1 Introduction

Most of the literature on voting relies on the dichotomous assumption that parties are strategic and voters are sincere (see Shepsle 1991 for a review of this literature), or parties' positions are given and voters are strategic (see, among others, Cox 1997). In a previous paper (De Sinopoli and Iannantuoni 2000) we investigated the consequence of strategic voting in a model of proportional representation[1] and showed the devastating effect of strategic voting: rational voters vote only for the two extremist parties. More precisely, given the assumption that the policy space is a closed interval of the real line, we analyzed proportional representation through a policy outcome defined as a linear combination of parties' position weighted with the share of votes that each party gets in the election. In this way we clearly wanted to capture the spirit of proportional representation, i.e., any party getting some votes is represented in the political process of policy determination, with a weight that is proportional to its share of votes. We proved that, for a large electorate, a unique Nash equilibrium exists, characterized by an equilibrium outcome (defined as cutpoint outcome) such that any voter to its right/left votes for the rightmost/leftmost party. The intuition of the result is clear: strategic voters misrepresent their preferences voting for the extreme parties in order to drag the policy outcome toward their preferred policy point.

In this paper, we want to extend the previous model to show that the main results hold eventhough several voters do not act strategically but vote sincerely.[2]

One may identify at least two reasons to undertake this analysis.

The first one concerns the assumption that every voter is strategic; one may consider it a requirement too strong. From this point of view the hypothesis that only a subset of voters votes strategically seems more reasonable.

Moreover, it is possible to better reconcile the results of our previous model (i.e. only the two extremist parties take votes under proportional rule) with the general agreement on the evidence that more than two parties take votes in proportional representation elections (see Cox 1997).

One may think of sincere voters as "ideological" voters, who simply vote for the party whose policy they prefer[3], while the strategic voters vote to maximize their utilities.

It is interesting to compare the solution of this game with the solution of the game in which all voters are strategic. Two natural questions arise in this context. The first one is if, and how, the sincere voters' behavior affects the outcome. The second one is how strategic voters will "respond" to that.

[1] We refer to that paper for motivations.

[2] We speak about sincere voters only for a matter of clarity. As a matter of fact, the analysis carried in this note holds for all the cases when the behavior of this subset of voters is given.

[3] In this interpretation of the non strategic voters, we assume that their most preferred party is unique.

We show that the behavior of sincere voters affects the outcome, in that the policy will, in general, be different with respect to the case in which all voters vote strategically and a subset of strategic voters changes its voting behavior to balance the sincere players' votes. However, they can only partially adjust.

As a matter of fact, strategic voters will vote in accord with this modified (i.e., from the sincere votes) cutpoint outcome: any strategic voter on its right will vote for the most rightmost party and any strategic voter on its left will vote for the most leftmost party.

Obviously, we can no longer claim that only two parties emerge at equilibrium: the number of parties now depends on the sincere voters' behavior.

The paper is organized as follows. In section 2 we describe the model. We analyze the pure strategy equilibria, and, then, the mixed strategy ones in section 3. We present an example in section 4, and section 5 concludes.

2 The Model

In this section we briefly introduce the model of proportional representation proposed in De Sinopoli and Iannantuoni (2000), as well as the modifications that allow us to consider also sincere voting.

Policy Space. The policy space $\mathbb{X}$ is a closed interval of the real line, and without loss of generalities we assume $\mathbb{X} = [0, 1]$.

Parties. Parties are fixed both in number and in their positions, in that there is no strategic role for them: there is an exogenously given set of parties $M = \{1, ..., k, ...m\}$ $(m \geq 2)$, indexed by k. Each party k is characterized by a policy $\zeta_k \in [0, 1]$.

Proportional Rule. Given the set of parties M, each voter can cast his vote for any party.[4] The pure strategy space of each player i is $S_i = \{1, ..., k, ..., m\}$ where each $k \in S_i$ is a vector of m components with all zeros except for a one in position k, which represents the vote for party k.

