

Generic Efficiency and Collusion-Proofness in Exchange Economies

Gaël Giraud
BETA, Université Louis Pasteur

Céline Rochon*
CORE, Université Catholique de Louvain

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Abstract

We define a new strategic equilibrium concept – called *strong collusion-proof contract* – designed to characterize stable communication agreements in games with differential information against non-binding, self-enforcing and incentive compatible deviations by coalitions. We then construct a strategic market mechanism which, for quasi-linear economies, is such that its strong collusion-proof contracts generically induce the incentive compatible *interim* efficient allocations. Moreover, it is such that these allocations can be achieved by strong collusion-proof contracts. We show that the internally consistent extension of the strong collusion-proof contracts generically yields the same set of efficient allocations.

Keywords: Coalition-proofness, Bayesian implementation, Communication equilibrium, *Interim* efficiency.

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1 Introduction

The objective of implementation theory, as stated for example in Jackson [18], is to characterize the set of social choice correspondences that are obtainable as equilibrium outcomes when individuals interact through some game-form making strategic use of their knowledge. Specifically, given a social choice correspondence F which maps profiles of characteristics to allocations, one asks whether there exists a game-form G such that for any profile R of characteristics, the players' outcome from the game is precisely $F(R)$.

Under incomplete information and when the map F is, say, some correspondence of Pareto efficient allocations, this problem is not at all an easy one. Many impossibility results have been shown, and among others, Myerson and Satterthwaite [24] and Hurwicz [17]. Moreover, in order to properly study the strategic behavior of agents in an environment with more than two agents, one needs to consider the formation of coalitions and the possibilities of communication between the agents. More specifically, one should allow coalitions of players to communicate prior to actual play, and make non-binding agreements on information sharing and strategy choice. The solution concepts applied to a given game must reflect these concerns.

In this paper, we present a new equilibrium concept, that we shall call a strong collusion-proof contract, which is inspired by Aumann's strong Nash equilibrium concept [5]. Indeed, Aumann's solution concept was made for complete information games and took *any* coalitional deviation into account. In an asymmetric information framework, a new issue arises, which could not be addressed by the strong Nash concept, namely whether the members of a deviating coalition do have any incentive to share their information.

A coalitional deviation resting on some information pooling although individuals would rather not disclose their information makes hardly any sense. Thus, our opinion is that, when extending Aumann's equilibrium notion to a Bayesian setup, one should require that any sensible coalitional deviation be such that any member of the coalition has some incentive to transmit the information on which the deviation is based. Observe that the same type of problem occurs when defining the core with incomplete information. Two approaches can be used in order to define the core with incomplete information. The first one does not consider incentive constraints: the mechanism allowing for the transmission of information between the players is taken as given (Allen[2], Koutsougeras and Yannelis [19]). Alternatively, Vohra [29] and Forges, Mertens and Vohra [12] consider explicitly incentive problems. They show that in a differential information economy with quasi-linear utilities, the core may be empty. Given this negative result, this work can be seen as a first step towards an alternative theory of collusion-proof efficiency with incomplete information. The main difference between Forges, Mertens and Vohra [12] and our work is that we here adopt a strictly non-cooperative framework and focus on *interim* efficiency while they take a cooperative point of view but allow coalitions to form *ex ante*. In their model, the endowments of the agents are independent of the types, while

we consider type-dependent endowments. Note, in addition, that the quasi-linear set-up with private values under which the incentive compatible core is shown in [12] to be non-empty is a particular case of the framework to which we aim at applying our concept of collusion-proofness.

From our point of view, the correct extension of Aumann’s concept to incomplete information economies is the strong Bayesian equilibrium (see Appendix). However, this concept does not allow to get implementation results, as a counterexample suggests. Therefore, the main goal of this paper is to define an alternative concept, the “strong collusion-proof contract”, as close as possible to the strong Bayesian equilibrium and allowing for mechanism design results. It turns out that when applied to 2-player games, we get back the equilibrium concept presented in Laffont and Martimort [20].

We also present a consistent version of the strong collusion-proof contract, the collusion-proof contract, which bears the same relationship with respect to the strong collusion-proof notion as does the coalition-proof Nash equilibrium with respect to Aumann’s strong Nash equilibrium. The collusion-proof and strong collusion-proof contracts are defined in such a way that they do not allow simultaneous deviations in types and actions, as is the case for strong Bayesian equilibrium. Once the types are announced and subsequently actions are recommended to the agents by the device, then a coalition can form to discuss actions through a new device, but at this stage, the coalition considers the revealed types as given. On the other hand, at each coalitional deviation stage, types precede actions, the latter being fixed once types are strategically revealed by the players. Within a stage of the game, players are myopic.

In this paper, we present a strategic market mechanism which is such that generically (for a dense set of utility functions), almost every incentive compatible *interim* efficient allocation can be achieved with a strong collusion-proof contract. Moreover, generically, every strong collusion-proof contract induces an outcome which is incentive compatible *interim* efficient. In addition, these results hold for collusion-proof contracts, as we define them here.

In order to understand the logic underlying the various notions of collusion-proofness used in this paper and in preceding ones, it may help the reader to think in terms of Bernheim *et al*’s “room” analogy [7] (p.5): actual payoff-relevant play occurs simultaneously in one shot, but the collusion-proofness of the strategies played is “rationalized” *via* an extensive-form involving coordinated defections discussed within coalitions, and further (individual or even coalitional) defections from such defections. Imagine that players meet in a room before actually playing the game but after they have been privately informed of their respective type, and subsequently leave the room in an arbitrary order after a non-binding agreement on strategies is reached. The agreement involves some communication device with the help of which players can strategically share (part of) their private information, as well as some transfer of numéraire. When a player leaves the room, the remaining players assume he will play as agreed. Each player, regardless of the order of exit, is concerned

that the remaining players in the room may reach a new agreement either by redistributing some numéraire among themselves, or by using some new communication device in order to share some more information. Note that, now that they have already shared part of their initially privately held information, the remaining players in the room may want to share what they have learned from the previous communication device (specifically the actions they have been told to perform by the communication device). As a consequence, even if the initial economy is with complete information, the agents remaining in the room truly play a game with incomplete information: even if their types are common knowledge, they can gain by sharing the information embodied in the action recommended to them by the previous correlation device. However, each player outside from the room also knows that any such further agreement is also suspect for three reasons: a) first, some players in the room may not have any incentive to tell the truth to the side-communication mechanism they are secretly trying to construct; b) second, some players in the room may not want to obey the recommendation made by the side-mechanism, but rather play the action originally recommended by the grand mechanism, or even a third one; c) eventually, even if the new agreement is incentive-compatible, it remains suspect because the next player to leave the room will have exactly the same concerns. The notion of collusion-proofness captures the idea of an agreement among the members of the grand coalition such that, regardless of the order of exit, the remaining players in the room never have any incentive to form a new agreement. Needless to say, given its internally consistent flavour, this concept is quite involved. Nonetheless, we show that our market mechanism does provide a generic, full mechanism design result in terms of collusion-proof contracts. By its very definition, the strong collusion-proof contract notion has the advantage of being easier to work with than the collusion-proof contract concept, as it neglects point c) *supra*. Of course, as a consequence, the strong collusion-proof contract is *not* a consistent equilibrium concept, but it is stronger than the collusion-proof contract notion (since it is easier for a coalition to build a side-communication mechanism inducing an improvement). On the other hand, it will turn out that, for the application we have in mind, both notions induce the same set of final outcomes.

1.1 Links with the literature

The comparison between our approach and related works available in the literature may help the reader to understand the contribution of this paper.

In a companion paper [13], we consider the case of an economy with complete information. Using essentially the same mechanism and the coalition-proof correlated equilibrium concept of Ray [28], we get an efficient implementation result.

In Ray [28], deviations by coalitions take place *ex ante*, before the players learn their actions recommended by the correlation device, and a coalition that deviates cannot construct a new correlation device. In contrast with Ray,

the deviations of coalitions in Einy and Peleg [10] are introduced after the concerned players are informed of the actions they should follow through the (grand) communication mechanism, and deviating coalitions are allowed to construct side-mechanisms for their own purposes.

The timing of our game is the following:

- 1- Nature draws the types of the players.
- 2- The principal proposes a grand communication device with transfers.
- 3- Players make reports into the communication device.
- 4- The communication device recommends actions privately to the players.
- 5- Players revise their beliefs in a strategic way.
- 6- A coalition S may form and create a new communication device.
- 7- Members of S transmit reports into the side communication device.
- 8- The new device recommends actions to the players in S .
- 9- Players in S revise their beliefs in a Bayesian way.
- 10- The original game is played.

