CORE DISCUSSION PAPER
2001/6

A Multi–Item Production Planning Model with Setup Times: Algorithms, Reformulations, and Polyhedral Characterizations for a Special Case

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January 2001

Abstract

We study a special case of a structured mixed integer programming model that arises in a number of applications. For the most general case of the model, called PI, we have earlier analyzed the polyhedral structure (Miller et al. [2000a]), including identifying facet–defining valid inequalities. PI is $\mathcal{NP}$–hard; in this paper we focus on a special case, called PIC, that is polynomially solvable. We describe a polynomial algorithm for PIC, and we then use this algorithm to derive an extended formulation of polynomial size for PIC. Projecting from this extended formulation onto the original space of variables, we show that the set of inequalities presented for PI in Miller et al. [2000a] is sufficient to enable us to solve the special case PIC by linear programming. Finally, for PIC, we describe fast combinatorial separation algorithms for these inequalities.

Keywords: Mixed integer programming, polyhedral combinatorics, production planning, capacitated lot–sizing, fixed charge network flow, setup times

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This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.
This research was also supported by NSF Grant No. DMI-9700285 and by Philips Electronics North America.

1 Introduction

In Miller et al. [2000a] we introduced a mixed integer programming model that occurs as a relaxation of a number of structured MIP problems, such as capacitated production planning problems with setup times and fixed–charge network flow problems. We originally derived this model as a single–period relaxation of the multi–item capacitated lot–sizing problem with setup times (MCL) (Miller et al. [2000b]). We call this relaxation PI, for *preceding inventory*, because when it is used as a single–period submodel of MCL, it incorporates stock or inventory entering the period in question from the period before. PI can be formulated as follows:

$$\min \quad \sum_{i=1}^{P} p^i x^i + \sum_{i=1}^{P} q^i y^i + \sum_{i=1}^{P} h^i s^i \tag{1}$$

subject to

$$x^i + s^i \geq d^i, i = 1, ..., P, \tag{2}$$

$$\sum_{i=1}^{P} x^i + \sum_{i=1}^{P} t^i y^i \leq c, \tag{3}$$

$$x^i \leq (c - t^i) y^i, i = 1, ..., P, \tag{4}$$

$$x^i, s^i \geq 0, i = 1, ..., P, \tag{5}$$

$$y^i \in \{0, 1\}, i = 1, ..., P. \tag{6}$$

Perhaps the simplest way to view PI is as a single–node multi–item flow model with two inbound arcs (see Figure 1). The first of these arcs has capacity c, which is jointly imposed on P items. Moreover, a fixed capacity usage $t^i \geq 0$ must be incurred if any amount of item i is shipped along this arc. (Assuming that capacity is measured in time units, t^i can be considered as the *setup time*.) The variable x^i is the amount of flow shipped along this arc with unit cost p^i, and the binary variable y^i indicates whether the fixed capacity usage for item i is incurred; the cost for setting $y^i = 1$ is q^i. The second arc can also be used by each of the items, and it is uncapacitated. The variables s^i represent the amount of flow on this second, uncapacitated arc for item i; the unit cost for shipping i along this arc is h^i. The total amount of flow shipped along these two arcs must be at least $d^i, i = 1, ..., P$.

In Miller et al. [2000a], we analyzed the general case of this model. Among other results, we characterized the extreme points of the convex hull and defined families of valid inequalities that induce facets of the convex hull. PI is $\mathcal{NP}$–hard, in general; therefore, we did not expect to be able to define a set of inequalities characterizing the convex hull.

In this paper we analyze a special case of PI that occurs when setup times and

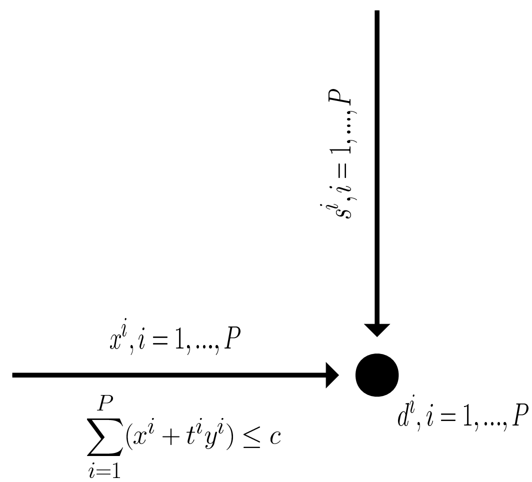

Figure 1: Single–node multi–item flow model with two arcs

demand are constant for all items, i.e., when $t^i = t \geq 0, d^i = d \geq 0, i = 1, ..., P$. For this case, called PIC, we give polynomial–time algorithms, including algorithms based on an extended formulation of PIC. By projecting onto the original (x, y, s) space from the extended space of this reformulation, we can show that the set of inequalities presented for the general case of PI in Miller et al. [2000a] is sufficient to allow us to solve the special case PIC by linear programming (LP).

Many researchers have endeavored to characterize simple MIP models in terms of linear inequalities. For example, in a pioneering paper on using cutting planes to solve structured mixed integer programs, Padberg, Van Roy, and Wolsey [1985] introduced flow cover inequalities for the single node fixed charge flow model (SNFC). They showed that these inequalities, along with trivial facets, suffice to define the convex hull of SNFC in the special case when arc capacities are constant.

When $t^i = 0, i = 1, ..., P$, PI resembles SNFC, but even in this case PI has a different, more complicated structure than SNFC. Both PI and SNFC have flow variables x^i, setup variables y^i, and a joint capacity constraint (3). Moreover, both have variable upper bounds similar to (4), although in SNFC these are generally tighter than in PI. PI, in addition, has constraints (2), which impose a demand on each of the flow variables, and variables s^i, which serve as slack variables for (2) and, for example, allow demand in excess of capacity to be satisfied. When there exists some items i with $t^i > 0$, the structure of PI is complicated even further.

Because of these additional structural complications, identifying a set of inequalities that characterize the convex hull of PIC appears to be much more difficult than identifying a set that characterizes the constant–capacity case of SNFC. We have, however, succeeded in characterizing a set of inequalities that, given certain conditions on the costs, allows us to solve PIC by LP. When these cost conditions hold, this characterization has the same algorithmic implications that an explicit description of the convex hull would have.

An outline of the remainder of the paper follows. In Section 2, we state results concerning the polyhedral structure of PIC that follow immediately from results for PI given in Miller et al. [2000a]. We will use these results in the development of later sections.