A mixed strategy of player i is a vector $\sigma_i = (\sigma_i^1, ...\sigma_i^k, ..., \sigma_i^m)$ where each σ_i^k represents the probability that player i votes for party k.

The policy outcome. The position of the government, i.e., the policy outcome, is a linear combination of parties' policies each coefficient being equal to the corresponding share of votes. Given a pure strategy combination $s = (s_1, s_2, ..., s_n)$, $v(s) = \frac{1}{n} \sum_{i \in N} s_i$ is the vector representing for each party its share of votes, hence the policy outcome can be written as:

$$X(s) = \sum_{k=1}^{m} \zeta_k v_k(s) . \quad (1)$$

[4] In this model we do not allow for abstention. We cannot claim that this assumption is neutral. In our proof we use the fact that, as the number of players goes to infinity, the weight of each player goes to zero, and this does not hold if a large number of voters abstains.

Strategic Voters. Each strategic voter i is characterized by a bliss point $\theta_i \in \Theta = [0,1]$. Voters' preferences are single peaked. We stress that this is the only assumption needed to reach the result for pure strategy equilibrium. To analyze also mixed strategy equilibria, we assume that it exists a fundamental utility function (à la Harsany) $u : \Re^2 \rightarrow \Re$, continuously differentiable with respect to the first argument[5], which represents the preferences, that is $u_i(X) = u(X, \theta_i)$. Since each voter can cast his vote for any party the pure strategy space of each player i is $S_i = \{1, ..., k, ..., m\}$. We will denote with N^ρ the set of strategic voters and with n^ρ its cardinality.

Sincere Voters. A natural way to model sincere voters is to assume that their strategy space is degenerate, coinciding with the vote for their most preferred party. We will denote with N_k^ι the set of sincere voters who vote for party k and with n_k^ι its cardinality, while $\bar{n} = \sum_{k=1}^{m} n_k^\iota$.

Given the set of parties and the utility function u, a finite game is characterized by the set of voters N, the subset of strategic voters and their bliss points, as well as the subsets of sincere voters voting for each party:

$$\Gamma = \left\{N, N^\rho, \{\theta_i\}_{i \in N^\rho}, \{N_k^\iota\}_{k \in M}\right\}$$

Finally, we define $H^\rho(\theta)$ the distribution of the strategic voters' bliss points.

3 The equilibrium

We analyze the equilibrium of the game defined above. First, given the voting behavior of sincere voters, we analyze strategic voters' behavior when only pure strategies are allowed. We show that in any pure strategy Nash equilibrium of the game, strategic voters vote only for the extreme parties, except for a neighborhood inversely related to the total number of players. We then define the cutpoint outcome, i.e., the outcome obtained with any strategic voter strictly on its right voting for the rightmost party and any strategic voter strictly on its left voting for the leftmost party. Such a strategy combination is a pure strategy Nash equilibrium of the game, if it does not coincide with a strategic voter's bliss point. Moreover, the cutpoint outcome so defined is in general different from the cutpoint outcome defined in De Sinopoli and Iannantuoni (2000), in that there is a "fixed effect" of the sincere voters' behavior for which strategic voters cannot fully adjust.

As nothing assures us that the sufficient condition above for the existence of a pure strategy equilibrium is satisfied, or that mixed strategy equilibria do not behave completely differently, we extend the analysis to the case in which voters are allowed to play mixed strategies. The result implies that in any mixed strategy equilibrium, except for a neighborhood inversely related to the total number of players, strategic voters vote for the extreme parties. Moreover, we

[5] Hence, by single-peakedness, $\forall \bar{x}_2 \in [0,1]$, $\frac{\partial u(x_1, \bar{x}_2)}{\partial x_1} \gtreqless 0$ for $x_1 \lesseqgtr \bar{x}_2$ and $x_1 \in [0,1]$.

prove the main result of this note: *in any equilibrium any strategic player on the right of the cutpoint outcome votes for the rightmost party, and any strategic player on the left of the cutpoint outcome votes for the leftmost party, except for a neighborhood whose lenght is inversely related to the total number of voters.*