Observe that between steps 9 and 10, nothing prevents a subcoalition of S to build a new communication device before playing the game.

In their pathbreaking paper, Laffont and Martimort [20] analyze a mechanism design problem in which the agents collude under asymmetric information, and which is closely related to this paper. The authors study a regulatory model of a duopoly where each firm has private information on its costs, unknown to the regulator.

The timing of the game in [20] is the following:

- 1- Nature draws types $t = (t_1, t_2)$ and each agent i learns his type t_i .
- 2- The Principal proposes a grand mechanism.
- 3- The agents accept or refuse the grand mechanism. If one of them refuses, they both get their reservation utility.
- 4- A third party offers a side contract.
- 5- The agents accept or refuse the side contract. If one of them refuses, the grand contract is played noncooperatively. Agents report their type to the grand mechanism and the next stages do not occur.
- 6- If the side contract is accepted, the agents transmit reports in the side contract.

- 7- The side contract manipulates transfers and payoffs in the grand mechanism.
- 8- The grand mechanism operates.

Their paper sheds invaluable light on collusion-proofness. The main differences between our paper and theirs are the following:

- Laffont and Martimort present a partial equilibrium model with two agents, two types per agent and quasi-linear utilities. In this paper, we consider a general equilibrium pure exchange economy model à la Arrow-Debreu, with any number of states of nature and of agents. However, as in Laffont and Martimort, we restrict ourselves to quasi-linear utilities. We need such a restriction for the same reasons as Forges, Mertens and Vohra [12], i.e., in order to use the incentive mechanism of d’Aspremont and Gérard-Varet [4].
- To extend a model such as theirs to more than two players — which was the primary goal of our inquiry — one needs a consistent solution concept, namely one that can be applied to all possible subcoalitions. Consider a game with 3 players: once a side contract for the three players is fixed, nothing precludes two of the three players to construct for themselves a new side contract. Hence a notion of coalition-proofness is required. In this paper, we use a variant of the coalition-proof equilibrium concept of Bernheim, Peleg and Whinston [7], with at least 3 agents.
- In [20], the agents do not revise their beliefs as the game is played. The authors implicitly consider an extensive-form game with trivial observation. Passive beliefs are justified by the use of sufficient conditions on the fundamentals of the economy that allow Bayesian implementation to be replaced by implementation in dominant strategies (Mookherjee and Reichelstein [23]). In contrast, in this paper the agents are able to revise their beliefs.
- All implementation criteria used by the authors are partial. For example, in the case of implementation in dominant strategies without collusion, they do not require that truthtelling be the only dominant strategy but that it be one. Implementation where truthtelling is the only equilibrium strategy is much more demanding. In this paper, we seek for “full implementation”¹ results.
- In [20], the agents do not share any information between them. As a consequence, they cannot pool any information. To take an analogy, the third party acts as a mediator in a correlated equilibrium while the appropriate

¹We are aware that we are not presenting implementation results in the strict sense of the word. The use of this term is to be understood in a vague sense.

concept under incomplete information is that of a communication equilibrium (Forges [11]). In this paper, we allow for communication among agents, but agents do not need to share their information. Information sharing is endogenous and submitted to standard incentive compatibility constraints. Agents share information only if they have an incentive to do so.

- Laffont and Martimort do not allow the agents to randomize over their strategies. In particular, their manipulation and transfer functions are all deterministic. Here we allow for any kind of correlation and randomization in the manipulation functions. (However, we keep deterministic transfer functions; this is without loss of any generality since players' utilities are quasi-linear.)
- The price to pay for our generality is that we only get a generic result.

Finally, our study is in some sense linked to the rational expectations equilibrium (REE) literature, initiated by Radner [27]. Indeed, in the scenario on which this paper is based, prices reveal some information that players use in order to refine their private information. Moreover, generically, we show that the equilibria are fully revealing. However, the classical paradoxes *à la* Grossman and Stiglitz do not occur here: prices are quoted by the players, and hence cannot transmit an information not already known to at least one agent. In this sense, our approach is closer to that of Dubey, Geanakoplos and Shubik [8], although these authors consider a continuum of players whereas we deal with a finite number of players. Moreover, we consider collusion-proof contracts while they restrict themselves to Bayesian equilibria.

The paper is organized as follows. In section 2, the economy and the game-form are presented. The solution concepts are defined in section 3. The last section is devoted to our mechanism design result.

2 The model

2.1 The economy

Consider a pure exchange economy $\mathcal{E} = (X_i, u_i, \omega_i)_{i \in I}$ with N agents $i \in I = \{1, \dots, N\}$ and L commodities $h \in \{1, \dots, L\}$. Commodity L serves as a numéraire whose price p^L is normalized to 1. Let T_i denote the finite set of types of player $i \in I$ and $T = \prod_{i \in I} T_i$. Nature draws at random a profile of types $t = (t_i)_i$ according to a probability $Q \in \Delta(T)$ with $Q(t) > 0$ for all $t \in T$.

Given t_i , we denote by $\omega_i(t_i) \in \mathbb{R}_+^L$ agent i 's initial endowment when he is of type t_i .² The aggregate endowment is assumed independent of $t = (t_i)_i$, and

²If ω_i was defined as a function of T and not constant with respect to t_i , then i would deduce, by observation of ω_i , some information about the type of the other players.

is denoted $\bar{\omega} = \sum_{i \in I} \omega_i(t_i)$. For each agent i , we assume that $X_i = \mathbb{R}_+^L$ is his consumption set, independently of his type. Let

$$Y_i := \left\{ x \in \mathbb{R}^L : -\varepsilon < x^h < \varepsilon + \sum_{k=1}^N \omega_k^h, 1 \leq h \leq L \right\}.$$

(ε is a positive number and superscripts on a vector give its component.) Let $Y = \prod_{i \in I} Y_i$. For any *ex post* feasible net trades in commodities z_1, \dots, z_N (see Definition 1 below), we have $\omega_i(t_i) + z_i \in Y_i$. The preferences of agent i are represented by a quasi-linear utility function $u_i : Y_i \times T_i \rightarrow \mathbb{R}$, defined for all $x \in Y_i$ and all $t_i \in T_i$ by $u_i(x, t_i) = v_i(x^1, \dots, x^{L-1}, t_i) + x^L$. Note that for every $t_i \in T_i$, the function $v_i(\cdot, t_i)$ is defined on

$$\Psi_i := \left\{ x \in \mathbb{R}^{L-1} : -\varepsilon < x^h < \varepsilon + \sum_{k=1}^N \omega_k^h, 1 \leq h \leq L-1 \right\}.$$

Assumption 1

- $N \geq 3$ and $L \geq 2$.
- For all $i \in I$ and all $t_i \in T_i$, $v_i(\cdot, t_i)$ is C^k for k sufficiently large, the restriction to Ψ_i of the Hessian $H(v_i(\cdot, t_i))$ is a negative definite quadratic form, the restriction to Ψ_i of the gradient $Dv_i(\cdot, t_i)$ satisfies $Dv_i(\cdot, t_i) \gg 0$.
- For all i and for all t_i , $\omega_i(t_i) > 0$.³

In the next definition, we recall the efficiency and incentive compatibility concepts presented in Holmström and Myerson [15].

Definition 1

- An allocation $x : T \rightarrow (\mathbb{R}^L)^N$ is *ex post efficient* if it is *ex post* feasible, namely, $\sum_{i \in I} x_i(t) = \sum_{i \in I} \omega_i(t_i)$ for all $t = (t_i)_i \in T$, and there does not exist another *ex post* feasible allocation x' such that for all $i \in I$ and all $t \in T$, $u_i(x'_i(t), t_i) \geq u_i(x_i(t), t_i)$ with strict inequality for some pair (i, t) .
- An allocation $x : T \rightarrow (\mathbb{R}^L)^N$ is *interim efficient* if it is *ex post* feasible, and there does not exist another *ex post* feasible allocation x' such that for all $i \in I$ and all $t_i \in T_i$, $U_i(x'_i|t_i) \geq U_i(x_i|t_i)$ with strict inequality for some pair (i, t_i) where

$$U_i(x_i|t_i) := \sum_{t_{-i} \in T_{-i}} Q(t_{-i}|t_i) u_i(x_i(t_i, t_{-i}), t_i).$$

³ $\omega_i(t_i) \geq 0$ and $\omega_i(t_i) \neq 0$

- An allocation $x : T \rightarrow (\mathbb{R}^L)^N$ is *incentive compatible* if and only if

$$\sum_{t_{-i} \in T_{-i}} Q(t_{-i}|t_i) u_i(x_i(t_{-i}, t_i), t_i) \geq \sum_{t_{-i} \in T_{-i}} Q(t_{-i}|t_i) u_i(x_i(t_{-i}, \hat{t}_i), t_i)$$

for all i , for all $t_i \in T_i$, for all $\hat{t}_i \in T_i$.

