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ON STRATEGIC COMPLEMENTARITY CONDITIONS IN BERTRAND OLIGOPOLY

Rabah AMIR[1] and Isabel GRILO[2]

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Abstract

For Bertrand duopoly with linear costs, we establish via a single counterexample that: (i) A new monotone transformation of the firms' profit functions may lead to the supermodularity of transformed profits when the standard log and identity transformations both fail, and (ii) Topkis's notion of critical sufficient condition for monotonicity of a Bertrand firm's best-reply correspondence cannot be extended to rely only on positive unit costs.

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[1]CORE, Louvain-la-neuve, Belgium (amir@core.ucl.ac.be)

[2]European Commission, Brussels; GREMARS, Université de Lille 3, France; and CORE, Belgium (E-mail: Grilo@core.ucl.ac.be)

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1. Introduction

This note deals with necessary and sufficient conditions for strategic complementarity in Bertrand duopoly with differentiated substitute products. Two related issues are examined. The first issue, of intermediate interest, is the relationship between the existing sufficient conditions for nondecreasing best-responses in Bertrand duopoly: The supermodularity of the profit functions and the log-supermodularity of the demand functions (see associated definitions below). These two conditions may be viewed as the supermodularity of the transformed profit function upon using the identity and the log transforms, respectively. Thus either condition alone is sufficient for the profit function to satisfy the single-crossing property in (own price; rival's price). It is then natural to ask whether these two conditions are comparable in some sense. We argue via simple examples that neither of these two conditions is weaker than the other. Moreover, we show that a new monotone transformation, $\log(ax + b)$, can bring about the supermodularity of (transformed) profits when the log and the identity transforms both fail to do so. We argue that this transformation has potential for wider applicability as it amounts to a weighted combination of the other two.

The second (and more central) issue raised in this note deals with critical sufficient conditions for Bertrand competition. Topkis (1995) shows that log-supermodularity of demand is critical for nondecreasing best-responses in Bertrand duopoly in the sense that, whenever the condition fails, there exists a firm, specified only by its unit cost in $(-\infty, \infty)$, such that the firm's best-response is not globally nondecreasing in the rival's price[1] . In view of the possibility of negative unit costs, this result is not fully satisfactory from an economic point of view, and an extension relying only on nonnegative unit costs (i.e. actual firms) would be of interest. Unfortunately, we establish via the same (counter-) example as above that such an extension is not possible without further assumptions.

[1] We observe that while Milgrom and Shannon's (1994) Monotonicity Theorem provides necessary and sufficient conditions for a monotone argmax, a crucial element for this conclusion is the treatment of the constraint set as a parameter (ordered with the strong set order). The main example given in this note will also show that the necessity part of this result cannot obtain when the constraint set is treated as in Topkis's formulation. As a consequence, the notion of critical sufficiency emerges as a good alternative.

2. Sufficient Conditions for Increasing Best-Responses

2.1 Set-up and Definitions

As we are concerned with the monotonicity of the reaction correspondence of one firm here, we restrict attention to the duopoly case and deal with (say) Firm 1. Consider two firms competing in prices in a differentiated-product market. Firm i produces product i at constant marginal cost c_i, charges price p_i, and faces a (direct) demand function $D^i(p_1, p_2), i = 1, 2$. Hence, Firm 1's profit function (say) is given by

$$\Pi^1(p_1, p_2) = (p_1 - c_1)D^1(p_1, p_2). \tag{1}$$

Since any price below marginal cost always yields negative profits for the firm, such a price is dominated by pricing at marginal cost, so that we can always restrict consideration to prices in $[c_1, \infty)$ for Firm 1. Furthermore, we assume that there is a natural upper bound $\overline{p}_1$ on price p_1, as a consequence e.g. of reservation prices or income limits, and denote the resulting price intervals $P_1 = [c_1, \overline{p}_1]$ and $P_2 = [c_2, \overline{p}_2]$. While other assumptions on D^1 will be added wherever needed, it is assumed throughout that

(A.1) D^1 *is twice continuously differentiable*[2] *, strictly decreasing in own price, and strictly increasing in rival's price.*

The smoothness assumption is convenient in view of the well-known differential characterization of supermodularity (see below), but it is not essential for our analysis. With monotonicity in own price being standard, the fact that demand increases in rival's price indicates that the two products are substitutes, also a common assumption for this model.