In order to simplify notation, we denote L the leftmost party and R the rightmost (i.e., $L = \arg\min_{k \in M} \zeta_k$, $R = \arg\max_{k \in M} \zeta_k$).[6]

3.1 Pure strategy equilibria

We start by analyzing the pure strategy equilibria in order to stress the intuition behind the result, that is, strategic voters have an incentive to vote for the extreme parties in order to drag the policy outcome toward their bliss policy. First, we underline that only the assumption of single peakedness of strategic voters' preferences is needed to get the result. We prove that every pure strategy equilibrium is such that (except for a neighborhood whose length is inversely proportional to the total number of players) every strategic voter votes for one of the two extreme parties.

Proposition 1 *Let s be a pure strategy equilibrium of a game Γ with n voters:*
(α) $\forall i \in N^\rho$, if $\theta_i \leq X(s) - \frac{1}{n}(\zeta_R - \zeta_L)$ then $s_i = L$,
(β) $\forall i \in N^\rho$, if $\theta_i \geq X(s) + \frac{1}{n}(\zeta_R - \zeta_L)$ then s_i=R.

Proof. [7](α) Notice that if $X(s_{-i}, L) \geq \theta_i$ then, by single-peakedness, L is the only best reply, for player i, to s_{-i} (i.e., $\forall k \neq L$, $X(s_{-i}, k) > X(s_{-i}, L)$). Since $X(s_{-i}, L) = X(s) - \frac{1}{n}(\zeta_{s_i} - \zeta_L) \geq X(s) - \frac{1}{n}(\zeta_R - \zeta_L)$, the assumption $\theta_i \leq X(s) - \frac{1}{n}(\zeta_R - \zeta_L)$, implies that L is the unique best reply, for player i, to s_{-i}. (β) A symmetric argument holds. ■

At this point, it is natural to analyze the case when any strategic voter strictly on the left of the policy outcome votes for L, and any strategic voter strictly on the right of the policy outcome votes for R.

Given a game Γ and the distribution of strategic voters' bliss points $H^\rho(\theta)$, let $\tilde{\theta}^\Gamma_\rho$, defined as cutpoint policy, be the unique policy outcome obtained with strategic voters strictly on its left voting for L and strategic voters strictly on its right voting for R, i.e. let $\tilde{\theta}^\Gamma_\rho$ be implicitly defined by:

$$\tilde{\theta}^\Gamma_\rho = \frac{n^\rho}{n^\rho + \bar{n}} \left(\zeta_L \bar{H}^\rho \left(\tilde{\theta}^\Gamma_\rho \right) + \zeta_R \left(1 - \bar{H}^\rho \left(\tilde{\theta}^\Gamma_\rho \right) \right) \right) + \frac{\bar{n}}{n^\rho + \bar{n}} \sum_{k=1}^{m} \frac{n^\iota_k}{\bar{n}} \zeta_k$$

where $\bar{H}^\rho$ is the correspondence defined by $\bar{H}^\rho(\theta) = \left[\lim_{y \to \theta^-} \bar{H}^\rho(y), \bar{H}^\rho(\theta) \right]$.

[6] We assume that there is only one party in ζ_L as well as in ζ_R. This assumption simplifies the notation, but it does not affect the result. Without this assumption, if we denote L and R the set of extreme parties, everything still holds.

[7] This proof, as well as the others, are completely analogous to the corresponding ones presented in De Sinopoli and Iannantuoni (2000). Nevertheless, we prefer to repeat them here for completeness and for some typos in the previous DP.

The first term of the right-hand side of the expression above[8] represents the effect of the strategic voters' behavior, weighted by the share of the strategic voters on the total number of voters, on the cutpoint ; the second stands for the "fixed" effect of the sincere voters' behavior, weighted by the share of sincere voters.