We begin Section 3 by noting that PI remains $\mathcal{NP}$–hard if either setup times or demand, but not both, are constant. We then give a combinatorial polynomial algorithm for PIC. In Section 4 we first reformulate PI with additional variables. These variables are motivated by the polynomial algorithm for PIC described in Section 3, and the LP relaxation of this reformulation is integral. By considering the projection of the polyhedron defined by the constraints of this linear program onto the space of the original variables, we show that the inequalities presented in Section 2 for PIC are sufficient to solve PIC by LP in the original variable space.

In Section 5 we give combinatorial, polynomial–time separation algorithms for cover and reverse cover inequalities. Finally, we summarize our contributions and discuss potential areas of future research.

2 Polyhedral Results for PIC

The results of this section follow from results for the general case of PI stated by Miller et al. [2000a], and are here restated for PIC for ease of reference.

Let $\mathcal{P} = \{1, ..., P\}$. Given an instance of PIC, we denote the set of points defined by (2)–(6) as X^{PIC}. (Since we are concerned with PIC, we consider only instances of (2)–(6) for which $t^i = t \geq 0, i = 1, ..., P$, and $d^i = d > 0, i = 1, ..., P$.)

If $h^i < 0$, PIC has unbounded optimum. We will therefore assume that $h^i \geq 0$. Also, we assume that $q^i > 0$, which ensures that in every optimal solution, $y^i = 1$ if and only if $x^i > 0$. In discussing a set of points $X \subset \{(x, y, s) : x^i, s^i \geq 0, 0 \leq y^i \leq 1, i = 1, ..., P\}$, we will use

Definition 1 *The point* $(\bar{x}, \bar{y}, \bar{s})$ *is a* dominant solution *of* X *if there exists some cost vector* (p, q, h)*, such that* $q^i > 0, h^i > 0, i = 1, ..., P$*, for which optimizing (1) over* X *yields* $(\bar{x}, \bar{y}, \bar{s})$ *as the* unique *optimal solution.*

Proposition 1 (Characterization of the dominant solutions of conv(X^{PIC})) *Given an extreme point* $(\bar{x}, \bar{y}, \bar{s})$ *of conv(*X^{PIC}*), let* $Q = \{i \in \mathcal{P} : \bar{y}^i = 1\}$*. Partition* Q *into* Q_d *and* Q_r *so that* $Q_d = \{i \in \mathcal{Q} : \bar{x}^i = d\}$ *and* $Q_r = \{i \in Q : \bar{y}^i = 1, \bar{x}^i \neq d\}$*. Then every extreme point* $(\bar{x}, \bar{y}, \bar{s})$ *of conv(*X^{PIC}*) has the form*

$$\begin{array}{c}\bar{x}^i = d^i, \bar{y}^i = 1, \bar{s}^i = 0, i \in Q_d \\ \bar{x}^{i'} = c - \sum_{i \in Q_d}(t+d) - t, \bar{y}^{i'} = 1, \bar{s}^{i'} = (d - \bar{x}^{i'})^+, i' \in Q_r \\ \bar{x}^i = \bar{y}^i = 0, \bar{s}^i = d, i \in \mathcal{P} \setminus Q,\end{array} \quad (7)$$

where either $Q_r = \emptyset$ *or* $|Q_r| = 1$.

Proposition 2 *If* $c > t + d$*, the inequality*

$$s^i + dy^i \geq d \quad (8)$$

is valid and facet–inducing for conv(X^{PIC}*),* $i = 1, ..., P$.

Note that (8) implies the bounds $s^i \geq 0, i = 1, ..., P$; therefore these bounds never induce facets. Also note that these inequalities are essentially (l, S) *inequalities*. These inequalities were introduced for the uncapacitated lot–sizing problem by Barany, Van Roy, and Wolsey [1984]. Van Roy and Wolsey [1987] generalized these inequalities to fixed charge network flow problems.

In order to define the next two sets of inequalities, we first define $M = \lfloor \frac{c}{t+d} \rfloor$. To avoid uninteresting cases, we assume throughout that $c > M(t + d)$. Let

$$\lambda = (M + 1)(t + d) - c;$$

observe that $\lambda > 0$. Now define a *cover* of PIC to be any set $S \subseteq \mathcal{P}$ such that $|S| \geq M+1$.

Proposition 3 (Cover Inequalities) *Given a cover* S *of PIC, let* $T = \mathcal{P} \setminus S$*, and let* (T', T'') *be any partition of* T*. Then*

$$\sum_{i \in S} s^i \geq (|S| - M)\lambda + \sum_{i \in S}(d - \lambda)(1 - y^i) + \frac{\lambda}{t + d} \sum_{i \in T'}(x^i - (d - \lambda)y^i) \quad (9)$$

is a valid inequality for X^{PIC}*. Moreover, if* $d > \lambda$*, it is facet–inducing for conv(*X^{PIC}*).*

Define a reverse cover of PIC to be any set S such that $\emptyset \neq S \subset \mathcal{P}$ with $|S| \leq M$.

Proposition 4 (Reverse Cover Inequalities) *Let* $S \neq \emptyset$ *be a reverse cover of PIC, let* $T = \mathcal{P} \setminus S$*, and let* (T', T'') *be any partition of* T*. Then*

$$\sum_{i \in S} s^i \geq |S|(t + d) \sum_{i \in T'} y^i - \sum_{i \in S} t(1 - y^i) - \sum_{i \in T'}((c - t)y^i - x^i) \quad (10)$$

is valid for X^{PIC}*. Moreover, if* $T' \neq \emptyset$ *and* $d > \lambda$*, it is facet–inducing for conv(*X^{PIC}*).*

3 Complexity

In this section, we first show that PI is $\mathcal{NP}$–hard if either demand or setup times, but not both, are assumed to be constant. We then describe a polynomial algorithm for PIC.

Proposition 5 *If either setup times or demand, but not both, are constant, then PI is $\mathcal{NP}$–hard.*

For the case in which setup times are all 0 and $h^i = h > p = p^i = 0, i = 1, ..., P$, there is a reduction from the continuous 0–1 knapsack problem (see Miller [1999], Marchand and Wolsey [1999]). For the case in which $d^i = 1, i = 1, ..., P$, and $h^i > 0, p = p^i = 0, i = 1, ..., P$, there is a reduction from the 0–1 knapsack problem (see Miller [1999]).

Note that this result implies that the general case of PI is $\mathcal{NP}$-hard. Since PI is $\mathcal{NP}$–hard, we cannot expect to find an explicit description of the convex hull in general, or expect to solve PI by optimizing (1) over a specific set of linear inequalities. Now we consider the case when demand and setup times are constant for all i, i.e., PIC.