We require an allocation to be *ex post* feasible. Observe that contrary to Prescott and Townsend [25] or Allen [3], but following Vohra [29] among others, we do not content ourselves with feasibility in expected terms.

2.2 The game-form

A *game-form* G is defined by $G = (A_i, \phi_i)_{i=1, \dots, N}$ where for all $i \in I$, A_i denotes the set of pure actions of player i , while $\phi_i : A = \prod_{h \in I} A_h \rightarrow \mathbb{R}_+^L$ denotes the strategic outcome function of player i .

A_i will be defined as the set of prices proposed by player i , $p_i \in \mathbb{R}_+^L$ such that $p_i^L = 1$, the vectors of net demands addressed by player i to player $j \neq i$, $z_i = (z_j^i)_{j \neq i} \in (\mathbb{R}^L)^{N-1}$, and the announced initial endowment $e_i \in (0, \omega_i(t_i))$:

$$A_i = \left\{ (p_i, z_i, e_i) \in \mathbb{R}_+^L \times (\mathbb{R}^L)^{N-1} \times (0, \omega_i(t_i)) : p_i^L = 1 \right\}.$$

The *strategic budget constraint* of player i is

$$\sum_{j \neq i} p_j (z_j^i - z_i^j) \leq 0. \quad (1)$$

Note that this constraint does not depend on the price quoted by player i . Let $\psi_i : A \rightarrow \mathbb{R}^L$ be the *auxiliary function* defined by

$$\psi_i(a) = \begin{cases} \sum_{j \neq i} (z_j^i - z_i^j) & \text{if (1) is satisfied} \\ -e_i & \text{otherwise} \end{cases}$$

We can now define the *strategic outcome function* of player i , $\phi_i : A \rightarrow \mathbb{R}_+^L$:

$$\phi_i(a) = \text{proj}_i \circ \text{proj}_F ((e_k + \psi_k(a))_{k=1, \dots, N})$$

where

$$F = \left\{ (x_i)_i \in \prod_{i \in I} X_i : \sum_i x_i \leq \sum_i e_i \right\}$$

is the *feasible set*, which is convex, compact and nonempty;

$$\text{proj}_F : (\mathbb{R}^L)^N \rightarrow F$$

is the orthogonal projection over F and

$$\text{proj}_i : (\mathbb{R}^L)^N \rightarrow \mathbb{R}^L$$

is the canonical projection on the i th coordinate. For all $x \in (\mathbb{R}^L)^N$, $proj_F(x)$ can be decomposed into two projections: first a projection on $\bar{\omega} - \mathbb{R}_+^L$ and then a projection on \mathbb{R}_+^L . As a consequence, if $\psi_i(a) + e_i \leq 0$, then $\phi_i^h(a) = 0$ for all $h = 1, \dots, L$.

The *payoff function* of player i with respect to the actions when he is of type t_i , $g_i(\cdot, t_i) : A \rightarrow \mathbb{R}$, is defined by $g_i(a, t_i) = u_i(\phi_i(a), t_i)$.

Remarks

- The mechanism is feasible, in the sense that any strategy profile induces an outcome such that demand does not exceed supply (given by the initial endowments) and the agents receive a bundle which belongs to their consumption set. Moreover, the mechanism is not balanced out of equilibrium if some player goes bankrupt. In this way, part of the initial resources are wasted. Finally, it is not individually rational out of equilibrium since if someone is kept in the red, he necessarily gets zero.
- We assume the planner knows that $\omega_i(t_i) \neq 0$ for all i . Moreover, it is assumed that the players cannot overstate their true endowments (implicitly, since they have to show physically their endowments). In addition, if player i does not announce the exact value of his initial endowments, then the difference $\omega_i(t_i) - e_i$ is wasted.⁴
- In equilibrium, through the use of proper transfers [4], we provide incentives for every player to reveal his true type and hence his true initial endowments. Consequently, the feasible set used to define the strategic outcome function of our game is equal to the true feasible set, defined with the true endowments.

2.3 The extensive-form game

The game-form G is played in extensive-form, as follows. Players first simultaneously quote prices, $p_i \in \mathbb{R}_+^L$, $i = 1, \dots, N$. Once the outcome of this first stage is publicly observed, the players simultaneously set quantities $(z_i)_i \in ((\mathbb{R}^L)^{N-1})^N$ and make an announcement about their initial endowments.

We assume perfect recall. Let B_i be the set of pure behavioral strategies of player i :

$$B_i = \left\{ b_i = (p_i, Z_i, E_i) : p_i \in \mathbb{R}_+^L, (Z_i, E_i) : (\mathbb{R}_+^L)^N \rightarrow (\mathbb{R}^L)^{N-1} \times (0, \omega_i(t_i)] \right\}$$

where for each player i , the function (Z_i, E_i) determines the action of agent i at the second stage once the outcome of the first stage is observed. Let $B = \prod_{i \in I} B_i$.

⁴As soon as strategic manipulation of endowments is taken into account, two possibilities arise: either $\omega_i - e_i$ is wasted, as in this paper or in Hong [16], or the difference is consumed by player i in addition to his final allocation [26].

The *payoff function* of player i of type t_i , $\tilde{g}_i(\cdot, t_i) : B \rightarrow \mathbb{R}$, is defined by $\tilde{g}_i(b, t_i) := g_i(a, t_i)$, where $a \in A$ is the unique action induced by b (namely, $a = (p_i, z_i, e_i)_{i \in I}$ where $z_i = Z_i((p_j)_{j \in I})$ and $e_i = E_i((p_j)_{j \in I})$ for all i). Similarly, define the outcome function $\tilde{\phi}_i : B \rightarrow \mathbb{R}^L$ by $\tilde{\phi}_i(b) := \phi_i(a)$ and the auxiliary function $\tilde{\psi}_i : B \rightarrow \mathbb{R}^L$ by $\tilde{\psi}_i(b) := \psi_i(a)$ for $a \in A$ the unique action induced by b .

In the sequel, we deal with actions instead of behavioral strategies, keeping in mind the correspondence between the two.

3 Collusion-proofness

In this section, we present the strategic solution concepts we shall use to get our mechanism design result. We introduce the notions of collusion-proof and strong collusion-proof contracts, by building on the construction of Einy and Peleg [10].

In the spirit of Laffont and Martimort [20], the players use contracts. In contrast with Einy and Peleg, a contract involves a communication device and some balanced transfers, namely some reallocations of the initial endowments, while Einy and Peleg [10] only allow for communication mechanisms. Within a coalition aiming to deviate from a grand contract, new transfers can be introduced, as long as they balance the transfers received from that grand contract. The possibility of transfers is consistent with Laffont and Martimort [20]. One important difference between our framework and theirs is that we allow coalitions to share the private information held by its members through communication devices. Indeed, in [20] no communication devices are allowed.

We restrict ourselves to direct communication devices, following Einy and Peleg. Hence the report to a communication device consists in telling one's type. Observe that this restriction aims at saving tedious notations and that the whole analysis could be carried out without imposing such a restriction, hence allowing for abstract message spaces.

The next definition captures the strategic environment in which a coalitional deviation may occur. The story behind the definition is the following: suppose that the central planner has proposed a grand contract for all the players. For reasons that should become clear in a moment, let us denote by H the set of all the players. Imagine that a coalition $I_m \subset H$ deviates. The reason for which we denote by H , in this definition, the set of players is that the definition aims at characterizing the following more complicated scenario: suppose a subcoalition $I_{m'} \subset I_m$ also deviates. Then by construction $I_{m'}$ will face the strategic environment obtained by replacing H by I_m , and I_m by $I_{m'}$. This definition provides a recursive description of the strategic environment faced by a deviating coalition.

