega\in[\psi]}\neg\omega\right)\rightarrow\left(\bigvee_{\omega\in[\psi]}\omega\rightarrow\psi\right)$ is a tautology of the Propositional Calculus, hence a theorem, it follows by Modus Ponens that $\vdash\bigvee_{\omega\in[\psi]}\omega\rightarrow\psi$, so, by Conjunction, we conclude $\vdash\psi\leftrightarrow\bigvee_{\omega\in[\psi]}\omega$.

(4) By 3 it suffices to show that $\omega\in\left[\bigwedge_{\varphi\in\Phi}\varphi\right]$ iff $\omega\in[\Phi]$. Let $\omega\in[\Phi]$.

Then, for every $\varphi \in \Phi$, we have $\vdash \omega \to \varphi$. By Conjunction, (A4) and Modus Ponens it follows that $\vdash \omega \to \bigwedge_{\varphi\in\Phi} \varphi$, so $\omega \in \left[\bigwedge_{\varphi\in\Phi} \varphi\right]$.

If $\omega \in \left[\bigwedge_{\varphi\in\Phi} \varphi\right]$, then $\vdash \omega \to \bigwedge_{\varphi\in\Phi} \varphi$. For every $\psi \in \Phi$ we have, by (A5), $\vdash \bigwedge_{\varphi\in\Phi} \varphi \to \psi$, so, by 3 of Lemma 2, it follows that $\vdash \omega \to \psi$, so $\omega \in [\Phi]$.

(5) By 2. we have $\Omega\backslash [\psi] = [\neg\psi]$. 5. follows now from 3.

(6) See the Proof of 5.

(7) See the Proof of 4.

■

Remark 2 *(1) The class of (Harsanyi) type spaces is nonempty.*

(2) The set Ω is nonempty.

Proof

(1) Let $M := \{m\}$, Σ the power set of M, for every $i \in I$: $T_i(m)$ the delta measure at m, and for every $x \in X \cup \{\top\}$: $v(m, x) := 1$. Then, $\langle M, \Sigma, (T_i)_{i\in I}, v\rangle$ forms a Harsanyi type space on X (for player set I).

(2) follows immediately from 1 of Proposition 2), 1. of this Remark and the Strong Soundness (Proposition 1).

■

Now, we are going to build the canonical model. The first step is to define a measurable space:

Definition 17 Let Σ be the $\sigma-$Field on Ω generated by the set $\{[\psi] \mid \psi \in \mathcal{L}_0\}$.

By Proposition 2, it follows that $\Omega = [\top]$ and $\Omega\backslash [\psi] = [\neg\psi]$, for every $\psi \in \mathcal{L}_0$. We have $\vdash \omega \to \varphi$ and $\vdash \omega \to \psi$ iff $\vdash \omega \to \varphi \wedge \psi$. Hence:

Remark 3 *The set* $\mathcal{F} := \{[\psi] \mid \psi \in \mathcal{L}_0\}$ *is a field on* Ω.

Definition 18

- For $\psi \in \mathcal{L}_0$ define

$$T_i'(\omega)([\psi]) := \sup\{\alpha \mid \vdash \omega \to p_i^\alpha(\psi)\}.$$

- Define $v(\omega, x) := \begin{cases} 1, & \text{if} \quad \omega \in [x], \\ 0, & \text{if} \quad \omega \notin [x]. \end{cases}$

Obviously, we have:

Remark 4 $v(\cdot, x)$ *is measurable in* Ω, *for every* $x \in X$.

Lemma 3 *Let* $\psi \in \mathcal{L}_0$ *and* $\omega \in \Omega$ *such that* $\vdash \omega \to \neg p_i^\alpha(\psi)$.
Then $T_i'(\omega)([\psi]) < \alpha$.

Proof Assume that $T_i'(\omega)([\psi]) \geq \alpha$. Then, for every $\beta' < \alpha$, there is a $\beta > \beta'$ with $\vdash \omega \to p_i^\beta(\psi)$. By (P7), $\vdash p_i^\beta(\psi) \to p_i^{\beta'}(\psi)$. Hence, by 3 of Lemma 2, $\vdash \omega \to p_i^{\beta'}(\psi)$. By Conjunction, (A4) and Modus Ponens, we have $\vdash \omega \to \bigwedge_{\beta<\alpha} p_i^\beta(\psi)$. By 3 of Lemma 2 and (P3), it follows that $\vdash \omega \to p_i^\alpha(\psi)$, a contradiction to 2 of Proposition 2. ■

Lemma 4 *For every* $i \in I$ *and* $\omega \in \Omega : T_i'(\omega)(\cdot)$ *is well-defined and a countably additive measure on* $\mathcal{F}$. *Furthermore*: *For every* $i \in I$ *and* $\omega \in \Omega$: $T_i'(\omega)(\Omega) = 1$.

Proof By (P1), $p_i^0(\varphi)$ is an axiom and by (A1), $p_i^0(\varphi) \to (\omega \to p_i^0(\varphi))$ is an axiom. By Modus Ponens, it follows that $\vdash \omega \to p_i^0(\varphi)$. Hence $T_i'(\omega)([\varphi]) \geq 0$.

Let φ, $\psi \in \mathcal{L}_0$ with $[\varphi] = [\psi]$. (Of course we, have by (A5) and Modus Ponens that $\vdash \varphi \leftrightarrow \varphi'$ implies $\vdash \varphi \to \varphi'$ and $\vdash \varphi' \to \varphi$, and by Conjunction follows the opposite direction.) Then, $\vdash \psi \leftrightarrow \bigvee_{\omega\in[\psi]} \omega$ and $\vdash \left(\bigvee_{\omega\in[\psi]} \omega\right) \leftrightarrow \varphi$, so by Lemma 2, $\vdash \varphi \leftrightarrow \psi$. By Necessitation, (P8) and Modus Ponens, it follows that $\vdash p_i^\alpha(\varphi) \to p_i^\alpha(\psi)$ and $\vdash p_i^\alpha(\psi) \to p_i^\alpha(\varphi)$, so $\sup\{\alpha \mid \vdash \omega \to p_i^\alpha(\varphi)\} = \sup\{\alpha \mid \vdash \omega \to p_i^\alpha(\psi)\}$, and $T_i'(\omega)$ is well-defined.

To show that $T_i'(\omega)$ is countably additive it is enough to show that it is finitely additive and continuous at $\varnothing$ (see Dudley (1989), Theorem 3.1.1). Let $\varphi, \psi \in \mathcal{L}_0$ with $[\varphi] \cap [\psi] = \varnothing$. Then, $[\varphi] \subseteq \Omega \setminus [\psi]$. It follows that $[\varphi] = ([\varphi] \cup [\psi]) \cap (\Omega \setminus [\psi])$ and $[\psi] = ([\varphi] \cup [\psi]) \cap [\psi]$. By 6 and 7 of Proposition 2, it follows that $([\varphi] \cup [\psi]) \cap (\Omega \setminus [\psi]) = [(\varphi \vee \psi) \wedge \neg\psi]$ and $([\varphi] \cup [\psi]) \cap [\psi] = [(\varphi \vee \psi) \wedge \psi]$.

Now, let $T_i'(\omega)([\varphi]) = r$ and $T_i'(\omega)([\psi]) = r'$.

Assume that $r + r' > 1$. Then there are rationals $\alpha, \beta \in [0, 1]$, such that $\alpha < r$, $\beta < r'$ and $\alpha + \beta > 1$. But then, we have $\vdash \omega \to p_i^\alpha((\varphi \vee \psi) \wedge \neg\psi)$ and $\vdash \omega \to p_i^\beta((\varphi \vee \psi) \wedge \psi)$. It is $\vdash (\varphi \vee \psi) \wedge \psi \to \neg((\varphi \vee \psi) \wedge \neg\psi)$, because this is a tautology of the Propositional Calculus. Necessitation, (P8) and Modus Ponens yield now $\vdash p_i^\beta((\varphi \vee \psi) \wedge \psi) \to p_i^\beta(\neg((\varphi \vee \psi) \wedge \neg\psi))$. By Lemma 2, we conclude that $\vdash \omega \to p_i^\beta(\neg((\varphi \vee \psi) \wedge \neg\psi))$. But since $\alpha + \beta > 1$, we have that $p_i^\alpha((\varphi \vee \psi) \wedge \neg\psi) \to \neg p_i^\beta(\neg((\varphi \vee \psi) \wedge \neg\psi))$ is an axiom, hence $\vdash \omega \to \neg p_i^\beta(\neg((\varphi \vee \psi) \wedge \neg\psi))$, by 2 of Proposition 2, a contradiction. So, it follows that $r + r' \leq 1$.