Proposition 6 *PIC is polynomially solvable.*

Proof: We describe an $\mathcal{O}(P^2)$ algorithm for PIC. To initialize the algorithm, we sort the items so that $dp^{[1]} + q^{[1]} - dh^{[1]} \leq ... \leq dp^{[P]} + q^{[P]} - dh^{[P]}$. Then, for each $i' \in \mathcal{P}$, we find an extreme point solution with $Q_r = i'$ that has the minimum objective function value among all extreme points with $Q_r = i'$, where Q_r is defined as in Proposition 1.

To do this, for each $i' \in \mathcal{P}$, we first set $y^{i'} = 1$, $x^{i'} = c - t$, and $s^{i'} = 0$, and we also set $y^i = 0$, $x^i = 0$, $s^i = d$, $i \in \mathcal{P} \setminus i'$. Then we put $i \in \mathcal{P} \setminus i'$ greedily into $Q_u = \{i : y^i = 1, x^i = d\}$, proceeding in the order in which we have sorted the elements, decreasing $x^{i'}$ by $t + d$ for each element we put into Q_u. We continue until doing so no longer decreases the value of the objective function (1), or until we do not have enough capacity left to include any more items in Q_u.

We then find the extreme point solution with $Q_r = \emptyset$ that has the minimum objective function value among all such extreme points, using a similar greedy procedure. Because of Proposition 1, the solution having the minimum value among these $P+1$ points must be an optimal solution. $\square$

4 Reformulations of PIC

The main result of this section is that the inequalities presented in Section 2 suffice to solve PIC by LP. To achieve this result, we first present an extended formulation of PIC, that is, a linear reformulation of PIC including additional variables that has integral extreme points. The projection of the extreme points of this formulation onto the original (x, y, s) space is the set of dominant solutions of X^{PIC}. We then modify this formulation algebraically to obtain formulations that have the same projection onto the (x, y, s) space. The last of these formulations incorporates the inequalities from Section 2, in addition to the extra variables introduced for the first extended formulation. By applying Farkas' lemma, we implicitly characterize the projection of this final formulation onto the (x, y, s) space. Moreover, we also show that all of the dominant solutions of the polyhedron defined by (2)–(5), (8), (9), (10), and the bounds $y^i \leq 1, i = 1, ..., P$, are in the projection just mentioned. From these facts we establish an equivalence between the set of dominant solutions of (2)–(5), (8), (9), (10), and $y^i \leq 1, i = 1, ..., P$, and the set of dominant solutions of X^{PIC}. This shows that the set of inequalities just listed suffices to solve PIC by LP.

Given an instance of PIC, let $\tilde{X}^{PIC}$ be the set of dominant solutions to X^{PIC}. We will assume that $d > \lambda$; if this does not hold, the development is similar but simpler.

To reformulate PIC, we introduce new binary variables based on the algorithm described in the proof of Proposition 6. In defining these variables, we say that capacity is met if (3) is tight, and we say that i is produced at demand if exactly d units of i are produced. The variables are

$$\Delta_m = \begin{cases} 1 & \text{if } m \text{ items are produced at demand} \\ 0 & \text{otherwise} \end{cases}$$

$$\delta^i_m = \begin{cases} 1 & \text{if } m \text{ items are produced at demand} \\ & \text{and } i \text{ is produced at demand} \\ 0 & \text{otherwise} \end{cases}$$

$$\rho^i_m = \begin{cases} 1 & \text{if } m \text{ items are produced at demand} \\ & \text{and } i \text{ is produced but } \textit{not} \text{ at demand} \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha^i_m = \begin{cases} 1 & \text{if } m \text{ items are produced at demand} \\ & \text{and demand for } i \text{ is satisfied entirely from inventory} \\ 0 & \text{otherwise} \end{cases}$$

Note that $\delta^i_0 = 0, i = 1, ..., P$, must hold by definition, but we include these variables in order to simplify the formulation that follows. In addition, we define the continuous variables $\gamma^i, i = 1, ..., P$ to be the amount of inventory of item i that is carried in excess of what is needed to satisfy demand.

We call the set of points defined by the following set of constraints $X^{PIC}_{(\Delta,\delta,\rho,\alpha,\gamma)}$:

$$x^i = d \sum_{m=0}^{M} \delta^i_m + \sum_{m=0}^{M} (d - \lambda + (M - m)(t + d))\rho^i_m, i = 1, ..., P \tag{11}$$

$$y^i = \sum_{m=0}^{M} (\delta_m^i + \rho_m^i), i = 1, ..., P \quad (12)$$

$$s^i = d(\sum_{m=0}^{M} \alpha_m^i) + \lambda \rho_M^i + \gamma^i, i = 1, ..., P \quad (13)$$

$$\sum_{m=0}^{M} \Delta_m = 1 \quad (14)$$

$$\delta_m^i + \rho_m^i + \alpha_m^i = \Delta_m, i = 1, ..., P, m = 0, ..., M \quad (15)$$

$$\sum_{i=1}^{P} \delta_m^i = m\Delta_m, m = 0, ..., M \quad (16)$$

$$\sum_{i=1}^{P} \rho_m^i \leq \Delta_m, m = 0, ..., M \quad (17)$$

$$\gamma^i \geq 0, i = 1, ..., P \quad (18)$$

$$\Delta \in \mathbb{B}^{M+1}, \delta \in \mathbb{B}^{P(M+1)}, \rho \in \mathbb{B}^{P(M+1)}, \alpha \in \mathbb{B}^{P(M+1)} \quad (19)$$

Now let $X^{PIC}_{(\Delta,\delta,\rho,\alpha)} = \{(x, y, s, \Delta, \delta, \rho, \alpha) \in X^{PIC}_{(\Delta,\delta,\rho,\alpha,\gamma)} : \gamma^i = 0, i = 1, ..., P\}$. Also, define $Proj_{(x,y,s)}X$ to be the projection of a set of points X onto the space of the (x, y, s) variables.

Proposition 7 *$Proj_{(x,y,s)}X^{PIC}_{(\Delta,\delta,\rho,\alpha)} = \tilde{X}^{PIC}$.*

Proof: From Proposition 1 and from the variable definitions.