Let us assume that no strategic player's preferred policy coincides with the cutpoint outcome. If all strategic players vote accordingly with the cutpoint, no strategic player on its left has an incentive to vote for any party different from L, because doing so would push the policy outcome further away from his preferred policy. The same argument holds for any strategic player on the right of the policy outcome. We can, then, state the following proposition:

Proposition 2 *If* $\theta_i \neq \tilde{\theta}_\rho^\Gamma$ $\forall i \in N^\rho$, *then the strategy combination given by*
$s_i = L \qquad \forall i \in N^\rho$ *with* $\theta_i < \tilde{\theta}_\rho^\Gamma$
$s_i = R \qquad \forall i \in N^\rho$ *with* $\theta_i > \tilde{\theta}_\rho^\Gamma$
$s_i = k \qquad \forall i \in N_k^\iota$
is a pure strategy Nash equilibrium of the game Γ.

It is clear that nothing assures us that pure strategy equilibria exist; moreover we have to check if mixed strategy equilibria prescribe a dramatically different behavior for strategic voters.

3.2 Mixed strategy equilibria

We analyze the case when strategic voters are allowed to play mixed strategies. In order to undertake this analysis we have to assume also that the utility function u is continuously differentiable with respect to the first argument.

We recall that, given the set of candidates M and the utility function u, a game Γ is characterized by the set of players, the set of strategic voters and their bliss points, as well as the set of sincere voters voting for each party. Let $\sigma = (\sigma_1, ..., \sigma_n)$ and $\bar{\mu}^\sigma = \sum_{i \in N} \frac{\sigma_i}{n}$. With abuse of notation, let $X(\bar{\mu}^\sigma) = \sum_{k=1}^{m} \zeta_k \bar{\mu}_k^\sigma$.

We can state the following proposition:

Proposition 3 $\forall \varepsilon > 0$, $\exists n_0$ *such that* $\forall n \geq n_0$ *if* σ *is a Nash equilibrium of a game* Γ *with* n *voters then:*
(α) $\forall i \in N^\rho$, *if* $\theta_i \leq X(\bar{\mu}^\sigma) - \varepsilon$ *then* $\sigma_i = L$
(β) $\forall i \in N^\rho$, *if* $\theta_j \geq X(\bar{\mu}^\sigma) + \varepsilon$ *then* $\sigma_j = R$.

[8] Obviously, it can be written also in terms of the share of votes. Let v^ρ be the share of strategic voters and v_k^ι be the share of sincere voters, on the total number of voters, who votes for party k, then the cutpoint outcome is:

$$\tilde{\theta}_\rho^\Gamma = v^\rho \left(\zeta_L \bar{H}^\rho \left(\tilde{\theta}_\rho^\Gamma \right) + \zeta_R \left(1 - \bar{H}^\rho \left(\tilde{\theta}_\rho^\Gamma \right) \right) \right) + \sum_{k=1}^{m} v_k^\iota \zeta_k$$

Proof. See Appendix. ■

In the appendix we will show that $\bar{\mu}^\sigma$ is the expected share of votes. The proposition above says that in any Nash equilibrium, except for a neighborhood whose length decreases as the total number of players increases, every strategic voter to the left of $X(\bar{\mu}^\sigma)$ votes for L, while every strategic voter to the right votes for R.

Using the definition of cutpoint policy outcome, we can state the main result of this paper, that is "basically" an unique Nash equilibrium of the game exists:

Corollary 4 *$\forall \eta > 0$, $\exists n_1$ such that $\forall n \geq n_1$ if σ is a Nash equilibrium of a game Γ with n voters then:*
(α) $\forall i \in N^\rho$, if $\theta_i \leq \tilde{\theta}_\rho^\Gamma - \eta$ then $\sigma_i = L$
(β) $\forall i \in N^\rho$, if $\theta_j \geq \tilde{\theta}_\rho^\Gamma + \eta$ then $\sigma_j = R$.