Definition 2 An *extended Bayesian game with transfers* is a system

$$\Gamma_y = (A_1, \dots, A_m; T_1, \dots, T_m, T_{m+1}, \dots, T_n; q_1, \dots, q_m; u_1^y, \dots, u_m^y)$$

where

- $I_m = \{1, \dots, m\}$ is the set of “inside” players,
- $J = \{m + 1, \dots, n\}$ is the set of “outside” players,
- A_i is the set of actions of $i \in I_m$,
- T_i is the set of possible types of $i \in H = I_m \cup J$,
- $q_i \in \Delta(T_H)$ is i 's prior, where $T_H = \prod_{i \in H} T_i$,
- For all $i \in I_m$, $y_i : T_H \rightarrow \mathbb{R}$ denotes the type-contingent transfer to player i , and $y = (y_i)_{i \in I_m}$,
- Γ_y is the modified game obtained from G by replacing $\psi_i(\cdot)$ for all $i \in I_m$ with $\psi_i^y(\cdot)$ defined for all $t \in T_H$ and $a \in A_{I_m} = \prod_{i \in I_m} A_i$ by

$$\psi_i^{y(t)}(a) = \begin{cases} \sum_{j \neq i} (z_j^i - z_j^j) & \text{if } \sum_{j \neq i} p_j (z_j^i - z_j^j) \leq y_i(t) \\ -e_i & \text{otherwise} \end{cases}$$

- The corresponding outcome function is defined for all $t \in T_H$ and $a \in A_{I_m}$ by

$$\phi_i^{y(t)}(a) = \text{proj}_i \circ \text{proj}_F((e_k + \psi_k^{y(t)}(a))_{k=1, \dots, N}),$$

- $u_i^y : A_{I_m} \times T_H \rightarrow \mathbb{R}$ is the payoff function of $i \in I_m$, defined by $u_i^y(a, t) := u_i(\phi_i^{y(t)}(a), t_i)$.

In the sequel, the set of “inside” players will be interpreted as the set of players belonging to a deviating coalition while the set of “outside” players will be interpreted as the set of players exterior to the deviating coalition. To say that coalition I_m deviates means that members of I_m build a new communication device in order to share information and to reallocate income.

Definition 3 Given an extended game Γ_y , a *communication contract* is a pair (μ, \tilde{y}) where $\mu : A_{I_m} \times T_{I_m} \rightarrow [0, 1]$ is such that $\sum_{a \in A_{I_m}} \mu(a|t_{I_m}) = 1$ for every $t_{I_m} \in T_{I_m} = \prod_{i \in I_m} T_i$, and $\tilde{y} : T_H \rightarrow (\mathbb{R}^L)^N$ denotes transfers, satisfying $\sum_{i \in I_m} \tilde{y}_i(t) = \sum_{i \in I_m} y_i(t)$ for all $t \in T_H$.

Remarks

- Throughout, all the communication devices μ will be assumed to have *finite* support.
- The last condition on transfers \tilde{y} captures the idea that the coalition I_m can at most redistribute among its members transfers up to the value of transfers induced on I_m by $y = (y_i)_{i \in I_m}$.

- The device μ works as follows: players in I_m simultaneously report their types $(t_i)_{i \in I_m} = t_{I_m}$ to μ ; then μ chooses an action-tuple $a \in A_{I_m}$ according to the probability $\mu(\cdot | t_{I_m})$.
- For all $i \in I$, if $Q = q_i$, $J = \emptyset$ and $y_i = 0$, then $\Gamma_y = G$: the extended Bayesian game with transfers is nothing but the original game played by the players.

Let $T_i^+ = \{t_i \in T_i : q_i(t_i) > 0\}$ for $i \in I_m$ with $q_i(t_i) := \sum_{\tau_{-i} \in T_{-i}} q_i(\tau_{-i}, t_i)$. Let (μ, \tilde{y}) be a communication contract, $i \in I_m$ and $t_i \in T_i^+$. The *expected utility* of t_i is

$$U_i^{\tilde{y}}(\mu | t_i) = \sum_{t_{-i} \in T_{-i}} q_i(t_{-i} | t_i) \sum_{a \in A_{I_m}} \mu(a | t_{I_m}) u_i^{\tilde{y}}(a, t).$$

Definition 4 A contract (μ, \tilde{y}) is a *Bayesian incentive compatible contract with transfers* (BIC) if for every $i \in I_m$, $t_i \in T_i^+$ and $s_i \in T_i$,

$$U_i^{\tilde{y}}(\mu | t_i) \geq \sum_{t_{-i} \in T_{-i}} q_i(t_{-i} | t_i) \sum_{a \in A_{I_m}} \mu(a | (t_{I_m} \setminus \{i\}, s_i)) u_i^{\tilde{y}}(a, t).$$

In other words, at a BIC, no individual player has an incentive to misreport his type. Notice that the incentive compatibility requirement embodied into this definition is responsible for our restriction to quasi-linear utilities. Indeed, the tool we shall use in order to provide such incentives is the mechanism of d'Aspremont and Gérard-Varet [4].

Since we want to build a consistent collusion-proof concept, we need to be able to consider a coalitional deviation by some subcoalition $S \subset I_m$. Recall that this deviation by S can only occur after players in I_m learn their type and the action recommended to them by the communication mechanism they have built for themselves. We first define a revised game.

Definition 5 Let Γ_y be an extended Bayesian game, (μ, \tilde{y}) a communication contract and $\emptyset \neq S \subset I_m$. The *revised game* $\Gamma_{\mu, \tilde{y}, S}$ is the extended Bayesian game with transfers

$$((A_i)_{i \in S}; T_1 \times A_1, \dots, T_m \times A_m, T_{m+1}, \dots, T_n; (\hat{q}_i)_{i \in S}; (\hat{u}_i^{\tilde{y}})_{i \in S}),$$

where

$$\hat{q}_i(t_1, a_1, \dots, t_m, a_m, t_{m+1}, \dots, t_n) = \hat{q}_i(t, a) = q_i(t) \mu(a | t_{I_m})$$

for all $i \in S$, $t \in T_H$ and $a \in A_{I_m}$, and

$$\hat{u}_i^{\tilde{y}}((b_j)_{j \in S}, (t, a)) = \hat{u}_i^{\tilde{y}}(b_S, (t, a)) = u_i^{\tilde{y}}((b_S, a_{I_m \setminus S}), t)$$

for all $i \in S$, $t \in T_H$, $b_S \in A_S$, $a \in A_{I_m}$.

The probability that $a_i \in A_i$ is played by $i \in I_m$ of type $t_i \in T_i^+$ is

$$\pi_i(a_i|t_i) = \sum_{t_{-i} \in T_{-i}} q_i(t_{-i}|t_i) \sum_{a_{-i} \in A_{-i}} \mu((a_{-i}, a_i)|t_{I_m}).$$

Let

$$(T_i \times A_i)^+ = \{(t_i, a_i) : t_i \in T_i^+, a_i \in A_i, \pi_i(a_i|t_i) > 0\}.$$

If $i \in I_m$, $t_i \in T_i^+$, $a_i \in A_i$, and $\pi_i(a_i|t_i) > 0$, then the expected payoff for t_i when playing a_i is

$$U_i^{\bar{y}}(\mu|(t_i, a_i)) = \sum_{t_{-i} \in T_{-i}} q_i(t_{-i}|t_i) \sum_{a_{-i} \in A_{-i}} \mu((a_{-i}, a_i)|t_{I_m}) \frac{u_i^{\bar{y}}(a, t)}{\pi_i(a_i|t_i)}.$$

Let $i \in S$, $(t_i, a_i) \in (T_i, A_i)^+$, $t_{-i} \in T_{-i}$, $a_{-i} \in A_{-i}$. Then

$$\hat{q}_i((t_{-i}, a_{-i})|(t_i, a_i)) = \frac{q_i(t_{-i}|t_i)\mu(a|t_{I_m})}{\pi_i(a_i|t_i)}.$$

If (η, y') is a communication contract for $\Gamma_{\mu, \bar{y}, S}$, $i \in S$, $(t_i, a_i) \in (T_i, A_i)^+$, then the expected utility of i knowing (t_i, a_i) is

$$U_i^{y'}(\eta|(t_i, a_i)) = \sum_{t_{-i} \in T_{-i}} \sum_{a_{-i} \in A_{-i}} \hat{q}_i((t_{-i}, a_{-i})|(t_i, a_i)) \sum_{b_S \in A_S} \eta(b_S|(t_S, a_S)) \hat{u}_i^{y'}(b_S, (t, a)).$$

Our notion of collusion-proofness requires that no coalition has any profitable deviation. We now define a coalitional improvement.

Definition 6 The strategy (η, y') is a BIC-improvement of $\emptyset \neq S \subset I_m$ upon (μ, \bar{y}) in Γ_y if

- a) (η, y') is a BIC for the revised game $\Gamma_{\mu, \bar{y}, S}$,
- b) for every $i \in S$ and $(t_i, a_i) \in (T_i, A_i)^+$ then

$$U_i^{y'}(\eta|(t_i, a_i)) \geq U_i^{\bar{y}}(\mu|(t_i, a_i)),$$

with at least one strict inequality.