Corollary 8 *The projection onto the (x, y, s) space of the dominant solutions of $X^{PIC}_{(\Delta,\delta,\rho,\alpha,\gamma)}$ is $\tilde{X}^{PIC}$.*

Proof: From Proposition 7 and from the fact that $\gamma^i = 0, i = 1, ..., P$, in every dominant solution of $X^{PIC}_{(\Delta,\delta,\rho,\alpha,\gamma)}$. □

The algorithm in Proposition 6 can be seen as identifying the best solution to the above formulation by searching among the $\mathcal{O}(P^2)$ choices of ρ_m^i. If it sets $\rho_{m'}^{i'} = 1$, for some i' and m', then clearly it would also set $\Delta_{m'} = 1$, $\Delta_m = 0, m \neq m'$. The algorithm would set $\delta_m^i = 0, i = 1, ..., P, m \neq m'$, and it would set $\delta_{m'}^{i'} = 0$. The values of $\delta_{m'}^i, i \neq i'$, would depend on the ordering defined in the algorithm: for the items with the m smallest values of $dp^i + q^i - dh^i$, the algorithm sets $\delta_{m'}^i$ to 1, and for the other items it sets $\delta_{m'}^i$ to 0. (Note that if the capacity constraint (3) is not tight, then this corresponds to the case $\rho_m^i = 0, i = 1, ..., P, m = 0, ..., M$. Also note that choosing values for $\rho_m^i, \delta_m^i, i = 1, ..., P, m = 0, ..., M$, and $\Delta_m, m = 0, ..., M$, uniquely defines the possible value for each $\alpha_m^i, i = 1, ..., P, m = 0, ..., M$, and that $\gamma^i = 0, i = 1, ..., P$ in every optimal solution.)

Now we consider the polyhedral structure of conv($X^{PIC}_{(\Delta,\delta,\rho,\alpha,\gamma)}$). Let m be any integer between 0 and M; dropping the subscript m for convenience, we obtain the subformulation

$$\delta^i + \rho^i + \alpha^i = 1, i = 1, ..., P, \tag{20}$$

$$\sum_{i=1}^{P} \delta^i = m \tag{21}$$

$$\sum_{i=1}^{P} \rho^i \leq 1 \tag{22}$$

$$\delta^i, \rho^i, \alpha^i \geq 0, i = 1, ..., P \tag{23}$$

Lemma 9 *The constraint matrix associated with (20)–(23) is a network flow matrix.*

Let $X^{PIC}_{LP(\Delta,\delta,\rho,\alpha,\gamma)}$ be defined by replacing (19) in $X^{PIC}_{(\Delta,\delta,\rho,\alpha,\gamma)}$ with

$$\delta_m^i, \rho_m^i, \alpha_m^i \geq 0, i = 1, ..., P, m = 0, ..., M; \tag{24}$$

Note that these bounds imply $0 \leq \Delta_m \leq 1, m = 0, ..., M$.

Proposition 10 *The polyhedron associated with (14)–(17), (24) is integral.*

Proof: Because of Lemma 9 and of (14), every fractional solution of (14)–(17), (24) is a convex combination of $M + 1$ solutions to the network flow subproblems (20)–(23) indexed by each m. Since every solution to the network flow subproblems is a convex combination of the subproblem extreme points, every fractional solution of (14)–(17), (24) is therefore a convex combination of extreme point solutions. □

Corollary 11 *The polyhedron associated with (14)–(18), (24) is integral.*

Proposition 12 $Proj_{(x,y,s)} X^{PIC}_{LP(\Delta,\delta,\rho,\alpha)} = conv(\tilde{X}^{PIC})$.

Proof: From Proposition 10, $X^{PIC}_{LP(\Delta,\delta,\rho,\alpha)}=$ conv($X^{PIC}_{(\Delta,\delta,\rho,\alpha)}$). From Proposition 7, the projection of conv($X^{PIC}_{(\Delta,\delta,\rho,\alpha)}$) onto the (x, y, s) space is conv($\tilde{X}^{PIC}$). □

The following corollary follows immediately from the above proposition, and from the fact that any unique optimal solution obtained by optimizing (1) over $X^{PIC}_{LP(\Delta,\delta,\rho,\alpha,\gamma)}$ will have $\gamma^i = 0, i = 1, ..., P$.

Corollary 13 *Optimizing (1) over* $X^{PIC}_{LP(\Delta,\delta,\rho,\alpha,\gamma)}$ *solves PIC.*

Thus we have shown how to solve PIC by LP. Moreover, since the formulation defining $X^{PIC}_{LP(\Delta,\delta,\rho,\alpha,\gamma)}$ has $\mathcal{O}(P^2)$ variables and constraints, we have shown how to solve PIC by LP in polynomial time.

In order to use these results to help us explicitly define the set of inequalities needed to solve PIC by LP in the original space of (x, y, s) variables, we will manipulate (11)–(18), (24), into a different formulation which yields the same projection onto the (x, y, s) space. We first eliminate the α variables in (11)–(18). By substituting first (15) and then (12) into (13), we obtain

$$s^i = d - dy^i + \lambda\rho^i_M + \gamma^i, i = 1, ..., P, \tag{25}$$

with which we can replace (13). By substituting (12) into (11), we obtain

$$x^i = dy^i + \sum_{m=0}^{M}(-\lambda + (M - m)(t + d))\rho^i_m, i = 1, ..., P, \tag{26}$$

which can replace (11). Now (15) is the only remaining constraint in which the α variables are present. Since $\alpha^i_m \geq 0, i = 1, ..., P, m = 0, ..., M$, we can simplify (15) to

$$\delta^i_m + \rho^i_m \leq \Delta_m, i = 1, ..., P, m = 0, ..., M, \tag{27}$$

and thus we have an equivalent formulation of $X^{PIC}_{LP(\Delta,\delta,\rho,\alpha,\gamma)}$ that does not contain α variables.

We continue by substituting (25) into (26) to obtain

$$x^i = d - s^i + \sum_{m=0}^{M-1}(-\lambda + (M - m)(t + d))\rho^i_m + \gamma^i, i = 1, ..., P. \tag{28}$$

Note that we can replace (28) (and thus (26)) with

$$x^i \geq d - s^i + \textstyle\sum_{m=0}^{M-1}(-\lambda + (M - m)(t + d))\rho^i_m + \gamma^i, i = 1, ..., P, \tag{29}$$

$$\textstyle\sum_{i=1}^{P} x^i \leq \sum_{i=1}^{P}(\sum_{m=0}^{M} d\delta^i_m + \sum_{m=0}^{M}(d - \lambda + (M - m)(t + d))\rho^i_m), \tag{30}$$

where the last inequality is obtained by substituting for $s^i, i = 1, ..., P$, using (25) and (12). We can also rewrite (25) and (29), respectively, as

$$s^i + dy^i - d = \lambda\rho^i_M + \gamma^i, i = 1, ..., P, \quad (31)$$

$$x^i + s^i - d \geq \sum_{m=0}^{M-1}(-\lambda + (M-m)(t+d))\rho^i_m + \gamma^i, i = 1, ..., P. \quad (32)$$