Proof. Fix η and in Proposition 3, take $\varepsilon = \frac{\eta}{2}$. For the corresponding n_0 it is easy to see that if $n \geq n_0$ and σ is a Nash equilibrium of Γ, then $\tilde{\theta}_\rho^\Gamma - \frac{\eta}{2} \leq X(\bar{\mu}^\sigma) \leq \tilde{\theta}_\rho^\Gamma + \frac{\eta}{2}$. In fact, suppose by contradiction that $\tilde{\theta}_\rho^\Gamma - \frac{\eta}{2} > X(\bar{\mu}^\sigma)$. Proposition 3 implies that all voters to the right of $\tilde{\theta}_\rho^\Gamma$ vote for the rightist party and hence $\tilde{\theta}_\rho^\Gamma \leq X(\bar{\mu}^\sigma)$, contradicting $\tilde{\theta}_\rho^\Gamma - \frac{\eta}{2} > X(\bar{\mu}^\sigma)$. Analogously for the second inequality. Hence $\tilde{\theta}_\rho^\Gamma - \eta \leq X(\bar{\mu}^\sigma) - \frac{\eta}{2}$ and $\tilde{\theta}_\rho^\Gamma - \eta \geq X(\bar{\mu}^\sigma) + \frac{\eta}{2}$, which, with Proposition 3, complete the proof. ■

Every equilibrium conforms to such a cutpoint, and hence, for n large enough, strategic voters basically vote only for the two extreme parties.

4 Example

We now present a very simple example in order to understand how the behavior of sincere voters affects the policy outcome.[9]

Let the distribution function of voters' bliss points be such that there are two voters at any bliss point.

First, let us consider the case when everybody is strategic. In this case, the cutpoint outcome is (De Sinopoli and Iannantuoni, 2000):

$$\tilde{\theta}^\Gamma = \zeta_L \bar{H}^\Gamma\left(\tilde{\theta}^\Gamma\right) + \zeta_R\left(1 - \bar{H}^\Gamma\left(\tilde{\theta}^\Gamma\right)\right).$$

Let now consider the case when one of the two player at each bliss point is strategic while the other is sincere. The cutpoint outcome now is the following:

$$\tilde{\theta}_\rho^\Gamma = \frac{n^\rho}{n^\rho + \bar{n}}\left(\zeta_L \bar{H}^\rho\left(\tilde{\theta}_\rho^\Gamma\right) + \zeta_R\left(1 - \bar{H}^\rho\left(\tilde{\theta}_\rho^\Gamma\right)\right)\right) + \frac{\bar{n}}{n^\rho + \bar{n}} \sum_{k=1}^{m} \frac{n_k^\iota}{\bar{n}} \zeta_k,$$

[9] This example is suitable to compare with our present model the two possible different cases, i.e., the one in which sincere voters are absent and the one in which they act strategically, in both cases the cutpoint being $\tilde{\theta}^\Gamma$.

which can be rewritten, considering that $n^{\rho} = \bar{n}$ and that $\bar{H}^{\rho}(\cdot) = \bar{H}^{\Gamma}(\cdot)$, in the following way:

$$\tilde{\theta}_{\rho}^{\Gamma} = \frac{1}{2}\left(\zeta_L \bar{H}^{\Gamma}\left(\tilde{\theta}_{\rho}^{\Gamma}\right) + \zeta_R\left(1 - \bar{H}^{\Gamma}\left(\tilde{\theta}_{\rho}^{\Gamma}\right)\right)\right) + \frac{1}{2}\sum_{k=1}^{m} v_k^{\iota}\zeta_k$$

The first term of the right-hand side of the above expression represents the effect on the policy outcome of the strategic voters' behavior. Clearly, this effect is completely analogous to the cutpoint when everybody is strategic, but it is now weighted by the share of strategic voters. The second term represents the fixed effect of sincere voters' behavior on the outcome.

This example clearly shows that the two cutpoints are not, in general, equal. Suppose that $\tilde{\theta}_{\rho}^{\Gamma}$ is on the right of $\tilde{\theta}^{\Gamma}$. There is a subset of strategic voters, those between $\tilde{\theta}^{\Gamma}$ and $\tilde{\theta}_{\rho}^{\Gamma}$, who votes for the leftmost party in order to adjust to the sincere voters' effect. Nevertheless, strategic voters cannot fully adjust.