In words, we restrict attention to coalitional deviations where members of the deviating coalition have some incentive to deviate (b), and some incentive to share their information with the other members of the deviating coalition (a). In the next definition, we focus on a smaller class of coalitional deviations, namely those which cannot be improved upon by any other incentive compatible and profitable deviation of any subcoalition.

Definition 7 Let Γ_y be an extended Bayesian game, (μ, \tilde{y}) a communication contract and $S \subset I_m$, $S \neq \emptyset$. A BIC *internally consistent improvement* (ICI) of S upon (μ, \tilde{y}) in Γ_y is defined by induction on the cardinality of S , as follows:

- If $|S| = 1$, then (η, y') is a BIC-ICI if it is a BIC-improvement of S upon (μ, \tilde{y}) . Note that in this case $\tilde{y} = y'$.
- If $|S| > 1$, then (η, y') is a BIC-ICI of S upon (μ, \tilde{y}) in Γ_y if
 - (η, y') is a BIC-improvement of S upon (μ, \tilde{y}) in Γ_y
 - if $\tilde{S} \subset S$ and $\tilde{S} \neq \emptyset$, then \tilde{S} has no BIC-ICI upon (η, y') in the game $\Gamma_{\mu, \tilde{y}, S}$.

Definition 8 Let Γ_y be an extended Bayesian game.

- A BIC (μ, \tilde{y}) is a *collusion-proof contract* if no coalition of players has a BIC-ICI upon (μ, \tilde{y}) .
- A BIC (μ, \tilde{y}) is a *strong collusion-proof contract* if no coalition of players has a BIC-improvement upon (μ, \tilde{y}) .

Remark A deviating coalition S deviates *ex post* (i.e. after having played in (μ, \tilde{y})) using (η, y') and does not modify the information given to the grand mechanism μ . Members of S can share information among themselves, thanks to η , and eventually play a strategy different from that recommended by the grand mechanism μ . In addition, thanks to transfers y' , the members of S can redistribute the transfers induced by \tilde{y} . Members outside of S do not modify their behavior.

4 Efficiency

In this section, we show that the collusion-proof and strong collusion-proof contracts of our strategic market mechanism induce efficient allocations and that efficient allocations can be achieved by such contracts.

Remarks

- A solution concept is fully revealing if and only if once the players are informed of their private types, they put a Dirac measure on the space of types.
- The space of utility functions is a topological space with the topology induced by the following norm:

$$\|u\| = \sup_D \sup_x |Du(x)|$$

where Du ranges over all derivatives of u of order $0, 1, \dots, k$ for k sufficiently large (Dubey and Rogawski [9]). The term “generically” means here “for a dense set for this topology”.

- The space of *interim* efficient allocations is equipped with the Euclidean topology and the Lebesgue measure, since it is a finite dimensional space.

Before stating the main theorem, we need to introduce a few more assumptions.

Outcome efficiency requirement For all $t \in T$, the outcome induced by a mechanism μ must belong to the set

$$\left\{ x^* = (x_i^*) : \sum_{i \in I} v_i(x_i^*, t_i) = \max_{x=(x_i)} \sum_{i \in I} v_i(x_i, t_i) \right\}.$$

Compatibility condition Let

$$C_i := \{(m_i, t_i) \in T_i \times T_i : m_i \neq t_i\}$$

and

$$\Lambda := \left\{ \lambda = (\lambda_i)_{i \in I} : \lambda_i \in \mathbb{R}_+^{C_i} \right\}.$$

For all $\kappa \in \mathbb{R}^T$, if $\kappa \neq 0$ then there is no $\lambda \in \Lambda$ such that for all $i \in I$, for all $t \in T$,

$$q_i(t_{-i}|t_i) \sum_{m_i \neq t_i, m_i \in T_i} \lambda(t_i, m_i) = \kappa(t) + \sum_{m_i \neq t_i, m_i \in T_i} \lambda_i(t_i, m_i) q_i(t_{-i}|m_i).$$

Non-degeneracy assumption There exists at least one agent i and one type t_i such that the incentive compatibility constraint for (i, t_i) is strict:

$$\sum_{t_{-i} \in T_{-i}} Q(t_{-i}|t_i) u_i(x_i(t_{-i}, t_i), t_i) > \sum_{t_{-i} \in T_{-i}} Q(t_{-i}|t_i) u_i(x_i(t_{-i}, \hat{t}_i), t_i)$$

for all $\hat{t}_i \in T_i$.

Theorem 1 *Assume Assumption 1, the compatibility condition and the non-degeneracy assumption hold. Then*

- *generically, every collusion-proof contract is fully revealing and induces an outcome which is interim efficient and incentive compatible;*
- *generically, almost every interim efficient incentive compatible allocation can be achieved with a fully revealing collusion-proof contract.*

Remarks

- The compatibility condition is not implied by the consistency condition $q_i = Q$ for all $i \in I$. The first of these conditions was presented in [4] and is needed to ensure the existence of incentive compatible allocations. The latter condition is required if we do not want to face the type of problems shown to happen with subjective correlated equilibria in Aumann [6].
- The non-degeneracy assumption ensures that there is a non-degenerate set of lotteries satisfying the incentive compatibility constraints. This kind of assumption is also required in [20].

Let $T_i = \{t_i^1, \dots, t_i^{T_i}\}$ be the set of types of i and $T = \{t(1), \dots, t(T)\}$. Let

$$\mathcal{U} = \left\{ u = (u_{1t_1^1}, \dots, u_{1t_1^{T_1}}; \dots; u_{Nt_N^1}, \dots, u_{Nt_N^{T_N}}) : u_{it_i^j} \in \mathcal{C}^r(Y_i, \mathbb{R}), \|u_{it_i^j}\| < \infty \right\},$$

where $u_{it_i^j}(x^1, \dots, x^L) = v_{it_i^j}(x^1, \dots, x^{L-1}) + x^L$. To prove Theorem 1, we use the following two lemmas. The proofs of these lemmas are given in the appendix.

Lemma 1 *Assume Assumption 1 holds. Generically on \mathcal{U} , the set of injective selections of Pareto optima is open and dense. Consequently, generically on \mathcal{U} , almost every selection of Pareto optima induces fully revealing prices.*

Lemma 2 *Assume the non-degeneracy assumption holds. If the allocation $(x_i)_i$ is incentive compatible but not interim efficient, then there exists a non-negligible set of incentive compatible interim efficient allocations that dominate $(x_i)_i$.*

Proof of Theorem 1 Take any incentive compatible *interim* efficient allocation $x^* = (x_i^*)_{i \in I}$; then x^* is *ex post* efficient. Since $Q(t) > 0$ for all t , then for every $t = (t_i)_i \in T$, $x^*(t)$ is a Pareto optimum for the economy $\mathcal{E}_t = (X_i, u_i(\cdot, t_i), \omega_i(t_i))_{i \in I}$ with no uncertainty when each agent i is of type t_i . By the constrained second welfare theorem [13], there exists a price vector $p^*(t) \neq 0$ and an assignment of wealth levels $w^* = (w_i^*)$ with $\sum_i w_i^* = p^*(t) \sum_i \omega_i(t_i)$ such that $(p^*(t), (x_i^*(t))_i)$ is a constrained price equilibrium. The price vector $p^*(t)$ is unique by the theorem of separation of strictly convex sets, and we can assume that the map $t \mapsto p^*(t)$ is one-to-one by Lemma 1, i.e. prices are fully revealing.

Consider the communication contract $(\bar{\mu}_I, \tilde{y})$ defined, for all announced vector of types $m = (m_i)_i$, by $\bar{\mu}_I(a|m) = 1$, where $a = (a_i)_{i \in I} = ((p^*(m), \frac{1}{N}(x_i^*(m) - \omega_i(m_i)), \omega_i(m_i)))_{i \in I}$, and $\tilde{y} = (\tilde{y}_i)$ is defined by $\tilde{y}_i = w_i^* - p^*(m) \omega_i(m_i)$. By construction, $\bar{\mu}_I$ associates to all m the incentive compatible *interim* efficient outcome $x^*(m) = (x_i^*(m))_{i \in I}$ and $\bar{\mu}_I$ is outcome efficient.