For every $\varepsilon > 0$, there are rationals $\alpha < r$ and $\beta < r'$ such that $\alpha \geq r - \frac{\varepsilon}{2}$ and $\beta \geq r' - \frac{\varepsilon}{2}$. For such α and β we have $\vdash \omega \to p_i^\alpha((\varphi \vee \psi) \wedge \neg\psi)$ and $\vdash \omega \to p_i^\beta((\varphi \vee \psi) \wedge \psi)$, so (by Conjunction, (A4) and Modus Ponens), $\vdash \omega \to p_i^\alpha((\varphi \vee \psi) \wedge \neg\psi) \wedge p_i^\beta((\varphi \vee \psi) \wedge \psi)$. Together with (P4) and Lemma 2, we conclude that $\vdash \omega \to p_i^{\alpha+\beta}(\varphi \vee \psi)$. This implies that $T_i'(\omega)([\varphi] \cup [\psi]) = T_i'(\omega)([\varphi \vee \psi]) \geq r + r'$.

If $r + r' = 1$, we have $T_i'(\omega)([\varphi \vee \psi]) = 1$. If $r + r' < 1$, then for all $\varepsilon > 0$ such that $\varepsilon + r + r' \leq 1$, there are rationals $\alpha, \beta \in [0, 1]$, such that $\alpha > r$, $\beta > r'$ and $\alpha + \beta \leq \varepsilon + r + r'$. For such α, β we have $\vdash \omega \to \neg p_i^\alpha((\varphi \vee \psi) \wedge \neg\psi)$ and $\vdash \omega \to \neg p_i^\beta((\varphi \vee \psi) \wedge \psi)$. This implies (like above, but with the use of (P5)) that $\vdash \omega \to \neg p_i^{\alpha+\beta}(\varphi \vee \psi)$. So, by Lemma 3, we have $T_i'(\omega)([\varphi] \cup [\psi]) = T_i'(\omega)([\varphi \vee \psi]) \leq r + r'$. Altogether, this shows that $T_i'(\omega)$ is finitely additive.

Since $\top$ is an axiom, we have for every $\omega \in \Omega$ that $\{\omega\} \vdash \top$, so, by Lemma 2, $\vdash \omega \to \top$, hence $[\top] = \Omega$. Since $\top$ is a theorem, Necessitation yields $\vdash p_i^1(\top)$, so, as above, we have for every $\omega \in \Omega$ that $\vdash \omega \to p_i^1(\top)$. This implies that $T_i'(\omega)(\Omega) = 1$, for every $\omega \in \Omega$.

It remains to show that $T_i'(\omega)$ is continuous at $\varnothing$: For $n \in \mathbb{N}$, let $E_n = [\varphi_n]$ with $\varphi_n \in \mathcal{L}_0$ and let $E_n \downarrow \varnothing$, that is, for all $n : E_{n+1} \subseteq E_n$ and $\bigcap_{n\in\mathbb{N}} E_n = \varnothing$. Then, by 7 of Proposition 2, we have $[\varphi_n] = \left[\bigwedge_{m\leq n} \varphi_m\right]$ and $\left[\bigwedge_{n\in\mathbb{N}} \varphi_n\right] = \bigcap_{n\in\mathbb{N}} [\varphi_n] = \varnothing$. We showed already that $[\top] = \Omega$, so $[\neg\top] = \varnothing$. It follows that $\Omega = \left(\left(\Omega \setminus \left[\bigwedge_{n\in\mathbb{N}} \varphi_n\right]\right) \cup [\neg\top]\right) = \left[\bigwedge_{n\in\mathbb{N}} \varphi_n \to \neg\top\right]$, by 6 and 7 of Proposition 2. By 3 and 1 of Proposition 2 and Modus Ponens, it follows that $\vdash \bigwedge_{n\in\mathbb{N}} \varphi_n \to \neg\top$. So, by the inference rule "Continuity at $\varnothing$", we have $\vdash \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}} \left(\bigwedge_{n\leq l} \varphi_n\right)$. Hence, for $\omega \in \Omega$, $\{\omega\} \vdash \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}} \left(\bigwedge_{n\leq l} \varphi_n\right)$ and, by 2 of Lemma 2, $\vdash \omega \to \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}} \left(\bigwedge_{n\leq l} \varphi_n\right)$. For $\varepsilon > 0$ fix $k \in \mathbb{N}$ with $\frac{1}{k} \leq \varepsilon$. By (A5) and 3 of Lemma 2, it follows that $\vdash \omega \to \bigvee_{l\in\mathbb{N}} \neg p_i^{\frac{1}{k}} \left(\bigwedge_{n\leq l} \varphi_n\right)$. But then there is a $l \in \mathbb{N}$ such that $\vdash \omega \to \neg p_i^{\frac{1}{k}} \left(\bigwedge_{n\leq l} \varphi_n\right)$, for if not, it follows by 2 of Proposition 2, Conjunction, (A4) and Modus Ponens that $\vdash \omega \to \bigwedge_{l\in\mathbb{N}} p_i^{\frac{1}{k}} \bigwedge_{n\leq l} \varphi_n$, a contradiction. By Lemma 3, it follows that $T_i'(\omega)\left(\left[\bigwedge_{n\leq l} \varphi_n\right]\right) < \frac{1}{k} \leq \varepsilon$. So, we have $\lim_{n\to\infty} T_i'(\omega)(E_n) = 0$. ■

Proposition 3 *(1) For every $i \in I$ and $\omega \in \Omega$ there is a unique extension of $T_i'(\omega)$ to a probability measure on (Ω, Σ), which we denote by $T_i(\omega)$.*

(2) This extension is a measurable mapping from Ω to $\Delta(\Omega, \Sigma)$, the space of probability measures on (Ω, Σ) which is endowed with the σ-field generated by the sets $\{\mu \in \Delta(\Omega, \Sigma) \mid \mu(E) \geq \alpha\}$, where $E \in \Sigma$ and rational $\alpha \in [0, 1]$.

(3)

$$\underline{\Omega} := \left\langle \Omega, \Sigma, (T_i)_{i\in I,}\, v \right\rangle$$

is a type space on X (for player set I).

(4) For every $\psi \in \mathcal{L}$:

$$\left(\left\langle \Omega, \Sigma, (T_i)_{i\in I,}\, v \right\rangle, \omega\right) \vDash \psi \textit{ iff } \omega \in [\psi].$$

(5) If the axiom $(I1)$ is added if $\aleph_\gamma = \aleph_0$ and the inference rule Uncountable Introspection if $\aleph_\gamma > \aleph_0$ (i.e. in the H-system case), then $\langle \Omega, \Sigma, (T_i)_{i\in I,}\, v\rangle$ is a Harsanyi type space on X (for player set I).

Proof

(1) Follows directly from Lemma 4 and Caratheodory's extension Theorem.

(2) Follows from Lemma 1. Since $\mathcal{F}$ is a field that generates Σ, by that Lemma, the σ-field on $\Delta(\Omega, \Sigma)$ generated by the sets $\{\mu \in \Delta(\Omega, \Sigma) \mid \mu(E) \geq \alpha\}$, with $E \in \mathcal{F}$ and rational $\alpha \in [0,1]$, is equal to the $\sigma-$field on $\Delta(\Omega, \Sigma)$ generated by the sets $\{\mu \in \Delta(\Omega, \Sigma) \mid \mu(E) \geq \alpha\}$, with $E \in \Sigma$ and rational $\alpha \in [0,1]$. Inverse images commute with arbitrary intersections and unions and with complements. So, it suffices to show that $\{\omega \mid T_i(\omega)([\psi]) \geq \alpha\} \in \Sigma$, for all $\psi \in \mathcal{L}_{0,}$ $i \in I$ and rational $\alpha \in [0,1]$. By Lemma 3, 2 of Proposition 2 and the definition of $T_i(\omega)$, it follows that $T_i(\omega)(\psi) \geq \alpha$ iff $\vdash \omega \to p_i^\alpha(\psi)$. But we have that $\vdash \omega \to p_i^\alpha(\psi)$ iff $\omega \in [p_i^\alpha(\psi)]$, and $[p_i^\alpha(\psi)] \in \mathcal{F} \subseteq \Sigma$.

(3) Follows from Remark 4, 2 of Remark 2, and 1. and 2. of this proposition.

(4) We proceed by induction on the formation of the formulas in $\mathcal{L}$: Let $\omega \in \Omega$. Then:

   (a) $(\underline{\Omega}, \omega) \vDash x$ iff $v(\omega, x) = 1$ iff $\omega \in [x]$.

   (b) $(\underline{\Omega}, \omega) \vDash \neg\varphi$ iff $(\underline{\Omega}, \omega) \nvDash \varphi$ iff $\omega \notin [\varphi]$ iff $\omega \in [\neg\varphi]$, the last equivalence follows from Proposition 2.