5 Conclusions

We have studied the case when there is only a subset of voters who vote strategically, whereas the others vote sincerely in a framework similar to the one developped in De Sinopoli and Iannantuoni (2000).

The main findings imply that there is "basically" an unique Nash equilibrium characterized by a cutpoint outcome such that any strategic voter on its left votes for the leftmost party and any strategic voter on its right votes for the rightmost party. Moreover, there is a "fixed" effect of the sincere voters' behavior on the equilibrium outcome for which strategic players cannot fully adjust.

Furthermore, this extension helps to better reconcile the results of our previous model (i.e. only the two extremist parties take votes under proportional rule, see De Sinopoli and Iannantuoni 2000) with the general agreement on the evidence that proportional representation systems are more likely characterized by multi-party[10] (see Cox 1997).

[10] Even if there are striking counterexamples to this general agreement.

6 Appendix

Proof of proposition 3:

(α) Given a mixed strategy σ_j, the player j's vote is a random vector[11] $\tilde{s}_j$ with $Pr(\tilde{s}_j = k) = \sigma_j^k$. Given $\sigma_{-i} = (\sigma_1, ...\sigma_{i-1}, \sigma_{i+1}, ...\sigma_n)$, let $\bar{\tilde{s}}^{-i} = \frac{1}{n-1} \sum\limits_{j \in N/i} \tilde{s}_j$ and $\bar{\mu}^{\sigma_{-i}} = \frac{1}{n-1} \sum\limits_{j \in N/i} \sigma_j$. The first step of the proof consists in proving the next lemma:

Lemma 5 *$\forall \phi > 0$ and $\forall \delta > 0$, if $n > \frac{m}{4\phi^2 \delta} + 1$, then $\forall \sigma, \forall i$*

$$\Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}\right) > 1 - \delta.$$

Proof. To prove the lemma we can use Chebichev's inequality component by component. Given σ_{-i}, it is easy to verify that $E(\tilde{s}_j^k) = \sigma_j^k$ and $Var(\tilde{s}_j^k) = \sigma_j^k(1 - \sigma_j^k) \leq \frac{1}{4}$, hence $E(\bar{\tilde{s}}_k^{-i}) = \bar{\mu}_k^{\sigma_{-i}}$ and $Var(\bar{\tilde{s}}_k^{-i}) \leq \frac{1}{4(n-1)}$. By Chebychev's inequality we know that $\forall k, \forall \phi$:

$$\Pr\left(\left|\bar{\tilde{s}}_k^{-i} - \bar{\mu}_k^{\sigma_{-i}}\right| > \phi\right) \leq \frac{1}{4(n-1)\phi^2}.$$

Hence

$$\Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}\right) \geq 1 - \sum_k \Pr\left(\left|\bar{\tilde{s}}_k^{-i} - \bar{\mu}_k^{\sigma_{-i}}\right| > \phi\right) \geq 1 - \frac{m}{4(n-1)\phi^2},$$

which is strictly greater than $1 - \delta$ for $n > \frac{m}{4\phi^2\delta} + 1$. ■

Now we show that $\forall \varepsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$, if $\theta_i < X(\bar{\mu}^{\sigma}) - \varepsilon$, then L is the only best reply for player $i \in N^{\rho}$ to σ^{-i}.

Fix $\varepsilon > 0$. Define $\forall \theta \in \left[0, 1 - \frac{\varepsilon}{2}\right]$

$$M_{\varepsilon}(\theta) = \max_{X \in \left[\theta + \frac{\varepsilon}{2}, 1\right]} \frac{\partial u(X, \theta)}{\partial X}.$$

By single-peakedness we know that $M_{\varepsilon}(\theta) < 0$. Moreover, given the continuity of $\frac{\partial u(X,\theta)}{\partial X}$ I can apply the theorem of the maximum[12] to deduce that the

[11] We remind readers that a vote is a vector with m components.