In order to show that $\bar{\mu}_I$ is a collusion-proof contract, we have to show that there is no coalition $S \subset I$ which has a BIC-ICI (η, y') upon $(\bar{\mu}_I, \tilde{y})$. As x^* is incentive compatible, the contract $(\bar{\mu}_I, \tilde{y})$ is a BIC. We claim that in order to

improve his payoff, some player $i_o \in S$ must change his price quoting strategy. Indeed, note that for all t , $\sum_{i \in S} y'_i(t) = \sum_{i \in S} \tilde{y}_i(t) = 0$. Hence if coalition S deviates by using a side mechanism η without modifying prices, there exists $i_o \in S$ and some t with $y'_{i_o}(t) < \tilde{y}_{i_o}(t)$. Since $(p^*(t), x^*(t))$ is a competitive equilibrium for the economy with transfers \tilde{y} , the final payoff of i_o following the deviation of S is necessarily smaller than that received prior to the deviation, a contradiction.

Suppose that for some t , some player $i_o \in S$ quotes a price $p_{i_o}(t)$ different from $p^*(t)$ with positive probability. Assume wlog $p_{i_o}^h(t) < p^{*h}(t)$ for some good h with positive probability. The 2-stage extensive-form nature of the game implies that there exists $i \in I \setminus S$ and $j \in S \setminus \{i_o\}$ such that $p_i(t) = p^*(t)$ and j faces an arbitrage opportunity with positive probability. Note indeed that S is a strict subset of I as $(x_i^*(t))_i$ is *ex post* Pareto optimal (the grand coalition has no improvement) and $\{i_o\}$ is a strict subset of S as a Walrasian equilibrium of \mathcal{E}_t is a Nash equilibrium of a complete information game where t is commonly known [13].

We now show that j can take advantage of this arbitrage opportunity. Let fix one vector of types $t \in T$ commonly known by all since $\bar{\mu}_I$ is fully revealing. Consider the following strategy of player j : $p_j > 0$ and if he observes $p_{i_o}^h < p^{*h}$,

- for all $k \neq \{i, i_o, j\}$, $z_k^j = \max_{\Upsilon} \left(\sum_{l \neq j, k} (z_l^k - z_k^l) + z_j^k \right) + \omega_k$
- $z_{i_o}^j = z^* + \max_{\Upsilon} \left(\sum_{l \neq j, i_o} (z_l^{i_o} - z_{i_o}^l) + z_j^{i_o} \right) + (\bar{\omega} - \omega_j + \omega_{i_o})$
- $z_i^j = -z^* + \max_{\Upsilon} \left(\sum_{l \neq j, i} (z_l^i - z_i^l) + z_j^i \right) + \omega_i$

where the maxima are taken with respect to all the finite supports of the random variables associated to the grand mechanism $\bar{\mu}_I$ and the side mechanism η of S , which we denote Υ .

We show now that there exists z^* chosen large enough so that player j satisfies his budget constraint with probability one and player i violates his budget constraint with probability one. As a consequence, every player will get zero a.s. but player j who will receive $\bar{\omega}$ a.s..

Let $z^* = (0, \dots, 0, z^{*h}, 0, \dots, 0) \in \mathbb{R}_+^L$ with $z^{*h} > 0$. The budget constraint of player i is

$$\sum_{k \neq j, i} p_k (z_k^i - z_i^k) + p_j (z_j^i - z_i^j) \leq \tilde{y}_i(t).$$

Whenever player i goes bankrupt, this inequality is not satisfied. A sufficient condition for player i to go bankrupt is to have $z^{*h} \geq W^{*h}$ where

$$W^{*h} \geq \frac{1}{p_j^h} \left[- \sum_{k \neq i, j} p_k (z_k^i - z_i^k) - p_j z_j^i + p_j \max_{\Upsilon} \left(\sum_{l \neq j, i} (z_l^i - z_i^l) + z_j^i \right) + p_j \omega_i \right] + \tilde{y}_i(t).$$

As for player j , he must never go bankrupt. Hence,

$$\max_{\Upsilon} \left[\sum_{k \neq i, j, i_o} p_k (z_k^j - z_j^k) + p_{i_o} (z_{i_o}^j - z_j^{i_o}) + p_i (z_i^j - z_j^i) \right] \leq \tilde{y}_j(t).$$

Using the definition of $z_{i_o}^j$ and z_i^j and $p_i = p^*$ we get

$$\begin{aligned} z^{*h} (p^{*h} - p_{i_o}^h) &\geq \max_{\Upsilon} \left[\sum_{k \neq i, j, i_o} p_k (z_k^j - z_j^k) + p_{i_o} \max_{\Upsilon} \left(\sum_{l \neq j, i_o} (z_l^{i_o} - z_{i_o}^l) + z_j^{i_o} \right) \right. \\ &\quad \left. + p_{i_o} (\bar{\omega} - \omega_j + \omega_{i_o} - z_j^{i_o}) + p^* \left(\max_{\Upsilon} \left(\sum_{l \neq j, i} (z_l^i - z_i^l) + z_j^i \right) + \omega_i - z_j^i \right) \right] \\ &\quad + \tilde{y}_j(t) = Y^{*h} \end{aligned}$$

In this case $z^{*h} \geq \frac{Y^{*h}}{p^{*h} - p_{i_o}^h} = \tilde{Y}^{*h}$ is required. Overall, it suffices to take $z^{*h} \geq \max\{\tilde{Y}^{*h}, W^{*h}\}$.

Straightforward calculations show that $\psi_{i_o}^{\tilde{y}} = -z^* - (\bar{\omega} - \omega_j) - \omega_{i_o} < 0$, $\psi_j^{\tilde{y}} = \bar{\omega} - \omega_j$ and for each other player k , $\psi_k^{\tilde{y}} = -\omega_k$. Hence the claim: $\phi_j^{\tilde{y}} = \bar{\omega}$ and $\phi_h^{\tilde{y}} = 0$ for all $h \neq j$. Therefore, the deviation of S is not an ICI and x^* can be decentralized as a collusion-proof contract.

Conversely, we show that every collusion-proof contract is incentive compatible *interim* efficient. Assume the outcome induced by the collusion-proof contract $\bar{\mu}_I$ is not incentive compatible *interim* efficient. Since it is a collusion-proof contract, then it is incentive compatible but not *interim* efficient. Hence by Lemma 2 there exists a non-negligible set of incentive compatible *interim* efficient allocations $(\hat{x}_i)_i$ that dominate the allocation induced by $\bar{\mu}_I$. By the constrained second welfare theorem, for all t there exists $p^*(t) \neq 0$ and an assignment of wealth levels w^* such that $(p^*(t), (\hat{x}_i(t))_i)$ is a constrained Walrasian equilibrium of \mathcal{E}_t . By the first part of the proof, the grand coalition can play $(p^*(t), (\hat{x}_i(t))_i)$ by means of a BIC contract, $\hat{\mu}$, which Pareto dominates *interim* $\bar{\mu}_I$. So far, the price system induced by $\hat{\mu}$ may be non injective. By Lemma 1, the set of such non injective price systems is negligible. Hence at least one of them is injective, and it is the one we select. \square

Remarks

- We require the compatibility condition to hold to ensure the existence of incentive compatible *interim* efficient allocations through appropriate (possibly negative) transfers [4]. We need to assume that each player is endowed with an initial quantity of the numéraire good greater than the maximal value of these transfers. As the utilities in non-numéraire consumption goods are continuous and defined on the non-numéraire part of the feasible set which is compact, a maximum exists. Note that there is no loss of generality in assuming that such initial endowments in numéraire

exist as, given some ordinal preferences, these can be obtained by a translation of the utilities.

- A careful reading of the proof of Theorem 1 shows that it is essential to assume that the game is in extensive-form. Assume on the contrary that price quoting strategies and quantity setting strategies were played simultaneously. It would be possible for some players to deviate from the equilibrium strategy by playing a mixed price quoting strategy, thereby creating an arbitrage opportunity with positive probability. However, no player could take advantage of this opportunity as players would set quantities at the same time as the price random variables would realize.
- One could imagine that players have the possibility to accept or refuse the grand mechanism at the start of the game, which would then require the addition of suitable individual rationality constraints. We do not consider this different setup here as the existence of *interim* individually rational incentive compatible efficient allocations cannot be ensured.

Corollary 1 *Whenever Assumption 1, the compatibility condition and the non-degeneracy assumption hold, then*

- *generically, every strong collusion-proof contract is fully revealing and induces an outcome which is incentive compatible interim efficient;*
- *generically, almost every incentive compatible interim efficient allocation can be achieved with a fully revealing strong collusion-proof contract.*

5 Conclusion

In this paper, we present a strategic market mechanism with the property that generically (for a dense set of utility functions), almost every incentive compatible *interim* efficient allocation can be achieved both with a collusion-proof and a strong collusion-proof contract. Moreover, this game is such that generically, every collusion-proof and strong collusion-proof contract induce an outcome which is incentive compatible *interim* efficient.