The following notions of, and facts about, complementarity will be invoked in this note (for details, the reader is referred to Milgrom and Shannon, 1994, or Topkis, 1998).

Π^1 is supermodular (or equivalently here, has nondecreasing differences) in (p_1, p_2) if

$$\Pi^1(p_1', p_2') - \Pi^1(p_1, p_2') \geq \Pi^1(p_1', p_2) - \Pi^1(p_1, p_2) \text{ for all } p_1' \geq p_1 \text{ and } p_2' \geq p_2.$$

[2] Throughout the paper, subscripts will denote partial differentiation with respect to the indicated variable.

If Π^1 is twice continuously differentiable in (p_1, p_2), supermodularity of Π^1 is equivalent to $\Pi^1_{p_1p_2}(p_1, p_2) \geq 0$ for all (p_1, p_2). We say Π^1 is log-supermodular if $\log \Pi^1$ is supermodular.

Π^1 satisfies the single-crossing property in $(p_1; p_2)$ if for all $p_1' \geq p_1$ and $p_2' \geq p_2$,

$$\Pi^1(p_1', p_2) - \Pi^1(p_1, p_2) \geq 0 \Longrightarrow \Pi^1(p_1', p_2') - \Pi^1(p_1, p_2') \geq 0.$$

Either of these two properties is sufficient for the reaction correspondence of firm 1 to be globally nonincreasing in p_2. If Π^1 is supermodular in (p_1, p_2), then Π^1 satisfies the single-crossing property in $(p_1; p_2)$ and in $(p_2; p_1)$. The reverse implication need not hold. While the single-crossing property is an ordinal property in the sense of being preserved by strictly increasing transformations of the objective function, supermodularity is not. Finally, if h is such a transformation and $h \circ \Pi^1$ is supermodular, then Π^1 has the single-crossing property.

2.2 On the Relationship Between Previous Results

This model has been successfully analysed with the tools of supermodular optimization. In particular we report two well-known results (Vives, 1990 and Milgrom and Roberts, 1990).

Lemma 1 Π^1 *is supermodular in* (p_1, p_2) *if*

$$\Delta_c \triangleq D^1_{p_2} + (p_1 - c_1)D^1_{p_1p_2} \geq 0 \text{ for all } (p_1, p_2) \in P_1 \times P_2. \tag{2}$$

Proof. Using the differential characterization of supermodularity, set $\Pi^1_{p_1p_2}(p_1, p_2) \geq 0.\square$

Lemma 2 Π^1 *is log-supermodular in* $(p_1, p_2) \in P_1 \times P_2$ *if* D^1 *is log-supermodular, or equivalently if* D^1 *satisfies*

$$\Delta_o \triangleq D^1 D^1_{p_1p_2} - D^1_{p_1} D^1_{p_2} \geq 0 \text{ for all } (p_1, p_2) \in P_1 \times P_2. \tag{3}$$

Proof. Since $\log \Pi^1(p_1, p_2) = \log(p_1 - c_1) + \log D^1(p_1, p_2)$, the result follows directly. With smoothness, the statement follows from setting the cross-partial derivative of $\log \Pi^1 \geq 0.\square$

Clearly, both conditions (2) and (3) are more easily satisfied if the products are substitutes, i.e. if D^1 is globally nondecreasing in p_2. However, this is clearly not a necessary condition for either (2) or (3). On the other hand, as is easily seen, log-supermodularity of demand is less (more) restrictive than supermodularity of demand if the products are substitutes (complements)[3] . Furthermore, log-supermodularity of demand has the precise economic interpretation that the price elasticity of demand is nondecreasing in the other price.