   (c) Let $|\Phi| \leq 2^{\aleph_\gamma}$. $(\underline{\Omega}, \omega) \vDash \bigwedge_{\varphi\in\Phi} \varphi$ iff for all $\varphi \in \Phi$: $(\underline{\Omega}, \omega) \vDash \varphi$ iff $\omega \in \bigcap_{\varphi\in\Phi} [\varphi]$ iff (7 of Prop. 2): $\omega \in \left[\bigwedge_{\varphi\in\Phi} \varphi\right]$.

   (d) Let $\varphi \in \mathcal{L}_0$. $(\underline{\Omega}, \omega) \vDash p_i^\alpha(\varphi)$ iff $\{\omega' \in \Omega \mid (\underline{\Omega}, \omega') \vDash \varphi\} \in \Sigma$ and $T_i(\omega)(\{\omega' \in \Omega \mid (\underline{\Omega}, \omega') \vDash \varphi\}) \geq \alpha$. But, by the induction hypothesis, $\{\omega' \in \Omega \mid (\underline{\Omega}, \omega') \vDash \varphi\} = [\varphi] \in \Sigma$. So, $(\underline{\Omega}, \omega) \vDash p_i^\alpha(\varphi)$ iff sup $\left\{\beta \mid \vdash \omega \to p_i^\beta(\varphi)\right\} \geq \alpha$, which is by Lemma 3 and Proposition 2 the case iff $\vdash \omega \to p_i^\alpha(\varphi)$.

(5)

   (a) Case $\aleph_\gamma = \aleph_0$: We show that, for all $\omega \in \Omega$ and $i \in I$: $[T_i(\omega)]$ is measurable and $T_i(\omega)([T_i(\omega)]) = 1$:

Since $\mathcal{L}_0$ is countable and since $\{\omega' \in \Omega \mid T_i(\omega')([\varphi]) \geq \alpha\} = [p_i^\alpha(\varphi)]$, for $\varphi \in \mathcal{L}_0$, (by 4. of this proposition), the set

$$[T_i(\omega)]_0 := \bigcap_{\alpha \in [0,1] \cap \mathbb{Q},\ \varphi \in \mathcal{L}_0,\ \text{s.t. } T_i(\omega)([\varphi]) \geq \alpha} \{\omega' \in \Omega \mid T_i(\omega')([\varphi]) \geq \alpha\}$$

is measurable. Since $\mathcal{F}$ is closed under complements, every $\omega' \in [T_i(\omega)]_0$ satisfies for all $A \in \mathcal{F}$: $T_i(\omega)(A) = T_i(\omega')(A)$. Since $\mathcal{F}$ is a field which generates Σ, by Caratheodory's extension Theorem, it follows that $T_i(\omega) = T_i(\omega')$. Hence, $[T_i(\omega)] = [T_i(\omega)]_0 \in \Sigma$. By (I1), $p_i^\alpha(\varphi) \to p_i^1(p_i^\alpha(\varphi))$ is an axiom, so, by 3. of Lemma 2, $\vdash \omega \to p_i^\alpha(\varphi)$ implies $\vdash \omega \to p_i^1(p_i^\alpha(\varphi))$, hence, by the definition of $T_i(\omega)$, it follows that $T_i(\omega)([\varphi]) \geq \alpha$ implies $T_i(\omega)([p_i^\alpha(\varphi)]) = 1$. Since $T_i(\omega)$ is a $\sigma-$additive probability measure, it follows that $T_i(\omega)([T_i(\omega)]) = 1$.

(b) Case $\aleph_\gamma > \aleph_0$: We have to show that, for all $\omega \in \Omega$, $i \in I$ and $A \in \Sigma$: $[T_i(\omega)] \subseteq A$ implies $T_i(\omega)(A) = 1$.
By the definition of $T_i(\omega)$ in the proof of Caratheodory's Theorem, it is enough to show that for $(\varphi_n)_{n\in\mathbb{N}}$, where $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$: $\bigcup_{n\in\mathbb{N}}[\varphi_n] \supseteq A$ implies $\sum_{n\in\mathbb{N}} T_i(\omega)([\varphi_n]) \geq 1$. We can assume without loss of generality that the $[\varphi_n]$ are pairwise disjoint. (That $\inf\{\sum_{n\in\mathbb{N}} T_i'(\omega)([\varphi_n]) \mid \varphi_n \in \mathcal{L}_0, n \in \mathbb{N} \text{ s.t. } \bigcup_{n\in\mathbb{N}}[\varphi_n] \supseteq A\} \leq 1$ is clear, because $A \subseteq \Omega$ and $T_i'(\omega)(\Omega) = 1$.) For $i \in I$ and $\omega \in \Omega$ define $\varphi_i(\omega) :=$

$$\bigwedge_{\alpha \in [0,1]\cap\mathbb{Q},\ \chi \in \mathcal{L}_0,\ \text{s.t. } \vdash \omega \to p_i^\alpha(\chi)} p_i^\alpha(\chi) \wedge \bigwedge_{\beta \in [0,1]\cap\mathbb{Q},\ \psi \in \mathcal{L}_0,\ \text{s.t. } \vdash \omega \to \neg p_i^\beta(\psi)} \neg p_i^\beta(\psi).$$

By the definition of $T_i(\omega)$, we have $[T_i(\omega)] = [\varphi_i(\omega)]$, where the formula on the right hand side is an i-formula. From $[T_i(\omega)] \subseteq \bigcup_{n\in\mathbb{N}}[\varphi_n] = [\bigvee_{n\in\mathbb{N}} \varphi_n]$ it follows that $\Omega = \left[\varphi_i(\omega) \to \bigvee_{n\in\mathbb{N}} \varphi_n\right]$. By 1 and 3 of Proposition 2 and Modus Ponens, it follows that the above formula is a theorem. By the inference rule "Uncountable Introspection", we can conclude that
$\varphi_i(\omega) \to \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}\left(\bigvee_{n\leq l} \varphi_n\right)$ is a theorem. Since $\vdash \omega \to \varphi_i(\omega)$, it follows that $\vdash \omega \to \bigwedge_{k\in\mathbb{N}\setminus\{0\}} \bigvee_{l\in\mathbb{N}} p_i^{1-\frac{1}{k}}\left(\bigvee_{n\leq l} \varphi_n\right)$, which implies that $\lim_{l\to\infty} T_i(\omega)([\bigvee_{n\in\mathbb{N}} \varphi_n]) = 1 \leq \sum_{n\in\mathbb{N}} T_i(\omega)([\varphi_n])$.

■

Theorem 1 *(1) The system P is strongly sound and strongly complete with respect to type spaces.*

(2) The system H is strongly sound and strongly complete with respect to Harsanyi type spaces.

Proof The soundness follows from Proposition 1.

According to Proposition 3, $\underline{\Omega} = \langle \Omega,\ \Sigma,\ (T_i)_{i \in I,}\ v \rangle$ is a type space (resp. a Harsanyi type space).

Let $\Gamma \vDash \varphi$ in the class of type spaces (resp. Harsanyi type spaces). Then, for all $\omega \in \Omega$, $i \in I$: $(\underline{\Omega}, \omega) \vDash \Gamma$ implies $(\underline{\Omega}, \omega) \vDash \varphi$. So, by 4 of Proposition 3, $[\Gamma] \subseteq [\varphi]$, which implies $\vdash \bigvee_{\omega \in [\Gamma]} \omega \rightarrow \varphi$, because for sets of formulas $A \subseteq B$ with $|B| \leq 2^{\aleph_\gamma}$: $\bigvee_{\varphi \in A} \varphi \rightarrow \bigvee_{\varphi \in B} \varphi$ is a tautology of the Propositional Calculus. Let $\omega \in \Omega$ and $\omega \notin [\Gamma]$. Then there is $\psi \in \Gamma$ with $\omega \notin [\psi]$, so $\vdash \omega \rightarrow \neg\psi$. But then, $\vdash \psi \rightarrow \neg\omega$ (since $(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)$ is a tautology of the Propositional Calculus). By Modus Ponens, it follows that $\Gamma \vdash \neg\omega$. By Conjunction and the fact that $|\Omega| \leq 2^{\aleph_\gamma}$, we have $\Gamma \vdash \bigwedge_{\omega \notin [\Gamma]} \neg\omega$. Then, since (as in the proof of 3. of Proposition 2) $\vdash \bigwedge_{\omega \notin [\Gamma]} \neg\omega \rightarrow \bigvee_{\omega \in [\Gamma]} \omega$ is a theorem, it follows that $\Gamma \vdash \bigvee_{\omega \in [\Gamma]} \omega$, hence, by Modus Ponens, we have $\Gamma \vdash \varphi$. ■

Corollary 1 *Let* $\Gamma \subseteq \mathcal{L}$. *Then* Γ *is consistent iff* Γ *has a model. Furthermore, if* Γ *is consistent, then there is a* $\omega \in \Omega$ *such that* $(\underline{\Omega}, \omega) \vDash \Gamma$.