From a general equilibrium perspective, quasi-linear economies with incomplete information are trivially equivalent to linear economies with complete information once only interior equilibria are considered. Indeed, whenever preferences are quasi-linear, we can consider that they are linear with respect to income, and hence, if there exists assets contingent on the realization of the state of nature, markets are complete (or effectively complete) as the assets are priced by their expected returns. The linearity of the indirect preferences implies that the agents are risk neutral.

However, competitive equilibria do not take into account incentive problems. Indeed, in a differential information economy with quasi-linear utilities, the

incentive compatible core may be empty [12], whereas the Walrasian equilibria may exist. This shows that the formal equivalence between the complete and incomplete information worlds breaks down once incentive constraints are taken into account.

McLean and Postlewaite [21] have introduced a notion of informational size that allows to show that the conflict between incentive compatibility and efficiency in a general equilibrium model with asymmetric information can be made small, if agents are small relative to this notion. Moreover, in an economy where agents are informationally small, the incentive compatible core is non-empty [22]. It would be interesting to extend our paper to these economies and construct a model that does not rely on the quasi-linearity assumption.

The next point on the research agenda is certainly to try to extend the present work to *ex ante* efficient allocations. This could possibly be achieved by allowing coalitions to form *ex ante* in our mechanism. Note, nevertheless, that it would require a non-trivial modification of the equilibrium concept. Indeed, one main advantage of adopting an *interim* standpoint is that, once the grand mechanism has been played, even if a coalition of players wants to construct a side-mechanism, they cannot revise the messages they sent to the grand mechanism. Thus, at each coalitional deviation, we can take the behavior of players in the previous stages of the extensive-form preceding the actual play as given. Unfortunately, this is no more so in an *ex ante* setting : there, if a coalition deviates, given the fact that a side-mechanism will be used after the grand mechanism, there is no guarantee any more that no player will have an incentive to misreport his type to the grand mechanism. This could be a way, for instance, to manipulate the posteriors of the members of the deviating coalition... We leave these fascinating aspects for further research.

6 Appendix

6.1 Incentives

Let μ be any outcome efficient mechanism. For all $i \in I$, for all $m \in T$, for all $t_i \in T_i$, let

$$\bar{v}_i(m; t_i) := v_i(x_i(m), t_i)$$

where $x(m) = (x_i(m))_i$ is the allocation induced by μ , and for all $m_i, t_i \in T_i$,

$$\hat{v}_i(m_i, t_i) := \sum_{t_{-i} \in T_{-i}} [\bar{v}_i(m_i, t_{-i}; t_i) - \bar{v}_i(t_i, t_{-i}; t_i)] q_i(t_{-i}|t_i).$$

Consider the following system of linear inequalities, where $f : T \rightarrow \mathbb{R}^N$ is taken as variable: for all $i \in I$, for all $(m_i, t_i) \in C_i$

$$\sum_{t_{-i} \in T_{-i}} \left(f_i(t_i, t_{-i}) - \frac{1}{N-1} \sum_{j \neq i} f_j(t_i, t_{-i}) \right) q_i(t_{-i}|t_i) -$$

$$\sum_{t_{-i} \in T_{-i}} \left(f_i(m_i, t_{-i}) - \frac{1}{N-1} \sum_{j \neq i} f_j(m_i, t_{-i}) \right) q_i(t_{-i}|t_i) \geq \hat{v}_i(m_i, t_i).$$

d'Aspremont and Gérard Varet [4] have shown that the above system has a solution f , and the contract (μ, θ) where for all $t \in T$ and for all $i \in I$

$$\theta_i(t) := f_i(t) - \frac{1}{N-1} \sum_{j \neq i} f_j(t)$$

is budget balancing and Bayesian incentive compatible.

d'Aspremont and Gérard Varet [4] have shown that under the *Independence condition* recalled below, the function $f_i(t)$ takes the following form:

$$f_i(t) = E_{T_{-i}} \left[\sum_{j \neq i} v_j(x_j(t), t_j) \right].$$

Independence condition For all $i \in I$, for all $t_i, t'_i \in T_i$, $q_i(\cdot|t_i) = q_i(\cdot|t'_i)$.

6.2 Proof of Lemma 1

Lemma 1 *Assume Assumption 1 holds. Generically on \mathcal{U} , the set of injective selections of Pareto optima is open and dense. Consequently, generically on \mathcal{U} , almost every selection of Pareto optima induces fully revealing prices.*

Proof Consider the function

$$F : \mathcal{U} \times \tilde{Y} \rightarrow \tilde{Y} \times \text{Mat}(T, L-1)$$

given by

$$(u, x) \mapsto \begin{pmatrix} \nabla v_{1t(1)}(x_1(t(1))) \\ x \\ \vdots \\ \nabla v_{1t(T)}(x_1(t(T))) \end{pmatrix}$$

where

$$\tilde{Y} := \left\{ x = (x_i(t))_{i \in I, t \in T} \in Y^T : \forall t \in T, \sum_{i \in I} x_i(t) = \sum_{i \in I} \omega_i(t_i) \text{ and } \forall i, j, \right. \\ \left. \exists \lambda_j > 0 \text{ such that } \nabla u_{it}(x_i(t)) = \lambda_j \nabla u_{jt}(x_j(t)) \right\}$$

is the set of selections of Pareto optimal allocations. With a slight abuse of notation, we write $t(j)$ (or t) in the utility subscript of a player without specifying which type in $t(j)$ (or t) corresponds to that player.

For a fixed u , note that the domain of F_u is a smooth submanifold of the feasible set, of dimension $T(N-1)(L-1)$.

Consider the set E^j of $T \times (L-1)$ matrices with rank j for some fixed $j < \min(T, L-1)$. Then $\dim E^j < T(L-1)$, the dimension of the space of all $T \times (L-1)$ matrices $\text{Mat}(T, L-1)$.

The infinite dimensional transversality theorem (Abraham and Robin [1]) applied to the Banach manifold \mathcal{U} of utility functions ensures that there exists a dense subset \mathcal{U}_o in \mathcal{U} such that for any u in \mathcal{U}_o , F_u is transverse to the submanifold $\tilde{Y} \times E^j$ of $\tilde{Y} \times \text{Mat}(T, L - 1)$. Hence

$$\text{codim}F_u^{-1}(\tilde{Y} \times E^j) = \text{codim}(\tilde{Y} \times E^j) > 0$$

and the complement of $F_u^{-1}(\tilde{Y} \times E^j)$ is an open and dense set of full dimension in \tilde{Y} (as $F_u^{-1}(\tilde{Y} \times E^j)$ is not of full dimension, it is of measure zero, closed with empty interior in \tilde{Y}).

This can be done for any j . Hence, the union over j of all the sets $F_u^{-1}(\tilde{Y} \times E^j)$ is a closed set with empty interior whose complement is an open and dense set of full dimension in \tilde{Y} . □

6.3 Proof of Lemma 2

Lemma 2 *Assume the non-degeneracy assumption holds. If the allocation $(x_i)_i$ is incentive compatible but not interim efficient, then there exists a non-negligible set of incentive compatible interim efficient allocations that dominate $(x_i)_i$.*

Proof We denote by D_{++} the subset of *ex post* feasible lotteries which are incentive compatible and everywhere strictly positive (i.e., $\forall t, \forall i, x_i(t) \gg 0$). This is a closed and convex subset of the space $L = R_{++}^{TLN}$ of strictly positive lotteries. Consider the fictitious, complete information economy $\tilde{\mathcal{E}}$ with $M := \sum_i |T_i|$ agents, and L as trade space. An agent t_i is characterized by his utility function $U_{t_i}(x) = u_i(x_i(t), t_i)$ and his initial (deterministic) endowment $\omega_i(t_i)$. The total resources are $\bar{\omega}$. Obviously, the restriction of each individual's utility function to L is smooth, smoothly increasing, smoothly concave, and verifies the usual boundary condition $\{x \in L : U_{t_i}(x) = c_i\}$ is closed in L .