We call $X^{PIC}_{LP(\Delta,\delta,\rho,\gamma)}$ the polyhedron defined by (12), (30)–(32), (14), (27), (16)–(18), (24). For ease of reference, we list the formulation of $X^{PIC}_{LP(\Delta,\delta,\rho,\gamma)}$ again:

$$y^i = \sum_{m=0}^{M}(\delta^i_m + \rho^i_m), i = 1, ..., P \quad (12)$$

$$\sum_{i=1}^{P} x^i \leq \sum_{i=1}^{P}(d\sum_{m=0}^{M}\delta^i_m + \sum_{m=0}^{M}(d - \lambda + (M-m)(t+d))\rho^i_m) \quad (30)$$

$$s^i + dy^i - d = \lambda\rho^i_M + \gamma^i, i = 1, ..., P \quad (31)$$

$$x^i + s^i - d \geq \sum_{m=0}^{M-1}(-\lambda + (M-m)(t+d))\rho^i_m + \gamma^i, i = 1, ..., P \quad (32)$$

$$\sum_{m=0}^{M}\Delta_m = 1 \quad (14)$$

$$\delta^i_m + \rho^i_m \leq \Delta_m, i = 1, ..., P, m = 0, ..., M \quad (27)$$

$$\sum_{i=1}^{P}\delta^i_m = m\Delta_m, m = 0, ..., M \quad (16)$$

$$\sum_{i=1}^{P}\rho^i_m \leq \Delta_m, m = 0, ..., M \quad (17)$$

$$\gamma^i \geq 0, i = 1, ..., P \quad (18)$$

$$\delta^i_m, \rho^i_m, \alpha^i_m \geq 0, i = 1, ..., P, m = 0, ..., M \quad (24)$$

Corollary 14 *Optimizing (1) over* $X^{PIC}_{LP(\Delta,\delta,\rho,\gamma)}$ *solves PIC.*

Now let $\bar{X}^{PIC} = Proj_{(x,y,s)}X^{PIC}_{LP(\Delta,\delta,\rho,\gamma)}$. From Corollary 14 we immediately have

Corollary 15 *The set of dominant solutions of* $\bar{X}^{PIC}$ *is* $\tilde{X}^{PIC}$.

Also, from Proposition 12 and the definition of $\gamma_i, i = 1, ..., P$, we have

Corollary 16 *$\bar{X}^{PIC} \subset conv(X^{PIC})$.*

We can obtain an implicit description of a set of inequalities which is sufficient to solve PIC by LP in the (x, y, s) space by considering the dual polyhedron of (12), (30)–(32), (14), (27), (16)–(18), (24). To see this, given a fixed point $(\bar{x}, \bar{y}, \bar{s})$, associate dual variables ω^i with constraints (12), β with constraint (30), σ^i with constraints (31), π^i with constraints (32), η with constraint (14), ν^i_m with constraints (27), τ_m with constraints (16), and κ_m with constraints (17).

Let $\hat{X}^{PIC}$ be the polyhedron define by (2)–(5), (8), (9), (10), and the bounds $y^i \leq 1, i = 1, ..., P$; we then have

Proposition 17 *Given any $(\bar{x}, \bar{y}, \bar{s}) \in \hat{X}^{PIC}$, $(\bar{x}, \bar{y}, \bar{s}) \in \bar{X}^{PIC}$ if and only if*

$$\eta + \sum_{i=1}^{P} \omega^i \bar{y}^i + \sum_{i=1}^{P} \sigma^i (\bar{s}^i + d\bar{y}^i - d) - \sum_{i=1}^{P} \pi^i (\bar{x}^i + \bar{s}^i - d) + \beta \sum_{i=1}^{P} \bar{x}^i \leq 0 \qquad (33)$$

for all extreme rays of the dual cone

$$(\delta^i_m) \qquad \omega^i + d\beta + \tau_m - \nu^i_m \leq 0, i = 1, ..., P, m = 0, ..., M \qquad (34)$$

$$(\rho^i_m) \qquad \omega^i + (d - \lambda + (M - m)(t + d))\beta - (-\lambda + (M - m)(d + t))\pi^i$$
$$-\kappa_m - \nu^i_m \leq 0, i = 1, ..., P, m = 0, ..., M - 1 \qquad (35)$$

$$(\rho^i_M) \qquad \omega^i + (d - \lambda)\beta + \lambda\sigma^i - \kappa_M - \nu^i_M \leq 0, i = 1, ..., P \qquad (36)$$

$$(\gamma^i) \qquad \sigma^i - \pi^i \leq 0, i = 1, ..., P \qquad (37)$$

$$(\Delta_m) \qquad \eta = m\tau_m - \kappa_m - \sum_{i=1}^{P} \nu^i_m, m = 0, ..., M \qquad (38)$$

$$\beta \geq 0; \pi^i \geq 0, i = 1, ..., P; \kappa_m \geq 0, m = 0, ..., M;$$
$$\nu^i_m \geq 0, i = 1, ..., P, m = 0, ..., M \qquad (39)$$

Proof: This follows from Farkas' Lemma. □

Thus, for a given $(\bar{x}, \bar{y}, \bar{s})$, if there exists an inequality that separates this point from $\bar{X}^{PIC}$, we can find it in polynomial time by solving the dual linear program given above. This allows us to define an ellipsoid algorithm for PIC that operates in the original space of (x, y, s) variables. However, the separation oracle in this algorithm is itself a linear program in a higher–dimensional space, and Proposition 17 does not by itself give us an explicit description of the valid inequalities needed to solve PIC by LP in the (x, y, s) space.

Because of Corollary 16 and Proposition 17, we know that all valid inequalities for X^{PIC} are solutions, or rays, of (34)–(39). In particular, we can express each of (2)–(5), (8), (9), (10), and $y^i \leq 1, i = 1, ..., P$, as rays of (34)–(39). For example, a given reverse cover inequality (that is, an inequality of the form (10)) in which $|S| = m$ for some m such that $0 < m \leq M$, can be expressed as the extreme ray

$$\begin{gathered}\beta = 1, \eta = -\bar{m}(t+d);\\ \tau_m = -(t+d), \kappa_m = (\bar{m}-m)(t+d), m = 0, ..., \bar{m}-1; \tau_m = \kappa_m = 0, m = \bar{m}, ..., M;\\ \pi^i = 1, \sigma^i = 0, \omega^i = t, \nu^i_m = (t+d), m = \bar{m}, ..., M, \nu^i_m = 0, m = 0, ..., \bar{m}-1, i \in S;\\ \omega^i = d - \lambda - (M-\bar{m})(t+d), \pi^i = \sigma^i = 0, \nu^i_m = 0, m = 0, ..., M, i \in T';\\ \pi^i = 1, \sigma^i = 1, \omega^i = -d, \nu^i_m = 0, m = 0, ..., M, i \in T''. \end{gathered} \tag{40}$$

It can be checked that the above values define an extreme ray of (34)–(39). It can also be checked that plugging the above values into (33) yields an inequality of the form (10).