Proof Assume that $(\underline{\Omega}, \omega) \nvDash \Gamma$, for every $\omega \in \Omega$. Hence, for every ω there is a $\varphi_\omega \in \Gamma$ such that $(\underline{\Omega}, \omega) \vDash \neg\varphi_\omega$. By 4 of Proposition 3, it follows that $\omega \in [\neg\varphi_\omega]$, that is $\vdash \omega \rightarrow \neg\varphi_\omega$, hence, $\vdash \varphi_\omega \rightarrow \neg\omega$. Since $|\Omega| \leq 2^{\aleph_\gamma}$, we have $\vdash \bigwedge_{\omega' \in \Omega} \varphi_{\omega'} \rightarrow \neg\omega$ (because $\vdash \bigwedge_{\omega' \in \Omega} \varphi_{\omega'} \rightarrow \varphi_\omega$ is an axiom). It follows (by Conjunction, (A4) and Modus Ponens) that $\vdash \bigwedge_{\omega' \in \Omega} \varphi_{\omega'} \rightarrow \bigwedge_{\omega \in \Omega} \neg\omega$. By Proposition 2, we have $[\neg (x \wedge \neg x)] = \Omega$, for $x \in X$. So, again by Proposition 2, it follows that $\vdash \neg (x \wedge \neg x) \rightarrow \bigvee_{\omega \in \Omega} \omega$ and hence $\vdash \bigwedge_{\omega \in \Omega} \neg\omega \rightarrow x \wedge \neg x$, which implies $\vdash \bigwedge_{\omega' \in \Omega} \varphi_{\omega'} \rightarrow x \wedge \neg x$, and we conclude $\Gamma \vdash x \wedge \neg x$. By (A5) and Modus Ponens, it follows that $\Gamma \vdash x$ and $\Gamma \vdash \neg x$, so Γ is inconsistent.

If Γ is not consistent, then there is a φ with $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$. So, by the strong soundness, for every type space (resp. Harsanyi type space) $\underline{M}$ and every $m \in M$: If $(\underline{M}, m) \vDash \Gamma$, then $(\underline{M}, m) \vDash \varphi$ and $(\underline{M}, m) \vDash \neg\varphi$. By the definition of the relation "$\vDash$", there is no $(\underline{M}, m)$ with this property. So Γ has no model. ■

4 Universality of the Canonical (Harsanyi) Type Space

In this section we shall prove that, by a fortunate coincidence, the canonical (Harsanyi) type space is (up to type isomorphism) the universal (Harsanyi) type space. This gives a characterization of the universal (Harsanyi) type space and shows that our language is rich enough to describe the states in the universal (Harsanyi) type space. (So, in some sense, the language is rich enough to capture "all relevant information".)

Definition 19 Let $\underline{M} = \langle M, \Sigma, (T_i)_{i \in I}, v\rangle$ and $\underline{N} = \langle N, \Sigma^N, \left(T_i^N\right)_{i \in I}, v^N\rangle$ be type spaces on X (for player set I). A function $f : M \to N$ is a *type morphism* if it satisfies the following conditions:

(1) f is Σ-Σ^N measurable,

(2) for all $m \in M$ and $x \in X : v(m, x) = v(f(m), x)$,

(3) for all $m \in M$, $E \in \Sigma^N$, and $i \in I : T_i^N(f(m))(E) = T_i(m)\left(f^{-1}(E)\right)$.

Definition 20 A type morphism f is a *type isomorphism* if it is one-to-one, onto, and the inverse of f is also a type morphism.

Lemma 5 *Type morphisms preserve the validity of formulas, i.e. if f is a type morphism from $\underline{M}$ to $\underline{N}$, $m \in M$, and $\varphi \in \mathcal{L}$, then*

$$(\underline{M}, m) \vDash \varphi \text{ iff } (\underline{N}, f(m)) \vDash \varphi.$$

Proof

(1) $(\underline{M}, m) \vDash x$ iff $v(m, x) = 1$ iff $v(f(m), x) = 1$ iff $(\underline{N}, f(m)) \vDash x$.

(2) Let $|\Phi| \leq 2^{\aleph_\gamma}$. Then, $(\underline{M}, m) \vDash \bigwedge_{\varphi \in \Phi} \varphi$ iff $(\underline{M}, m) \vDash \varphi$, for all $\varphi \in \Phi$, iff $(\underline{N}, f(m)) \vDash \varphi$, for all $\varphi \in \Phi$, iff $(\underline{N}, f(m)) \vDash \bigwedge_{\varphi \in \Phi} \varphi$.

(3) $(\underline{M}, m) \vDash \neg\varphi$ iff $(\underline{M}, m) \nvDash \varphi$ iff $(\underline{N}, f(m)) \nvDash \varphi$ iff $(\underline{N}, f(m)) \vDash \neg\varphi$.

(4) Let $\psi \in \mathcal{L}_{0.}$ As remarked in the definition of the relation "$\vDash$", $[\psi]^{\underline{M}}$ is measurable in $\underline{M}$ and $[\psi]^{\underline{N}}$ is measurable in $\underline{N}$. By the induction hypothesis $f^{-1}([\psi]^{\underline{N}}) = [\psi]^{\underline{M}}$, so we have $(\underline{M}, m) \vDash p_i^\alpha(\psi)$ iff $\alpha \leq T_i(m)([\psi]^{\underline{M}})$ iff $(\underline{N}, f(m)) \vDash p_i^\alpha(\psi)$ $\big($note that $T_i(m)([\psi]^{\underline{M}}) = T_i(m)(f^{-1}([\psi]^{\underline{N}})) = T_i^N(f(m))([\psi]^{\underline{N}})\big)$.

■

An easy check shows:

Remark 5
- *The type spaces on X (for player set I) - as the objects - together with the type morphisms - as the morphisms - form a category.*
- *The Harsanyi type spaces on X (for player set I) - as the objects - together with the type morphisms - as the morphisms - form a category.*

Definition 21 A type space (resp. Harsanyi type space) $\underline{M}$ is *universal* if for every type space (resp. Harsanyi type space) $\underline{N}$ there is exactly one type morphism from $\underline{N}$ to $\underline{M}$.

It is obvious that a type morphism $f : \underline{M} \to \underline{N}$ is a type isomorphism iff there is a type morphism $g : \underline{N} \to \underline{M}$ such that $g \circ f = \mathrm{id}_{\underline{M}}$ and $f \circ g = \mathrm{id}_{\underline{N}}$. Hence, type isomorphisms coincide with the isomorphisms of the category of type spaces on X. By the Yoneda Lemma, it follows (but is also easily seen directly) that:

Remark 6 *If there exists a universal type space (resp. Harsanyi type space), then it is unique up to type isomorphism.*

Theorem 2 *The (Harsanyi) type space*

$$\underline{\Omega} = \left\langle \Omega, \Sigma, (T_i)_{i \in I,} v \right\rangle$$

is universal.

Proof Let $\underline{M} = \langle M, \Sigma^M, \left(T_i^M\right), v^M \rangle$ be a type space (resp. Harsanyi type space). By Lemma 5, 2 of Proposition 2, and 4 of Proposition 3, it follows that there is at most one type morphism from $\underline{M}$ to $\underline{\Omega}$.

Let $\Phi_m := \{\varphi \in \mathcal{L}_0 \mid (\underline{M}, m) \vDash \varphi\}$, for $m \in M$, and define $f(m) := \bigwedge_{\varphi \in \Phi_m} \varphi$. By Corollary 1, Φ_m is consistent and by the definition of the relation "$\vDash$", it follows for $\psi \in \mathcal{L}_0$, that $\psi \in \Phi_m$ iff $\neg\psi \notin \Phi_m$. This implies that $f(m) \in \Omega$. It remains to show that $f : M \to \Omega$ is a type morphism:

(1) It is enough to show that for every $\psi \in \mathcal{L}_0 : f^{-1}([\psi]) \in \Sigma^M$, since the set $\{[\varphi] \mid \varphi \in \mathcal{L}_0\}$ is a field that generates Σ. We have $f(m) \in [\psi]$ iff $\vdash f(m) \to \psi$ iff $\psi \in \Phi_m$ iff $m \in [\psi]^{\underline{M}}$. But $[\psi]^{\underline{M}} \in \Sigma^M$ (see, the definition of "$\vDash$").

(2) $v^M(m, x) = 1$ iff $x \in \Phi_m$ iff $\vdash f(m) \to x$ iff $f(m) \in [x]$ iff $v(f(m), x) = 1$.