One of the issues raised in the present note is the relationship between supermodularity of profits and log-supermodularity of profits. Since (2) corresponds to the supermodularity of profits and (3) corresponds to the log-supermodularity of profits and hence implies that the profit function has the single-crossing property, it might seem natural to conjecture that (2) holds whenever (3) holds, as a consequence of the general fact that supermodularity implies the single-crossing property. However, (2) and (3) are generally not comparable. Indeed, consider the demand function $D^1(p_1, p_2) = e^{-p_1} + p_2/p_1$ and assume $c_1 = 0$. Then $\Delta_c \equiv 0$ while $\Delta_0 = (p_1 - 1)e^{-p_1}/p_1^2 \geq 0$ if and only if $p_1 \geq 1$. Conversely, the demand function $D^1(p_1, p_2) = e^{p_2 - p_1}$ has $\Delta_0 \equiv 0$, and $\Delta_c = (1 - p_1)e^{p_2 - p_1} \geq 0$ if and only if $p_1 \leq 1$. Other examples are easily constructed establishing that neither condition implies the other.

These examples establish that neither one of Conditions (2) and (3) is stronger than the other, in spite of the fact that (2) corresponds to the supermodularity of profits while (3) leads to the single-crossing property for profits. The two conditions emerge then as alternative sufficient conditions for the reaction correspondence of the firms to be upward-sloping, so that the weakest assumption on demand yielding this desired conclusion is that demand globally satisfies either (2) or (3).

The rest of this note is devoted to an illustrative exploration of the nature of sufficient conditions for upward-sloping price best responses, all based on one interesting example. We examine this issue along two angles, with a view to determine whether it is possible, in some circumstances, to improve in any essential way on Conditions (2) and (3). We begin with an illustration based on a well-known trick in supermodular optimization.

[3] Two products are complements if the demand for each is globally nonincreasing in the price of the other.

2.3 Alternative Monotone Transforms of the Profit Function

While the log transformation, as reflected by Condition (3), seems to be ideally suited to bring about the single-crossing property for this particular model, we will argue by way of example below that other monotone transformations may achieve supermodularity of profits in some cases when the log transformation fails to do so. In fact, the upcoming example shows that such a transformation can be found even when both Conditions (2) and (3) fail.

Consider the monotone transformation $h(x) = \log(ax + b)$, with $a > 0, b \geq 0$. Taking the cross-partial of $h \circ \Pi^1$ with respect to p_1 and p_2 and setting it ≥ 0, we can conclude that $h \circ \Pi^1$ is supermodular if

$$\Delta_h \triangleq a(p_1 - c_1)^2 \Delta_o + b\Delta_c \geq 0 \text{ for all } (p_1, p_2) \in P_1 \times P_2. \quad (4)$$

The complementarity test implicit in (4) is based on finding *one pair* of values (a, b) with $a > 0, b \geq 0$, for which Δ_h is globally ≥ 0. Since Δ_h is an additive combination of Δ_o and Δ_c, it is clearly possible for Δ_h to be globally ≥ 0 (for some suitable choice of a and b), when both Δ_o and Δ_c are not globally ≥ 0 . Here is the key example in this note:

Example 1. Consider the following demand function: $D^1(p_1, p_2) = 10e^{p_2 - p_1} + 2 - p_1$. Then

$$D^1_{p_1} = -10e^{p_2 - p_1} - 1, \; D^1_{p_2} = 10e^{p_2 - p_1} \;\text{ and }\; D^1_{p_1 p_2} = -10e^{p_2 - p_1},$$

so that

$$\Delta_o = 10(p_1 - 1)e^{p_2 - p_1} \qquad \text{and} \qquad \Delta_c = 10(1 - p_1 + c_1)e^{p_2 - p_1}.$$