[12] Because there are various versions of the theorem of the maximum, we prefer to state explicitly the version we are using. Let $f : \Psi \times \Phi \rightarrow \Re$ be a continuous function and $g : \Phi \rightarrow P(\Psi)$ be a compact-valued, continuous correspondence, then $f^*(\phi) := \max\{f(\psi, \phi) \mid \psi \in g(\phi)\}$ is continuous on Φ.

function $M_\varepsilon(\theta)$ is continuous, hence it has a maximum on $\left[0, 1-\frac{\varepsilon}{2}\right]$, which is strictly negative. Let

$$M_\varepsilon^* = \max_{\theta \in \left[0, 1-\frac{\varepsilon}{2}\right]} M_\varepsilon(\theta).$$

Let M denote the upper bound[13] of $\left|\frac{\partial u(X,\theta)}{\partial X}\right|$ on $[0,1]^2$, and let $\delta_\varepsilon^* = \frac{-M_\varepsilon^*}{M - M_\varepsilon^*} > 0$ and $\phi^* = \frac{\left(-2+\sqrt{6}\right)\varepsilon}{m}$. We prove that if $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$, then every strategy other than L cannot be a best reply for player i, which, setting n_0 equal to the smallest integer strictly greater than $\frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$, directly implies the claim.[14]

Take a party $c \neq L$. By definition $c \in BR_i(\sigma) \Longrightarrow$

$$\sum_{s_{-i} \in S_{-i}} \sigma(s_{-i}) \left[u\left(X(s_{-i}, c), \theta_i\right) - u\left(X(s_{-i}, L), \theta_i\right)\right] \geq 0, \tag{2}$$

which can be written as:

$$\sum_{s_{-i} \in S_{-i}} \sigma(s_{-i}) \left[u\left(X(s_{-i}, c), \theta_i\right) - u\left(X(s_{-i}, c) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right] \geq 0. \tag{3}$$

Because the outcome function $X(s)$ depends only upon $v(s)$, denoting with V_n^{-i} the set of all vectors representing the share of votes obtained by each party with $(n-1)$ voters, (3) can be written as:

$$\sum_{v_n^{-i} \in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i}) \left[u\left(X\left(v_n^{-i}, c\right), \theta_i\right) - u\left(X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right] \geq 0 \tag{4}$$

where, with abuse of notation, $X\left(v_n^{-i}, c\right) = \frac{\zeta_c}{n} + \frac{n-1}{n}\sum_{k=1}^{m} \zeta_k v_{n(k)}^{-i}$. Multiplying both sides of (4) by $\frac{n}{\zeta_c - \zeta_L} > 0$ we have:

$$\sum_{v_n^{-i} \in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i}) \frac{\left[u\left(X\left(v_n^{-i}, c\right), \theta_i\right) - u\left(X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right]}{\frac{1}{n}(\zeta_c - \zeta_L)} \geq 0. \tag{5}$$

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[13] The continuity of $\frac{\partial u(X,\theta)}{\partial X}$ assures that such a bound exists.

[14] This is the same bound we found without sincere voters. Because if j is a sincere player $Var(\tilde{s}_j^k) = 0$, we have that the variance of $\bar{\tilde{s}}_k^{-i}$ decreases with sincere voters, we could perhaps find a better bound. As a matter of fact if $\frac{(n-1)^2}{n^\rho - 1} > \frac{m}{4\phi^2\delta}$ then

$$\Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}\right) > 1 - \delta.$$

However a preliminary cost-benefit analysis discouraged us from such a project.