In order to explicitly describe a set of inequalities that suffices to solve PIC by LP in the (x, y, s) space, we could attempt to characterize the extreme rays of (34)–(39). However, this has proven to be difficult, and therefore we first alter $X^{PIC}_{LP(\Delta,\delta,\rho,\gamma)}$ slightly in order to obtain another polyhedron with the same projection onto the (x, y, s) space, but with a dual cone of slightly simpler structure than (34)–(39).

Define $X^{PIC'}_{LP(\Delta,\delta,\rho,\gamma)}$ to be the polyhedron defined by (12), (30)–(32), (14), (27), (16)–(18), (24), (2)–(5), (8), (9), (10), and the bounds $y^i \leq 1, i = 1, ..., P$. Recall that (8) is the family corresponding to the (l, S) inequalities, (9) is the family of cover inequalities, and (10) is the family of reverse cover inequalities.

Corollary 18 $Proj_{(x,y,s)} X^{PIC'}_{LP(\Delta,\delta,\rho,\gamma)} = \bar{X}^{PIC}$.

Proof: The constraints (2)–(5), (8), (9), (10), and the bounds $y^i \leq 1, i = 1, ..., P$, are implied by the constraints (12), (30)–(32), (14), (27), (16)–(18), and (24), from Corollary 16. □

Now replace (31) with

$$s^i + dy^i - d \geq \lambda\rho^i_M + \gamma^i, i = 1, ..., P, \tag{41}$$

$$\sum_{i=1}^{P}(s^i + dy^i - d) \leq \lambda\sum_{i=1}^{P}\rho^i_M + \sum_{i=1}^{P}\gamma^i. \tag{42}$$

Define $X_{LP(\Delta,\delta,\rho,\gamma)}^{PIC''}$ to be the polyhedron defined by (12), (30), (42), (32), (14), (27), (16)–(18), (24), (2)–(5), (8), (9), (10), and the bounds $y^i \leq 1, i = 1, ..., P$. Note that this polyhedron is obtained from $X_{LP(\Delta,\delta,\rho,\gamma)}^{PIC'}$ by dropping the constraints (41).

Proposition 19 $Proj_{(x,y,s)} X_{LP(\Delta,\delta,\rho,\gamma)}^{PIC''} = \bar{X}^{PIC}$.

Proof: It is clear that $X_{LP(\Delta,\delta,\rho,\gamma)}^{PIC'} \subset X_{LP(\Delta,\delta,\rho,\gamma)}^{PIC''}$, and so

$$\bar{X}^{PIC} = Proj_{(x,y,s)} X_{LP(\Delta,\delta,\rho,\gamma)}^{PIC'} \subset Proj_{(x,y,s)} X_{LP(\Delta,\delta,\rho,\gamma)}^{PIC''}.$$

We will show that

$$Proj_{(x,y,s)} X_{LP(\Delta,\delta,\rho,\gamma)}^{PIC''} \subset Proj_{(x,y,s)} X_{LP(\Delta,\delta,\rho,\gamma)}^{PIC'},$$

which will prove the desired result. To do this, choose any point $(\bar{x}, \bar{y}, \bar{s}, \bar{\Delta}, \bar{\delta}, \bar{\rho}, \bar{\gamma}) \in X_{LP(\Delta,\delta,\rho,\gamma)}^{PIC''}$. We will show that there exists some point $(\hat{x}, \hat{y}, \hat{s}, \hat{\Delta}, \hat{\delta}, \hat{\rho}, \hat{\gamma}) \in X_{LP(\Delta,\delta,\rho,\gamma)}^{PIC''}$ with $\bar{x} = \hat{x}, \bar{y} = \hat{y}, \bar{s} = \hat{s}$, that satisfies (41).

First assume $\bar{\gamma}^i = 0, i = 1, ..., P$. From (30) and (32) we have that

$$\sum_{i=1}^{P} \bar{s}^i \geq d - d \sum_{i=1}^{P} (\sum_{m=0}^{M} \bar{\delta}_m^i + \sum_{m=0}^{M-1} \bar{\rho}_m^i) - (d - \lambda) \sum_{i=1}^{P} \bar{\rho}_M^i.$$

Substituting (12) into the above inequality yields

$$\sum_{i=1}^{P} (\bar{s}^i + d\bar{y}^i - d) \geq \lambda \sum_{i=1}^{P} \bar{\rho}_M^i,$$

which, with (42), implies

$$\sum_{i=1}^{P} (\bar{s}^i + d\bar{y}^i - d) = \lambda \sum_{i=1}^{P} \bar{\rho}_M^i. \tag{43}$$

Now, set

$$\hat{\rho}_M^i = \frac{\bar{s}^i + d\bar{y}^i - d}{\lambda}, i = 1, ..., P.$$

We know that (8) is satisfied for $i = 1, ..., P$; therefore $\hat{\rho}^i_M \geq 0, i = 1, ..., P$. Moreover, because of (43), we know that

$$\sum_{i=1}^{P} \hat{\rho}^i_M = \sum_{i=1}^{P} \bar{\rho}^i_M. \tag{44}$$

Now set

$$\hat{\delta}^i_M = \bar{y}^i - \hat{\rho}^i_M - \sum_{m=0}^{M-1} (\bar{\delta}^i_m + \bar{\rho}^i_m), i = 1, ..., P.$$

Note that $\hat{\delta}^i_M \geq 0, i = 1, ..., P$. Also, note that

$$\hat{\delta}^i_M + \hat{\rho}^i_M = \bar{\delta}^i_M + \bar{\rho}^i_M, i = 1, ..., P, \tag{45}$$

$$\textstyle\sum_{i=1}^{P} \hat{\delta}^i_M = \sum_{i=1}^{P} \bar{\delta}^i_M. \tag{46}$$

We also set

$$\hat{\delta}^i_m = \bar{\delta}^i_m, \hat{\rho}^i_m = \bar{\rho}^i_m, i = 1, ..., P, m = 0, ..., M-1.$$

Because of (44), the right hand side of (42) is the same for $\hat{\rho}$ and $\bar{\rho}$; therefore the point $(\bar{x}, \bar{y}, \bar{s}, \bar{\Delta}, \hat{\delta}, \hat{\rho}, \bar{\gamma})$ satisfies (30). Because of (45), $(\bar{x}, \bar{y}, \bar{s}, \bar{\Delta}, \hat{\delta}, \hat{\rho}, \bar{\gamma})$ satisfies (12) for $i = 1, ..., P$. Because of (44) and (46), the right hand side of (30) is the same for $(\hat{\delta}, \hat{\rho})$ and $(\bar{\delta}, \bar{\rho})$; therefore $(\bar{x}, \bar{y}, \bar{s}, \bar{\Delta}, \hat{\delta}, \hat{\rho}, \bar{\gamma})$ satisfies (30) as well.