(3) It is again (by Caratheodory's extension Theorem) enough to show, that for $\varphi \in \mathcal{L}_0$: $T_i^M(m)\left(f^{-1}([\varphi])\right) = T_i(f(m))([\varphi])$. We have $f^{-1}([\varphi]) = [\varphi]^{\underline{M}}$, hence $T_i^M(m)\left(f^{-1}([\varphi])\right) = T_i^M(m)\left([\varphi]^{\underline{M}}\right) =$ $\sup\{\alpha \mid (\underline{M}, m) \vDash p_i^\alpha(\varphi)\} = \sup\{\alpha \mid p_i^\alpha(\varphi) \in \Phi_m\} =$ $\sup\{\alpha \mid \vdash f(m) \to p_i^\alpha(\varphi)\} = T_i(f(m))([\varphi])$.

■

5 Product Type Spaces

The aim of this section is to show that the canonical space is (up to isomorphism) a product type space. Since this space is then a universal type space in the category of product type spaces on X (for player set I), this implies that - in the case of Harsanyi type spaces (i.e. the H-system case) - our canonical model is (up to isomorphism) the universal Harsanyi type space constructed by Heifetz and Samet (1998b).

We define:

Definition 22 A *product type space on* X (for player set I) is a 4-tuple

$$\underline{M} := \langle M, \Sigma, (T_i)_{i \in I}, v \rangle$$

such that there are measurable spaces (M_j, Σ_j), for $j \in I_0$, such that (up to type isomorphism):

- $M_0 = \mathrm{Pow}(X)$,
- Σ_0 is the σ-field on $\mathrm{Pow}(X)$ generated by the sets $[x]^0 := \{m_0 \subseteq X \mid x \in m_0\}$, where $x \in X$,
- $M = \mathrm{Pow}(X) \times \Pi_{i \in I} M_i$, where all the M_i are nonempty,
- Σ is the product σ-field on M which is generated by the σ-fields Σ_i, $i \in I_0$,
- for $x \in X$: $v(m_0, x) = \begin{cases} 1, & \text{if} \quad x \in m_0, \\ 0, & \text{if} \quad x \notin m_0, \end{cases}$ and $v(m_0, \top) = 1$ in any case,
- for all $i \in I$: T_i is a measurable mapping $T_i : (M_i, \Sigma_i) \to \Delta(M, \Sigma)$, where $\Delta(M, \Sigma)$ is the space of probability measures on (M, Σ), (endowed with the σ-field generated by all the sets $\{\mu \in \Delta(M, \Sigma) \mid \mu(E) \geq \alpha\}$, where $E \in \Sigma$ and α rational with $\alpha \in [0, 1]$).

Obviously, T_i can be viewed as a measurable mapping from (M, Σ) to $\Delta(M, \Sigma)$ and $v(., x)$ can be viewed as a measurable mapping from (M, Σ) to $(\{0, 1\}, \mathrm{Pow}(\{0, 1\}))$, for every $x \in X \cup \{\top\}$. So every product type space on X is a type space on X.

Definition 23

- For $i \in I_0$ define $\Omega_i :=$
$$\left\{ \bigwedge_{\varphi \in \Phi_0^i} \varphi \wedge \bigwedge_{\varphi \in \mathcal{L}_0^i \setminus \Phi_0^i} \neg\varphi \mid \Phi_0^i \subseteq \mathcal{L}_0^i, \text{ s.t. } \Phi_0^i \cup \{\neg\varphi | \varphi \in \mathcal{L}_0^i \setminus \Phi_0^i\} \text{ is consistent} \right\}.$$

- For $i \in I_0$ and $\psi_i \in \mathcal{L}_0^i$ define $[\psi_i]^i := \{\omega_i \in \Omega_i \mid \vdash \omega_i \to \psi_i\}$.
- Denote by Σ_i the σ-field on Ω_i generated by the $[\psi_i]^i$.
- Define $\Omega^* := \Pi_{i \in I_0} \Omega_i$.
- Denote by Σ^* the product σ-field of the σ-fields Σ_i, $i \in I_0$.
- Define $\Omega_{-i} := \Pi_{j \in I_0 \setminus \{i\}} \Omega_j$, for $i \in I$.
- For $i \in I$ denote by Σ_{-i} the product σ-field of the σ-fields Σ_j, $j \in I_0 \setminus \{i\}$.
- By 4. of Proposition 3 and Corollary 1, for every $\omega_i \in \Omega_i$, where $i \in I_0$, there is a $\omega \in \Omega$ such that $\vdash \omega \to \omega_i$. By definition of the $\omega_i \in \Omega_i$, two such formulas contradict each other, so for every $\omega \in \Omega$, there is exactly one $\omega_i \in \Omega_i$ (the existence is clear, since $\left(\bigwedge_{\varphi_i \in \mathcal{L}_0^i \cap \Psi_\omega} \varphi_i\right) \in \Omega_i$) such that $\vdash \omega \to \omega_i$. Denote this ω_i by $\omega(i)$.
- For $i \in I_0$ and $E_i \in \Sigma_i$ define $E_i^* := \Pi_{j \in I_0} U_j$, where $U_j = \Omega_j$, for $j \neq i$ and $U_i = E_i$. We have $E_i^* \in \Sigma^*$. Observe that $\mathcal{L}_0^i \cap \mathcal{L}_0^j = \emptyset$, for $i \neq j \in I_0$. Hence, for $\varphi_i \in \mathcal{L}_0^i$, the following is well-defined: $[\varphi_i]^* := ([\varphi_i]^i)^*$ and by the definition of the $\omega_i \in \Omega_i$, we have $\Omega^* \setminus [\varphi_i]^* = [\neg\varphi_i]^*$ and $[\varphi_i]^* \cap [\psi_i]^* = [\varphi_i \wedge \psi_i]^*$, for $\varphi_i, \psi_i \in \mathcal{L}_0^i$. We define now recursively (starting with the i-formulas) for $\varphi, \psi \in \mathcal{L}_0$: $[\neg\varphi]^* := \Omega^* \setminus [\varphi]^*$ and $[\varphi \wedge \psi]^* := [\varphi]^* \cap [\psi]^*$. It is still well-defined for the finitary i -formulas and it is well-defined for the other finitary formulas by the unique readability of finitary formulas, which can be proved in the usual way (what we don't know at the moment is that logically equivalent finitary formulas define the same sets in Ω^*).

  It is obvious that these sets form a field $\mathcal{F}^*$ on Ω^* which generates Σ^*.

Lemma 6 *Let $i \in I_0$, $\omega_i \in \Omega_i$ and $\varphi_i \in \mathcal{L}^i$. Then, either* $\vdash \omega_i \to \varphi_i$ *or* $\vdash \omega_i \to \neg\varphi_i$.

Proof The "either" follows by the consistency of ω_i, while the "or" follows by an easy induction on the formation of the formulas in $\mathcal{L}^i$. ■

Remark 7 • Σ_0 *is generated by the sets* $[x]^0$, *where* $x \in X$.

- *For every* $m_0 \subseteq X$, *there is exactly one* $\omega_0 \in \Omega_0$ *such that for every* $x \in X : \vdash \omega_0 \to x$ *iff* $x \in m_0$.

Proof The first assertion is clear.

For the second: Existence: Take a type space $\underline{M}$ consisting of one point (i.e. $m = \{m\}$) such that $v(m, x) = 1$ iff $x \in m_0$. The above lemma shows now that $\omega_0 := \bigwedge_{\varphi^0 \in \mathcal{L}_0^0 : (\underline{M}, m) \models \varphi^0} \varphi^0$ does the job.

Uniqueness: An easy induction on the formation of the formulas in $\mathcal{L}_0^0$ shows: For all $\varphi^0 \in \mathcal{L}_0^0$: $\vdash \bigwedge_{x \in m_0} x \wedge \bigwedge_{y \in X \setminus m_0} \neg y \to \varphi^0$ or $\vdash \bigwedge_{x \in m_0} x \wedge \bigwedge_{y \in X \setminus m_0} \neg y \to \neg \varphi^0$. By the inference rule Conjunction and the consistency of ω_0, this implies $\vdash \omega_0 \leftrightarrow \bigwedge_{x \in X : \vdash \omega_0 \to x} x \wedge \bigwedge_{y \in X : \vdash \omega_0 \to \neg y} \neg y$. ■

Lemma 7 *Let* $(\omega_i)_{i \in I_0} \in \Omega^*$ *and* $\{\omega_i \mid i \in I_0\}$ *be consistent. Then there is exactly one* $\omega \in \Omega$ *such that* $\vdash \left(\bigwedge_{i \in I_0} \omega_i\right) \leftrightarrow \omega$, *and furthermore* $\omega_i = \omega(i)$, *for all* $i \in I_0$. *Conversely, for every* $\omega \in \Omega : \vdash \left(\bigwedge_{i \in I_0} \omega(i)\right) \leftrightarrow \omega$.

Proof An easy induction on the formation of the formulas shows that for all $\varphi \in \mathcal{L}$: Either $\vdash \left(\bigwedge_{i \in I_0} \omega_i\right) \to \varphi$ or $\vdash \left(\bigwedge_{i \in I_0} \omega_i\right) \to \neg\varphi$. The rest is now obvious. ■

The above lemma shows that $h : \Omega \to \Omega^*$, defined by $h(\omega) := (\omega(i))_{i \in I_0}$ is one to one. We are now justified to identify with some abuse of notation $h(\Omega)$ with Ω.