By the *mean value theorem* we know that $\forall v_n^{-i}$,
$\exists X^* \in \left[X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), X\left(v_n^{-i}, c\right)\right]$ such that

$$\frac{\left[u\left(X\left(v_n^{-i}, c\right), \theta_i\right) - u\left(X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right]}{\frac{1}{n}(\zeta_c - \zeta_L)} = \left.\frac{\partial u(X, \theta_i)}{\partial X}\right|_{X=X^*}.$$

Hence we have:

$$\sum_{v_n^{-i} \in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i}) \frac{\left[u\left(X\left(v_n^{-i}, c\right), \theta_i\right) - u\left(X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right]}{\frac{1}{n}(\zeta_c - \zeta_L)} \leq$$

$$\Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*) M_n^*(\vec{\phi}^*, \theta_i) + (1 - \Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*)) M$$

where

$$M_n^*(\vec{\phi}^*, \theta_i) = \max_{X \in \left[X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, c) - \frac{1}{n}(\zeta_c - \zeta_L), 1\right]} \frac{\partial u(X, \theta_i)}{\partial X}.$$

Now we prove that, for $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$, $M_n^*(\vec{\phi}^*, \theta_i) \leq M_\varepsilon^*$. From the definition of M_ε^*, it suffices to prove that $M_n^*(\vec{\phi}^*, \theta_i) \leq M_\varepsilon(\theta_i)$, which is true if $X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, c) - \frac{1}{n}(\zeta_c - \zeta_L)$ is greater than $\theta_i + \frac{\varepsilon}{2}$.

$$X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, c) - \frac{1}{n}(\zeta_c - \zeta_L) = \frac{n-1}{n} \sum_k \bar{\mu}_k^{\sigma_{-i}} \zeta_k - \frac{n-1}{n} \sum_k \phi^* \zeta_k + \frac{1}{n}\zeta_L =$$

$$X(\bar{\mu}^\sigma) - \frac{1}{n} \sum_k \sigma_i^k \zeta_k + \frac{1}{n}\zeta_L - \frac{n-1}{n} \sum_k \phi^* \zeta_k >$$

$$X(\bar{\mu}^\sigma) - \frac{1}{n}(\zeta_R - \zeta_L) - m\phi^* \zeta_R \geq \theta_i + \varepsilon - \frac{1}{n} - m\phi^*.$$

Hence this step of the proof is concluded by noticing that δ_ε^* is by definition less than $\frac{1}{2}$, hence[15]

$$\theta_i + \varepsilon - \frac{1}{n} - m\phi^* > \theta_i + \varepsilon - \frac{2\phi^{*2}}{m} - m\phi^* =$$

$$\theta_i + \varepsilon - \frac{(20 - 8\sqrt{6})\varepsilon^2}{m^3} - \varepsilon\left(-2 + \sqrt{6}\right) \geq \theta_i + \varepsilon(1 - \frac{(20 - 8\sqrt{6})}{8} + 2 - \sqrt{6}) =$$

[15] In the following we assume that $\varepsilon \leq 1$, since otherwise the proposition is trivially true.

$$\theta_i + \frac{1}{2}\varepsilon.$$

By Lemma 5, we know that, for $n > \frac{m}{4\phi^{*2}\delta^*_\varepsilon} + 1$,

$$\Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*)M^*_n(\vec{\phi}^*, \theta_i) + (1 - \Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*))M <$$

$$(1-\delta^*_\varepsilon)M^*_\varepsilon + \delta^*_\varepsilon M = (1 - \frac{-M^*_\varepsilon}{M - M^*_\varepsilon})M^*_\varepsilon + \frac{-M^*_\varepsilon}{M - M^*_\varepsilon}M = 0.$$

Summarizing, we have proved that for $n > \frac{m}{4\phi^{*2}\delta^*_\varepsilon} + 1$, for every strategy $c \neq L$

$$\sum_{v_n^{-i} \in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i}) \frac{\left[u\left(X\left(v_n^{-i}, c\right), \theta_i\right) - u\left(X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right]}{\frac{1}{n}(\zeta_c - \zeta_L)} \leq$$

$$\Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*)M^*_n(\vec{\phi}^*, \theta_i) + (1 - \Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*))M <$$

$$(1-\delta^*_\varepsilon)M^*_\varepsilon + \delta^*_\varepsilon M = 0,$$

which implies that c is not a best reply for player $i \in N^\rho$.

(β) A symmetric argument holds.

References

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