It can be easily checked that $(\bar{x}, \bar{y}, \bar{s}, \bar{\Delta}, \hat{\delta}, \hat{\rho}, \bar{\gamma})$ satisfies the remaining constraints of $X^{PIC''}_{LP(\Delta,\delta,\rho,\gamma)}$. Moreover, by definition it satisfies (41), and thus $(\bar{x}, \bar{y}, \bar{s}) \in Proj_{(x,y,s)} X^{PIC'}_{LP(\Delta,\delta,\rho,\gamma)}$.

If $\bar{\gamma}^i > 0$ for some $i \in \mathcal{P}$, one must use an equality like (43) to define new values for $\hat{\gamma}$ as well as for $(\hat{\delta}, \hat{\rho})$, but the approach is similar. $\square$

Recall that $\hat{X}^{PIC}$ is defined by (2)–(5), (8), (9), (10), and the bounds $y^i \leq 1, i = 1, ..., P$. We see immediately that $\bar{X}^{PIC} \subset \hat{X}^{PIC}$. Moreover, we also have

Proposition 20 *Given any* $(\bar{x}, \bar{y}, \bar{s}) \in \hat{X}^{PIC}$, $(\bar{x}, \bar{y}, \bar{s}) \in \bar{X}^{PIC}$ *if and only if*

$$\eta + \sum_{i=1}^{P} \omega^i \bar{y}^i + \sigma \sum_{i=1}^{P} (\bar{s}^i + d\bar{y}^i - d) - \sum_{i=1}^{P} \pi^i (\bar{x}^i + \bar{s}^i - d) + \sum_{i=1}^{P} \beta x^i \leq 0 \tag{47}$$

holds for all extreme rays of the dual cone

$$
\begin{aligned}
&(\delta_m^i) && \omega^i + d\beta + \tau_m - \nu_m^i \le 0, i = 1, ..., P, m = 0, ..., M && (48)\\
&(\rho_m^i) && \omega^i + (d - \lambda + (M-m)(t+d))\beta - (-\lambda + (M-m)(t+d))\pi^i && (49)\\
& && -\kappa_m - \nu_m^i \le 0, i = 1, ..., P, m = 0, ..., M-1 && (50)\\
&(\rho_M^i) && \omega^i + (d-\lambda)\beta + \lambda\sigma - \kappa_M - \nu_M^i \le 0, i = 1, ..., P && (51)\\
&(\gamma^i) && \sigma - \pi^i \le 0, i = 1, ..., P && (52)\\
&(\Delta_m) && \eta = m\tau_m - \kappa_m - \sum_{i=1}^{P} \nu_m^i, m = 0, ..., M && (53)\\
& && \beta \ge 0, \sigma \ge 0; \pi^i \ge 0, i = 1, ..., P; \kappa_m \ge 0, m = 0, ..., M; && \\
& && \quad \nu_m^i \ge 0, i = 1, ..., P, m = 0, ..., M. && (54)
\end{aligned}
$$

Note that (47) and (48)–(54) are obtained from $X^{PIC''}_{LP(\Delta,\delta,\rho,\gamma)}$ in the same way that (33) and (34)–(39) are obtained from $X^{PIC}_{LP(\Delta,\delta,\rho,\gamma)}$. The major difference in the two cones is that we have one variable σ associated with (42) instead of $\sigma^i, i = 1, ..., P$, associated with (31).

The next lemma is necessary to prove the main results of this section.

Lemma 21 *Let $(\bar{x}, \bar{y}, \bar{s})$ be a dominant solution of $\hat{X}^{PIC}$. Then (47) holds for all extreme rays of the dual cone (48)–(54).*

The proof of Lemma 21 proceeds by considering the rays in question by cases, which are defined by values of β and σ. The proof is long, technical, and not intuitive, so it is not given here. The interested reader is referred to Miller [1999].

Proposition 22 *Given the conditions $q^i > 0, i = 1, ..., P$, and $h^i \ge 0, i = 1, ..., P$, the dominant solutions of $\hat{X}^{PIC}$ and of $\bar{X}^{PIC}$ are the same.*

Proof: It is obvious that $\bar{X}^{PIC} \subseteq \hat{X}^{PIC}$. Thus any dominant solution $(\bar{x}, \bar{y}, \bar{s})$ of $\hat{X}^{PIC}$ is also a dominant solution of $\bar{X}^{PIC}$ if and only if $(\bar{x}, \bar{y}, \bar{s}) \in \bar{X}^{PIC}$. Moreover, if all the dominant solutions of $\hat{X}^{PIC}$ are also dominant solutions of $\bar{X}^{PIC}$, then there exist no other dominant solutions of $\bar{X}^{PIC}$.

So, to prove Proposition 22, we need to show that if $(\bar{x}, \bar{y}, \bar{s})$ is a dominant solution of $\hat{X}^{PIC}$, then $(\bar{x}, \bar{y}, \bar{s}) \in \bar{X}^{PIC}$. This holds because of Proposition 20 and Lemma 21. □

Theorem 23 *$\tilde{X}^{PIC}$ is given by the dominant solutions of the polyhedron defined by (2)–(5), (8), (9), (10), and the bounds $y^i \leq 1, i = 1, ..., P$.*

Proof: This follows from Corollary 15, from Proposition 22, and from the definition of $\hat{X}^{PIC}$. □

Corollary 24 *Optimizing (1) over (2)–(5), (8), (9), (10), and the bounds $y^i \leq 1, i = 1, ..., P$, solves PIC.*

A consequence of these results is that the inequalities that we presented for PI in Miller et al. [2000a] suffice to solve the special case PIC by LP.

5 Separation for Cover and Reverse Cover Inequalities for PIC

While separation for inequalities of the forms (2)–(5), (8), is trivial, there are clearly an exponential number of both cover and reverse cover inequalities. As a consequence of our results in the previous section, we know that the separation problem for PIC can be solved in polynomial time by solving an LP in a higher–dimensional space. However, it would be better to have fast combinatorial algorithms to solve these separation problems. In this section we describe combinatorial, polynomial–time separation algorithms for inequalities of the forms (9) and (10).