Lemma 8 *For every* $\varphi \in \mathcal{L}_0 : [\varphi]^* \cap \Omega = [\varphi]$. *More exact*: $[\varphi]^* \cap h(\Omega) = h([\varphi])$.

Proof The assertion is true for i-formulas, according to the above two Lemmas 6 and 7. The rest follows from the definition of $[.]^*$ and 6 and 7 of Proposition 2. ■

Definition 24 • Define $T_j^*(\omega_j)([\varphi]^*) := \sup \left\{\alpha \mid \vdash \omega_j \to p_j^\alpha(\varphi)\right\}$, for $\varphi \in \mathcal{L}_0$.

- Define $v^*(\omega_0, x) := \begin{cases} 1, & \text{if} \quad \omega_0 \in [x]^0, \\ 0, & \text{if} \quad \omega_0 \notin [x]^0, \end{cases}$ and $v^*(\omega_0, \top) := 1$ in any case.

Obviously, for every $x \in X$, $v^*(\cdot, x)$ is measurable in Ω_0, hence viewed as a mapping from Ω^* to $\{0,1\}$, it is measurable in Ω^*.

Lemma 9 *For every $j \in I$ and $\omega_j \in \Omega_j$: $T_j^*(\omega_j)(\cdot)$ is well-defined and a countably additive measure on $\mathcal{F}^*$. Furthermore: For every $j \in I$ and $\omega_j \in \Omega_j$: $T_j^*(\omega_j)(\Omega^*) = 1$.*

Proof Choose $\omega \in \Omega$ such that $\omega(j) = \omega_j$. By the definitions and Lemma 6, $T_j^*(\omega_j)([\varphi]^*) = T_j'(\omega)([\varphi])$, for all $\varphi \in \mathcal{L}_0$. This implies in particular that $T_j^*(\omega_j)(\cdot)$ is non-negative.

If for $\varphi, \psi \in \mathcal{L}_0$: $[\varphi]^* = [\psi]^*$, then $[\varphi] = [\psi]$ (by Lemma 8). It follows that $T_j^*(\omega_j)([\varphi]^*) = T_j'(\omega)([\varphi]) = T_j'(\omega)([\psi]) = T_j^*(\omega_j)([\psi]^*)$, hence $T_j^*(\omega_j)(\cdot)$ is well-defined.

If for $\varphi, \psi \in \mathcal{L}_0$: $[\varphi]^* \cap [\psi]^* = \emptyset$, then $[\varphi] \cap [\psi] = \emptyset$ and, by the definition of $[.]^*$, it follows that $[\varphi]^* \cup [\psi]^* = [\varphi \vee \psi]^*$. Hence, $T_j^*(\omega_j)([\varphi]^*) + T_j^*(\omega_j)([\psi]^*) = T_j'(\omega)([\varphi]) + T_j'(\omega)([\psi]) = T_j'(\omega)([\varphi \vee \psi]) = T_j^*(\omega_j)([\varphi \vee \psi]^*)$, it follows that $T_j^*(\omega_j)(\cdot)$ is additive on $\mathcal{F}^*$.

We have $\Omega_0 = [\top]^0$ and, by the definition of the ω_j, $\vdash \omega_j \rightarrow p_j^1(\top)$, for $j \in I$, and it follows that $T_j^*(\omega_j)(\Omega^*) = 1$.

Let $\varphi_n \in \mathcal{L}_0$, for $n \in \mathbb{N}$, and let $[\varphi_n]^* \downarrow \emptyset$. It follows (by Lemma 8) that $[\varphi_n] \downarrow \emptyset$ and therefore $\lim_{n\to\infty} T_j^*(\omega_j)([\varphi_n]^*) = \lim_{n\to\infty} T_j'(\omega)([\varphi_n]) = 0$. ■

Proposition 4 *(1) For every $j \in I$ and $\omega_j \in \Omega_j$ there is a unique extension of $T_j^*(\omega_j)$ to a probability measure on (Ω^*, Σ^*), which we denote also by $T_j^*(\omega_j)$.*

(2) This extension is then a measurable mapping from (Ω_j, Σ_j) to $\Delta(\Omega^*, \Sigma^*)$, the space of probability measures on (Ω^*, Σ^*) endowed with the σ-field generated by the sets $\{\mu \in \Delta(\Omega^*, \Sigma^*) \mid \mu(E) \geq \alpha\}$, where $E \in \Sigma^*$ and rational $\alpha \in [0,1]$.

(3)

$$\underline{\Omega}^{*} := \langle \Omega^{*}, \Sigma^{*}, (T_{i}^{*})_{i\in I}, v^{*} \rangle$$

is a product type space on X (for player set I).

(4) For every $\varphi \in \mathcal{L}_0$:

$$\left(\langle \Omega^{*}, \Sigma^{*}, (T_{i}^{*})_{i\in I}, v^{*} \rangle , (\omega_i)_{i\in I_0} \right) \vDash \varphi \text{ iff } (\omega_i)_{i\in I_0} \in [\varphi]^{*}.$$

(5) If the axiom $(I1)$ is added in the case of $\aleph_\gamma = \aleph_0$ and the inference rule Uncountable Introspection in the case $\aleph_\gamma > \aleph_0$, then $\langle \Omega^{*}, \Sigma^{*}, (T_{i}^{*})_{i\in I}, v^{*} \rangle$ is a Harsanyi product type space on X (for player set I).

Proof

(1) Follows from Caratheodory's extension Theorem and Lemma 9.

(2) Follows from Lemma 1: It suffices to show that for every $\varphi \in \mathcal{L}_0$, rational $\alpha \in [0,1]$ and $j \in I : \{\omega_j \mid T_j^{*}(\omega_j)([\varphi]^{*}) \geq \alpha\} \in \Sigma_j$. By the definition, Lemma 3 and Lemma 6, it follows that $T_j^{*}(\omega_j)([\varphi]^{*}) \geq \alpha$ iff $\vdash \omega_j \rightarrow p_j^{\alpha}(\varphi)$. Hence, $\{\omega_j \mid T_j^{*}(\omega_j)([\varphi]^{*}) \geq \alpha\} = [p_j^{\alpha}(\varphi)]^{j} \in \Sigma_j$.

(3) Follows from Remark 7 and 1 and 2 of this proposition.

(4) We proceed by induction on the formation of the formulas in $\mathcal{L}_0$: $(\underline{\Omega}^{*}, (\omega_i)_{i\in I_0}) \vDash x$ iff $v^{*}(\omega_0, x) := 1$ iff $\omega_0 \in [x]^{0}$ iff $(\omega_i)_{i\in I_0} \in [x]^{*}$, the induction steps "$\neg\varphi$" and "$\varphi \wedge \psi$" are clear by the definition of "$\vDash$" and "$[\cdot]^{*}$", so it remains the step "$p_j^{\alpha}(\varphi)$": By the induction hypothesis, we have $[\varphi]^{*} = \{(\omega_i')_{i\in I_0} \mid (\underline{\Omega}^{*}, (\omega_i')_{i\in I_0}) \vDash \varphi\}$. It follows that, for $j \in I : (\underline{\Omega}^{*}, (\omega_i)_{i\in I_0}) \vDash p_j^{\alpha}(\varphi)$ iff $T_j^{*}((\omega_i)_{i\in I_0})([\varphi]^{*}) \geq \alpha$ iff $\sup\left\{\beta \mid \vdash \omega_j \rightarrow p_j^{\beta}(\varphi)\right\} \geq \alpha$ iff (by the axioms (P7) and (P3)) $\vdash \omega_j \rightarrow p_j^{\alpha}(\varphi)$ iff $\omega_j \in [p_j^{\alpha}(\varphi)]^{j}$ iff $(\omega_i)_{i\in I_0} \in [p_j^{\alpha}(\varphi)]^{*}$.