As is seen by inspection, neither Δ_o nor Δ_c is globally ≥ 0. In other words, this demand is not log-supermodular and the resulting profit function is not supermodular in $(p_1, p_2) \in [c_1, \infty) \times [c_2, \infty)$. Hence, from the sufficient conditions (2) and (3), one cannot conclude that (the max and min selections of) the optimal price are nonincreasing in p_2. Now, consider the strictly increasing transformation h. Applying h to the profit function upon taking $a = b = 1$, and then evaluating (4), we get

$$
\begin{aligned}
\Delta_h &= 10e^{p_2-p_1}[(p_1-c_1)^2(p_1-1)+(1-p_1+c_1)] \\
&= 10e^{p_2-p_1}[(p_1-1)\{(p_1-c_1)^2-1\}+c_1] \\
&= 10e^{p_2-p_1}[(p_1-c_1-1+c_1)\{(p_1-c_1-1)(p_1-c_1+1)\}+c_1] \\
&= 10e^{p_2-p_1}[(p_1-c_1-1)^2(p_1-c_1+1)+c_1\{(p_1-c_1-1)(p_1-c_1+1)\}+c_1] \\
&= 10e^{p_2-p_1}[(p_1-c_1-1)^2(p_1-c_1+1)+c_1(p_1-c_1)^2] \\
&\geq 0, \text{ for all } c_1 \geq 0,\ p_1 \geq c_1, \text{ and } p_2 \geq 0.
\end{aligned}
$$

Hence, the original profit function satisfies the single-crossing property in $(p_1; p_2)$, and thus, we know that (the max and min selections of) the optimal price are nonincreasing in p_2, in spite of the failure of (2) and (3) to hold globally.[4]

There are other parametrized monotone transformations that may serve the same purpose as $h(x) = \log(ax + b)$, as illustrated above, for suitable values of the parameter(s). These include, among others: (i) $e^{-\alpha x}$, $-\infty < \alpha < \infty$; (ii) $ax + b \log x, a \geq 0, b \geq 0$ and (iii) x^α/α, $-\infty < \alpha < \infty$. The latter transformation encompasses a broad scope of possibilities. In particular, the special cases $\alpha = 0$ and $\alpha = 1$ correspond respectively to the log and identity transforms, thereby leading back to Conditions (3) and (2) respectively.

3. On Critical Sufficient Conditions

As the prospects for deriving monotonicity conditions that would be both necessary and sufficient are virtually nonexistent for most comparative statics models, an alternative notion of necessity has been proposed in lattice-theoretic comparative statics, that of a critical sufficient condition. The key idea is to specify a broader context including the exact model under consideration as a special case, and then require that the desired monotonicity condition hold for *all* elements (or models) of the context. A sufficient condition that is necessary for the latter (broader) requirement is termed "critical": See Milgrom and Shannon (1994).

4 It is worthwhile to point out that having Δ_o and Δ_c not globally ≥ 0 (while the monotonicity conclusion actually holds) here means that the standard approach to comparative statics (based on differentiating first-order conditions upon invoking the Implicit Function Theorem) would also fail to produce the monotonicity conclusion here, although the profit function is strictly quasi-concave in own price, as is easily verified!

For the present context, an interesting notion of criticality is proposed by Topkis (1995): A given sufficient condition is critical for nondecreasing best-responses in the Bertrand duopoly if, whenever the condition fails, there exists a firm, specified only by its unit-cost $c_1 \in (-\infty, \infty)$, such that its best-response in not globally nondecreasing in the price of the rival. An important methodological finding in Topkis (1995) is that log-supermodularity of demand, or (3), is a critical sufficient condition for nondecreasing best-responses (see Topkis (1998), Theorem 3.3.7, p. 124). Unfortunately, the result is of limited economic value as c_1 can be < 0, so that there is no meaningful economic interpretation for the "firm" in question.

An important question then is whether Topkis's criticality result can be extended to rely only on nonnegative unit costs (or actual firms). As given, his proof makes crucial use of the full flexibility of choosing $c_1 \in (-\infty, \infty)$. Unfortunately, the answer is negative as shown by the above example. Indeed this follows immediately from the fact that $\Delta_h \geq 0$ since, as we saw before, (3) fails to hold globally while the best-response is nevertheless nondecreasing in the rival's price *for all* $c_1 \geq 0$, as a consequence of the supermodularity of h-transformed profits for all $c_1 \geq 0$. We conclude that log-supermodularity of demand is a critical sufficient condition (in the sense of Topkis) for monotone best-responses in the price game if one takes a firm to be specified by its unit cost in $(-\infty, \infty)$, but not in $[0, \infty)$.

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