Proposition 25 (Separation for Cover Inequalities) *Given a point $(\bar{x}, \bar{y}, \bar{s})$, there exists a violated inequality of the form (9) if and only if the optimal solution to the IP*

$$\max \quad \sum_{i=1}^{P}((d-\lambda)(1-\bar{y}^i)+\lambda-\bar{s}^i)\chi^i + \frac{\lambda}{t+d}\sum_{i=1}^{P}(\bar{x}^i-(d-\lambda)\bar{y}^i)\xi^i \tag{55}$$

$$\text{subject to} \quad \chi^i + \xi^i \leq 1, i = 1, ..., P \tag{56}$$

$$\sum_{i=1}^{P}\chi^i \geq M+1 \tag{57}$$

$$\chi^i, \xi^i \in \{0,1\}, i = 1, ..., P \tag{58}$$

is strictly greater than $M\lambda$. Moreover, if the optimal solution is strictly greater than $M\lambda$, then a most violated inequality of the form (9) is obtained by taking $S = \{i \in \mathcal{P} : \chi^i = 1\}$, $T' = \{i \in \mathcal{P} : \xi^i = 1\}$, and $T'' = \{i \in \mathcal{P} : \chi^i = \xi^i = 0\}$.

The IP (55)–(58) can be solved by finding a maximum weight perfect matching on a complete bipartite graph G. In G the vertex set V is partitioned into (A, B), with $|A| = |B| = P$. The nodes in A represent items in $\mathcal{P}$, while the nodes in B are used to represent positions in S, T' or T''.

We partition B further into B' and B'' such $|B'| = M+1$. For each $i \in A$ and $j \in B'$, we let the edge weight w_{ij} be

$$w_{ij} = (d - \lambda)(1 - \bar{y}^i) + \lambda - \bar{s}^i;$$

that is, w_{ij} is the value added to the objective function (55) by setting $\chi^i = 1$. For each $i \in A$ and $j \in B''$, we let the edge weight w_{ij} be

$$w_{ij} = \max\{(d - \lambda)(1 - \bar{y}^i) + \lambda - \bar{s}^i, \frac{\lambda}{t+d}(\bar{x}^i - (d - \lambda)\bar{y}^i, 0\};$$

that is, w_{ij} is the value added to (55) by setting to 1 the choice between χ^i or ξ^i that increases the value of (55) the most, or by setting $\chi^i = \xi^i = 0$ if both $(d-\lambda)(1-\bar{y}^i)+\lambda-\bar{s}^i$ and $\frac{\lambda}{t+d}(\bar{x}^i - (d - \lambda)\bar{y}^i)$ are nonpositive.

A maximum weight perfect matching in G yields the optimal solution to (55)–(58). The most violated cover inequality is then defined as prescribed in Proposition 25.

It is well known that finding a maximum weight matching in a bipartite graph can be accomplished by using a maximum flow algorithm (see e.g. Cook et al. [1998]). For a general graph with unit capacities, it is possible to solve the maximum flow problem in $\mathcal{O}(E^{\frac{3}{2}})$, where E is the number of edges (see e.g. Even and Tarjan [1975]). Since G has P^2 edges, we have

Proposition 26 *The separation problem for cover inequalities for PIC can be solved in* $\mathcal{O}(P^3)$ *time.*

Separation for reverse cover inequalities can be accomplished in a manner similar to cover inequality separation.

Proposition 27 (Separation for Reverse Cover Inequalities) *Given a fractional solution* $(\bar{x}, \bar{y}, \bar{s})$ *of* X_{LP}^{PIC}*, there is a violated inequality of the form (10) if and only if, for some* m*,* $0 \leq m \leq M$*, the value of the optimal solution to the IP*

$$\max \quad \sum_{i=1}^{P}(t\bar{y}^i - \bar{s}^i)\chi^i + \sum_{i=1}^{P}((m(t+d) - c + t)\bar{y}^i + \bar{x}^i)\xi^i \tag{59}$$

$$\text{subject to} \quad \chi^i + \xi^i \leq 1 \tag{60}$$

$$\sum_{i=1}^{P}\chi^i = m \tag{61}$$

$$\chi^i, \xi^i \in \{0, 1\}, \tag{62}$$

is strictly greater than tm. Moreover, if the optimal solution is strictly greater than tm for some such m, then a most violated inequality of the form (10) with $|S| = m$ is defined by putting i into S if $\chi^i = 1$, putting i into T' if $\xi^i = 1$, and putting i into T'' otherwise.

Thus the separation problem for all reverse cover inequalities amounts to solving M IP's of the form (59)–(62), which are identical except for the value of m. Each of these IP's can be solved by finding a maximum weight perfect matching in a complete bipartite graph similar to the graph described in the discussion on cover inequality separation. Therefore, for each $m = 1, ..., M$, the separation problem for reverse cover inequalities in which $|S| = m$ can be solved in $\mathcal{O}(P^3)$ time. This fact yields

Proposition 28 *The separation problem for reverse cover inequalities for PIC can be solved in $\mathcal{O}(P^4)$ time.*

Given the separation algorithms that we have described in this section, we can now define an ellipsoid algorithm for PIC that runs entirely in the space of (x, y, s) variables. In addition, these separation algorithms can clearly be applied to problems for which PIC is a substructure. Moreover, while these separation algorithms are not directly applicable to the general case of PI, we can use the results of this section to help to develop separation heuristics for PI, and for problems for which PI provides a relaxation. For example, the analysis in this section contributed to the development of the separation heuristics for MCL discussed in Miller et al. [2000b].

6 Conclusions and Future Directions

In this paper we have studied PIC, which is a special case of the more general model PI. We have given three nontrivial classes of facet–inducing valid inequalities for PIC, one of which corresponds to a well–known class of inequalities for fixed–charge network flow problems. The other two were originally introduced for PI.

We then gave polynomial–time algorithms for PIC. One result of the developments necessary to define these algorithms is that we are able to show that the families of inequalities mentioned above enable us to solve PIC in polynomial time by linear programming in the original space of variables. That is, the inequalities that we have given for PI in Miller et al. [2000a], are sufficient to solve PI by LP in the known case in which PI is polynomially solvable.

An interesting question that remains open is a characterization of the convex hull of X^{PIC} in terms of linear inequalities. Not only would such a description enable us to solve problems with the same constraint set but more general cost conditions, it could also provide intuition into the structure of the convex hull of closely related problems. One such problem was studied by Goemans [1989]:

$$\begin{gathered}\sum_{j\in N} x_j + a\sum_{j\in N} y_j \leq d \\ 0 \leq x_j \leq m y_j, j \in N \\ y_j \in \{0,1\}, j \in N.\end{gathered}$$

Here a and m are positive integers. This model is similar to PIC, and it appears as a submodel in a number of fixed–charge network flow problems, but a description of the convex hull in terms of inequalities is still not known. Progress toward complete descriptions of such models could help to solve problems in which they appear as substructures.

Given the results that we have established for PI and PIC, we expect that the application of the general forms of (9) and (10) would be computationally effective for many problems for which PI provides a relaxation. This has been confirmed for multi–item capacitated lot–sizing problems (Miller et al. [2000b]); we hope to perform similar tests on other structured MIP problems as well.

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