(5) We have to show that , for $j \in I : A \in \Sigma^{*}$ and $[T_j^{*}(\omega_j)]^{*} := \{(\omega_i')_{i\in I_0} \mid T_j^{*}(\omega_j') = T_j^{*}(\omega_j)\} \subseteq A$ imply $T_j^{*}(\omega_j)(A) = 1$. By definition of $\mathcal{F}^{*}, \Sigma^{*}$, and $T_j^{*}(\omega_j)$, it suffices to show that $(\varphi_n)_{n\in\mathbb{N}} \in \mathcal{L}_0$ and $\bigcup_{n\in\mathbb{N}}[\varphi_n]^{*} \supseteq A$ imply $\lim_{l\to\infty} T_j^{*}(\omega_j)\left([\bigvee_{n\leq l}\varphi_n]^{*}\right) = 1$. Choose $\omega \in \Omega$ with $\vdash \omega \rightarrow \omega_j$. By an easy induction on the formation of the j-formulas $\varphi^{j} \in \mathcal{L}^{j}$, we

have that for $\omega', \widetilde{\omega} \in \Omega$ and $j \in I : T_j(\omega') = T_j(\widetilde{\omega})$ implies $(\underline{\Omega}, \omega') \vDash \varphi^j$ iff $(\underline{\Omega}, \widetilde{\omega}) \vDash \varphi^j$. Hence, by 4 of Proposition 3, $T_j(\omega') = T_j(\widetilde{\omega})$ implies $\vdash \omega' \to \omega_j$ iff $\vdash \widetilde{\omega} \to \omega_j$. It follows that $h^{-1}\left([T_j^*(\omega_j)]^*\right) \supseteq [T_j(\omega)]$. Hence, by Lemma 8, $\bigcup_{n\in\mathbb{N}}[\varphi_n] \supseteq [T_j\left(\omega\right)]$. This implies $\lim_{l\to\infty} T_j^*\left(\omega_j\right)\left([\bigvee_{n\leq l}\varphi_n]^*\right)$ $= \lim_{l\to\infty} T_j\left(\omega\right)\left([\bigvee_{n\leq l}\varphi_n]\right) = 1$.

■

Theorem 3 *$h : \Omega \to \Omega^*$ defined by $h(\omega) := (\omega(i))_{i\in I_0}$ is a type isomorphism from $\underline{\Omega}$ to $\underline{\Omega}^*$.*

Proof By Lemma 7, h is one-to-one.

Let $i \in I_0$ and $(\omega_j)_{j\in I_0} \in \Omega^*$. By 4 of Proposition 4, it follows that for $\omega_i \in \Omega_i$ and $\varphi^i \in \mathcal{L}_0^i$: $\vdash \omega_i \to \varphi^i$ iff $\langle\underline{\Omega}^*, (\omega_j)_{j\in I_0}\rangle \vDash \varphi^i$. By definition of "$\vDash$", this implies that $\langle\underline{\Omega}^*, (\omega_j)_{j\in I_0}\rangle \vDash \omega_i$. And hence $\langle\underline{\Omega}^*, (\omega_j)_{j\in I_0}\rangle \vDash \bigwedge_{j\in I_0}\omega_j$. So $\bigwedge_{j\in I_0}\omega_j$ is consistent for every $(\omega_j)_{j\in I_0}$, hence, by Lemma 7, h is onto.

Let $\vdash \omega \leftrightarrow \bigwedge_{i\in I_0}\omega_i$. Then it follows that $\langle\underline{\Omega}^*, (\omega_i)_{i\in I_0}\rangle \vDash \omega$. Hence, h^{-1} is the type morphism from the proof of Theorem 2.

h is a type morphism:

h is measurable, because $\mathcal{F}^*$ generates Σ^* and for $[\varphi]^*$, with $\varphi \in \mathcal{L}_0$, we have, by Lemma 8, $h^{-1}([\varphi]^*) = [\varphi] \in \Sigma$. By Lemma 6, Lemma 7 and the definitions, we have $v^*(h(\omega), x) = v^*((\omega(i))_{i\in I_0}, x) = v^*(\omega(0), x)$, and $v^*(\omega(0), x) = 1$ iff $\vdash \omega(0) \to x$ iff $\vdash \omega \to x$ iff $v(\omega, x) = 1$.

By Caratheodory's extension Theorem, it is enough to show that for $\varphi \in \mathcal{L}_0$ and $i \in I$: $T_i^*\left(\omega(j)_{j\in I_0}\right)([\varphi]^*) = T_i\left(\omega\right)\left(h^{-1}([\varphi]^*)\right) = T_i\left(\omega\right)([\varphi])$. But this is clear by Lemma 6, Lemma 7 and the definitions. ■

That the canonical space is a product space implies the following: For states $u_i \in \Omega$, where $i \in I_0$, there is one state $u \in \Omega$ such that: $v\left(u_0, x\right) = v\left(u, x\right)$, for all $x \in X$, and $T_i\left(u_i\right) = T_i\left(u\right)$, for $i \in I$. This fact is reflected by the axioms in the following way: There is no axiom and also no inference rule that relates the believes of one player with the beliefs of other players or with nature. So, whatever a player in a state of the world believes about other players or nature might be wrong (as long as this is nothing tautological, of course). This is not the case for the canonical knowledge space and the corresponding $S5$ axiom system, where there is an axiom "$k_i\varphi \to \varphi$". So, if for example, $\varphi = k_j x$ and if $k_i\varphi$ is true in a state, then the fact that i knows that j knows x forces j to know x, and this forces x to be true in that state.

6 Beliefs Completeness

The aim of this section is to prove the following - somewhat surprising - theorem of appealing measure-theoretic taste, which, in some topological cases, was proved by Mertens and Zamir (1985), Brandenburger and Dekel (1993), Heifetz (1993) and Mertens, Sorin and Zamir (1994). The general measure-theoretic case proved here is original. The theorem says that, in the P-system case, the component space of each player is - up to isomorphism of measurable spaces - the space of probability measures on the space of states of the world, and in the H-system case, for each player $i \in I$, the component space of i is - up to isomorphism of measurable spaces - the space of probability measures on Ω_{-i}.

Theorem 4 • *In the P-system, let $\mu \in \Delta(\Omega^*, \Sigma^*)$. For every $i \in I$, there is exactly one $\omega_i \in \Omega_i$ such that $T_i^*(\omega_i) = \mu$. Furthermore,*

$$T_i^* : \Omega_i \to \Delta(\Omega^*, \Sigma^*)$$

is an isomorphism of measurable spaces.

- *In the H-system, let $\mu_i \in \Delta(\Omega_{-i}, \Sigma_{-i})$. Then there is exactly one $\omega_i \in \Omega_i$ such that the marginal of $T_i^*(\omega_i)$ on Ω_{-i} is μ_i. Furthermore,*

$$marg_{\Omega_{-i}} \circ T_i^* : \Omega_i \to \Delta(\Omega_{-i}, \Sigma_{-i})$$

is an isomorphism of measurable spaces.

Proof We prove the P-system case and sketch the differences for the proof of the H-system case. For the P-system case:

Let $i \in I$. For $\omega_i \in \Omega_i$ define $\varphi_i(\omega_i) :=$

$$\left(\bigwedge_{\chi \in \mathcal{L}_0,\ \alpha \in [0,1] \cap \mathbb{Q},\ \text{s.t. } \vdash \omega_i \to p_i^\alpha(\chi)} p_i^\alpha(\chi) \right) \wedge \left(\bigwedge_{\psi \in \mathcal{L}_0,\ \beta \in [0,1] \cap \mathbb{Q},\ \text{s.t. } \vdash \omega_i \to \neg p_i^\beta(\psi)} \neg p_i^\beta(\psi) \right).$$

An easy induction on the formation of the i-formulas shows that for every i-formula $\chi_i \in \mathcal{L}^i$: Either $\vdash \varphi_i(\omega_i) \to \chi^i$ or $\vdash \varphi_i(\omega_i) \to \neg\chi^i$. Since

$\vdash \omega_i \to \varphi_i(\omega_i)$, by the consistency of ω_i, it follows that $\vdash \omega_i \leftrightarrow \varphi_i(\omega_i)$. This implies that from $\omega_i' \neq \omega_i'' \in \Omega_i$ follows that $\varphi_i(\omega_i') \neq \varphi_i(\omega_i'')$. By the definitions of $T_i^*(\omega_i')$ and $T_i^*(\omega_i'')$, we have $T_i^*(\omega_i)([\varphi]^*) \neq T_i^*(\omega_i')([\varphi]^*)$, for some $\varphi \in \mathcal{L}_0$. We conclude that $T_i^* : (\Omega_i, \Sigma_i) \to \Delta(\Omega^*, \Sigma^*)$ is one-to-one.

By 2 of Proposition 4, T_i^* is measurable.

Let $\mu \in \Delta(\Omega^*, \Sigma^*)$ and fix $i \in I$. Consider the following set of formulas:

$$\begin{aligned} \Phi^\mu \;&:=\; \{p_i^\alpha(\varphi) \,|\varphi \in \mathcal{L}_0, \alpha \in [0,1] \cap \mathbb{Q} \text{ s.t. } \mu([\varphi]^*) \geq \alpha\} \\ &\cup \left\{\neg p_i^\beta(\psi) \,|\psi \in \mathcal{L}_0, \beta \in [0,1] \cap \mathbb{Q} \text{ s.t. } \mu([\psi]^*) < \beta\right\}. \end{aligned}$$

If this set of formulas is consistent in the system P, then by Corollary 1, there is a $\omega \in \Omega$ (the Ω belonging to the system P) such that $(\underline{\Omega}, \omega) \vDash \Phi^\mu$. But then, from 4 of Proposition 3, the definition of $\omega(i)$, the fact that $\vdash \omega \to \omega(i)$, and the consistency of ω, it follows that $\vdash \omega(i) \to \psi$, for all $\psi \in \Phi^\mu$. The definition of $T_i^*(\omega(i))$ implies then that $T_i^*(\omega(i))([\varphi]^*) = \mu([\varphi]^*)$, for all $\varphi \in \mathcal{L}_0$. Hence, since $T_i^*(\omega(i))$ and μ are σ-additive probability measures on Σ^* that coincide on the field $\mathcal{F}^*$, and since $\mathcal{F}^*$ generates Σ^*, Caratheodory's extension Theorem implies then that $T_i^*(\omega(i)) = \mu$. So, the consistency of Φ^μ implies that T_i^* is onto.
In the following, we show that Φ^μ is consistent in the system P.
Let $u \notin \Omega_i$. Define

$$\begin{aligned} \Omega_i^\mu \;&:=\; \Omega_i \cup \{u\}, \\ \Omega_j^\mu \;&:=\; \Omega_j, \quad \text{for } j \in I_0 \setminus \{i\}, \\ \Sigma_i^\mu \;&:=\; \Sigma_i \cup \{E \cup \{u\} \mid E \in \Sigma_i\}, \\ \Sigma_j^\mu \;&:=\; \Sigma_j, \quad \text{for } j \in I_0 \setminus \{i\}, \\ \Omega^\mu \;&:=\; \Pi_{j \in I_0} \Omega_j^\mu, \\ \Sigma^\mu \;&:=\; \text{the product } \sigma\text{-field of the } \Sigma_j^\mu,\ j \in I_0. \end{aligned}$$

Note that Σ_i^μ is a σ-field, $\Sigma^* \subseteq \Sigma^\mu$, and $E \cap \Omega^* \in \Sigma^*$, for $E \in \Sigma^\mu$.

For $j \in I$, $\omega_j \in \Omega_j^\mu$ and $E \in \Sigma^\mu$ define

$$\begin{aligned} T_j^\mu(\omega_j)(E) \;&:=\; T_j^*(\omega_j)(E \cap \Omega^*), \quad \text{if } j \neq i \text{ or if } i = j \text{ and } \omega_i \neq u, \\ T_i^\mu(\omega_i)(E) \;&:=\; \mu(E \cap \Omega^*), \quad \text{if } \omega_i = u. \end{aligned}$$

By this definition, $T_j^\mu(\omega)$ is a σ-additive probability measure on (Ω^μ, Σ^μ). For $x \in X$ and $\omega \in \Omega^\mu$ define:

$$\begin{array}{rcl} v^\mu(\omega, x) & := & v^*(\omega(0), x), \\ v^\mu(\omega, \top) & := & 1, \text{ in any case.} \end{array}$$

By this definition, it is clear that $v^\mu(\cdot, x)$ is measurable.

Let $E \in \Sigma^\mu$, $j \in I \setminus \{i\}$ and $b^\alpha(E) := \{\nu \in \Delta(\Omega^\mu, \Sigma^\mu) \mid \nu(E) \geq \alpha\}$. Then, by the definitions:

$$(T_j^\mu)^{-1}(b^\alpha(E)) = (T_j^*)^{-1}(b^\alpha(E \cap \Omega^*)) \in \Sigma_j.$$

Hence,

$$T_j^\mu : \Omega_j \to \Delta(\Omega^\mu, \Sigma^\mu)$$

is Σ_j-$\Sigma_{\Delta(\Omega^\mu, \Sigma^\mu)}$-measurable.

Note that $T_i^\mu(u)(E) = \mu(E \cap \Omega^*) = T_i^\mu(u)(E \cap \Omega^*)$, for $E \in \Sigma^\mu$. So, for all $j \in I$, $\omega_j \in \Omega_j^\mu$ and $E \in \Sigma^\mu$: $T_j^\mu(\omega_j)(E) = T_j^\mu(\omega_j)(E \cap \Omega^*)$. By definition, we have

$$(T_i^\mu)^{-1}(b^\alpha(E)) = \begin{cases} (T_i^*)^{-1}(b^\alpha(E \cap \Omega^*)) \in \Sigma_i \subseteq \Sigma_i^\mu, & \text{if } \mu(E \cap \Omega^*) < \alpha, \\ \{u\} \cup (T_i^*)^{-1}(b^\alpha(E \cap \Omega^*)) \in \Sigma_i^\mu, & \text{if } \mu(E \cap \Omega^*) \geq \alpha. \end{cases}$$

We have now proved that

$$\underline{\Omega}^\mu := \langle \Omega^\mu, \Sigma^\mu, (T_j^\mu)_{j \in I}, v^\mu \rangle$$

is a product type space on X.

Next, we show by induction on the formation of the formulas in $\varphi \in \mathcal{L}_0$, that, for $\omega \in \Omega^*$:

$$(\underline{\Omega}^\mu, \omega) \models \varphi \text{ iff } (\underline{\Omega}^*, \omega) \models \varphi.$$

An equivalent statement is $[\varphi]^* = [\varphi]^\mu \cap \Omega^*$, where $[\varphi]^\mu := \{\omega \in \Omega^\mu \mid (\underline{\Omega}^\mu, \omega) \models \varphi\}$. (Recall that, by 4 of Proposition 4, $[\varphi]^{\underline{\Omega}^*} = [\varphi]^*$, for $\varphi \in \mathcal{L}_0$.) Since $T_i^\mu(u)([\varphi]^\mu) = \mu([\varphi]^\mu \cap \Omega^*)$, it follows then that

$$(\underline{\Omega}^\mu, u) \models \Phi^\mu.$$

Let $\omega \in \Omega^*$. By definition, we have

$$(\underline{\Omega}^\mu, \omega) \models x \text{ iff } (\underline{\Omega}^*, \omega) \models x, \text{ for } x \in X \cup \{\top\}.$$

The steps "$\wedge$" and "$\neg$" are trivial.
Let $j \in I$.

$$\begin{array}{rcl} [p_j^\alpha(\varphi)]^* & = & \{\omega \in \Omega^* \mid T_j^*(\omega_j)([\varphi]^*) \geq \alpha\} \\ & = & \{\omega \in \Omega^* \mid T_j^\mu(\omega)([\varphi]^*) \geq \alpha\} \\ & = & \{\omega \in \Omega^* \mid T_j^\mu(\omega)([\varphi]^\mu \cap \Omega^*) \geq \alpha\} \\ & = & \{\omega \in \Omega^* \mid T_j^\mu(\omega)([\varphi]^\mu) \geq \alpha\} \\ & = & \{\omega \in \Omega^\mu \mid T_j^\mu(\omega)([\varphi]^\mu) \geq \alpha\} \cap \Omega^* \\ & = & [p_j^\alpha(\varphi)]^\mu \cap \Omega^*. \end{array}$$

It remains to prove that $(T_i^*)^{-1}$ is measurable:
The $[p_i^\alpha(\varphi)]^i$, with $\varphi \in \mathcal{L}_0$ and $\alpha \in [0,1] \cap \mathbb{Q}$, generate the σ-field Σ_i. So it is enough to show that $T_i^*([p_i^\alpha(\varphi)]^i)$ is a measurable set in $\Delta(\Omega^*, \Sigma^*)$. But we have

$$T_i^*([p_i^\alpha(\varphi)]^i) = \{\nu \in \Delta(\Omega^*, \Sigma^*) \mid \nu([\varphi]^*) \geq \alpha\} \in \Sigma_\Delta^*.$$

The the H-system case is proved similarly, but for the "onto part" one starts with $\mu \in \Delta(\Omega_{-i}, \Sigma_{-i})$ and defines $T^\mu := \delta_u \times \mu$, where "$\times$" denotes here the product of measures and δ_u is the delta measure at $u \in \Omega_i^\mu$.

And for the "one-to-one" part, one uses the following fact, which holds in general for σ-additive probability measures on product spaces (endowed with the product σ-field): If $\nu \in \Delta(\Omega^*, \Sigma^*)$ and for $\omega_i \in \Omega_i$: $\text{marg}_{\Omega_i}(\nu) = \delta_{\omega_i}$, then $\nu = \delta_{\omega_i} \times \text{marg}_{\Omega_{-i}}(\nu)$. This fact together with Caratheodory's extension Theorem is then used to show by induction on the formation of the i-formulas that $\text{marg}_{\Omega_{-i}} \circ T_i^*$ is one-to-one. The measurability of $\text{marg}_{\Omega_{-i}} \circ T_i^*$ is straightforward, while, using Lemma 1, the measurability of $\left(\text{marg}_{\Omega_{-i}} \circ T_i^*\right)^{-1}$ follows also by induction on the formation of the i-formulas. ■

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