For the ease of notation, we relabel the individuals by $i = 1, \dots, M$, where 1 is the individual satisfying our non-degeneracy assumption. A lottery $x \in D_{++}$ is incentive compatible and *interim* efficient (BICIE) associated with the utility levels $\bar{v} = (v_2, \dots, v_M) \in \mathbb{R}^{M-1}$ if it solves the following maximization problem:

$$\text{Max } U_1(x)$$

subject to the incentive compatibility constraint (*),

$$\sum_{t_{-1} \in T_{-1}} Q(t_{-1}|t_1) u_1(x_1(t_{-1}, t_1), t_1) \geq \sum_{t_{-1} \in T_{-1}} Q(t_{-1}|t_1) u_1(x_1(t_{-1}, \hat{t}_1), t_1),$$

for all $t_1, \hat{t}_1 \in T_1$, the *ex post* feasibility constraint,

$$\sum_{i \in M} x_i(t) = \sum_{i \in M} \omega_i(t_i), \quad \text{for all } t = (t_i)_i \in T,$$

and subject to $U_i(x) = v_i \forall i \geq 2$. We denote by \mathcal{V}_1 the set of elements $\bar{v} \in \mathbb{R}^{M-1}$ such that the preceding optimization problem admits a solution. \mathcal{V}^* is the image of the mapping $U : D_{++} \rightarrow \mathbb{R}^M$ defined by:⁵

$$U(x) = (U_1(x), \dots, U_M(x)).$$

The BICIE frontier is the boundary of the set \mathcal{V} .

Observe, first, that the set $L_i(v_i) := \{x \in D_{++} : U_i(x) \geq v_i\}$ is compact and bounded away from zero or empty. Next, for $\bar{v} = (v_2, \dots, v_M)$, we consider the sets:

$$K(\bar{v}) = \{x \in D_{++} : \forall i \geq 2, U_i(x) \geq v_i\}$$

$$L(v_1, \bar{v}) = \{x \in D_{++} : \forall i \geq 1, U_i(x) \geq v_i\}.$$

Given the concavity of the utility functions and the previous remark on $L_i(v_i)$, it should be clear that $K(\bar{v})$ and $L(v_1, \bar{v})$ are both convex, that $L(v_1, \bar{v})$ is compact, and that \mathcal{V}_1 is the subset of \mathbb{R}^{M-1} consisting of the $(M-1)$ -tuples $\bar{v} = (v_2, \dots, v_M)$ such that $K(\bar{v}) \neq \emptyset$. On the other hand, obviously, $K(\bar{v}) \subset K(v')$ as soon as $v' \leq \bar{v}$. It follows that \mathcal{V}_1 verifies the free-disposal property, i.e., that:

$$\forall \bar{v} \in \mathcal{V}_1, \bar{v} - \mathbb{R}^{M-1} \subset \mathcal{V}_1.$$

We are now ready to show that \mathcal{V}_1 is open in \mathbb{R}^{M-1} . Let $\bar{v} = (v_2, \dots, v_M) \in \mathcal{V}_1$. By definition of \mathcal{V}_1 , there exists a lottery $x \in D$ such that $U_i(x) \geq v_i \forall i \geq 2$. Since the incentive constraints are verified as strict inequality for individual 1 and since we restrict ourselves to strictly positive lotteries in D_{++} , it is possible to find $\varepsilon > 0$ such that the vector x' defined by:

$$\forall i, \forall l \leq L, x_i^l = x_i^l$$

$$\text{and } x_1^L = x_1^L - (M-1)\varepsilon, \quad x_i^L = x_i^L + \varepsilon, \quad \forall i \geq 2,$$

still belongs to D_{++} . Indeed, one verifies that, for ε small enough, $x' \in L$ and x' still verifies the incentive constraints (*). By construction, $U_i(x') > U_i(x)$ for all $i \geq 2$, which proves that there exists a point $\bar{v}' = (U_2(x'), \dots, U_M(x'))$ with $\bar{v}' > \bar{v}$. The orthant $\bar{v}' - \mathbb{R}^{M-1}$ being a subset of \mathcal{V}_1 , it follows that \mathcal{V}_1 is a neighborhood of all its element, hence is open.

The next step is to show that \mathcal{V}_1 is convex: take $\bar{v} = (v_2, \dots, v_M)$ and $\bar{v}' = (v'_2, \dots, v'_M)$ in \mathcal{V}_1 and some $\gamma \in [0, 1]$. There exist x, x' in D_{++} such that $\forall i \geq 2, U_i(x) \geq v_i$ and $U_i(x') \geq v'_i$. One has $x'' := \gamma x + (1-\gamma)x' \in D_{++}$ and $\forall i, U_i(x'') \geq v''_i := \gamma v_i + (1-\gamma)v'_i$. Hence, $\bar{v}'' = (v''_2, \dots, v''_M) \in \mathcal{V}_1$, so that \mathcal{V}_1 is indeed convex.

⁵As in Holmström & Myerson [15], we use the superscript * to emphasize that incentive constraints are taken into account.

Being open and convex, \mathcal{V}_1 is *diffeomorphic* to \mathbb{R}^{M-1} ([14], Exercise 6, p. 20).

Note that the image by U of a subset of measure zero in L is of measure zero, as U is differentiable⁶. Hence, the inverse image by U of a non-negligible subset in \mathbb{R}^M is non-negligible.

The boundary of the set of utility levels feasible and incentive compatible is equal to the set of utility levels induced by an *interim* efficient and Bayesian incentive compatible lottery and is a submanifold of \mathbb{R}^M diffeomorphic to \mathbb{R}^{M-1} . Let \bar{u} be the utility level induced by the allocation $(x_i)_i$. Since $(x_i)_i$ is not *interim* efficient, \bar{u} does not belong to the boundary of the set \mathcal{V} . Hence, there is an open set in the relative interior of $\partial\mathcal{V}$ which Pareto dominates \bar{u} . This open set is non-negligible. Consequently there exists a non-negligible subset of utility levels Bayesian incentive compatible and *interim* efficient that dominate \bar{u} and whose inverse image by U is non-negligible. Hence the set of *interim* efficient and Bayesian incentive compatible allocations that dominate $(x_i)_i$ is non-negligible. □

6.4 Strong Bayesian and collusion-proof communication equilibria

We now present a new equilibrium concept, the strong Bayesian equilibrium, which seems to be the correct extension of Aumann's strong Nash equilibrium concept to an incomplete information setting. We also introduce the auxiliary concept of a collusion-proof communication equilibrium, which is the internally consistent extension of the strong Bayesian equilibrium. Through an example, we show the impossibility of implementing efficient allocations with these concepts which allow simultaneous deviations in types and actions.

In the definitions that follow, we use the same notations as in section 3.

Definition 4' A contract (μ, \tilde{y}) is a *communication equilibrium with transfers* (ComE) if for every $i \in I_m$, $t_i \in T_i^+$, $s_i \in T_i$ and $g_i : A_i \rightarrow A_i$,

$$U_i^{\tilde{y}}(\mu|t_i) \geq \sum_{t_{-i} \in T_{-i}} q_i(t_{-i}|t_i) \sum_{a \in A_{I_m}} \mu(a|(t_{I_m \setminus \{i\}}, s_i)) u_i^{\tilde{y}}((a_{-i}, g_i(a_i)), t).$$

At a ComE, no individual player has an incentive to misreport his type or not follow the recommendation of μ . Except for transfers that enter the payoff functions, this concept is precisely Forges [11] definition of a communication equilibrium. Note moreover that every ComE is a BIC, while the converse is false.

⁶This property of U is crucial. There are well known examples of continuous functions (i.e. the Peano curve, the Schoenberg curve) that map negligible sets (i.e. the Cantor set) to non-negligible sets.

We now define a coalitional improvement.

Definition 6' The strategy (η, y') is a ComE-improvement of $\emptyset \neq S \subset I_m$ upon (μ, \tilde{y}) in Γ_y if

- a) (η, y') is a ComE for the revised game $\Gamma_{\mu, \tilde{y}, S}$,
- b) for every $i \in S$ and $(t_i, a_i) \in (T_i, A_i)^+$ then

$$U_i^{y'}(\eta|(t_i, a_i)) \geq U_i^{\tilde{y}}(\mu|(t_i, a_i)),$$

with at least one strict inequality.

In words, we restrict attention to coalitional deviations where members of the deviating coalition have some incentive to deviate (b), and some incentive to share their information with the other members of the deviating coalition (a). In the next definition, we focus on a smaller class of coalitional deviations, namely those which cannot be improved upon by any other incentive compatible and profitable deviation of any subcoalition.

Definition 7' Let Γ_y be an extended Bayesian game, (μ, \tilde{y}) a communication contract and $S \subset I_m$, $S \neq \emptyset$. A ComE *internally consistent improvement* (ICI) of S upon (μ, \tilde{y}) in Γ_y is defined by induction on the cardinality of S , as follows:

- If $|S| = 1$, then (η, y') is a ComE-ICI if it is a ComE-improvement of S upon (μ, \tilde{y}) . Note that in this case $\tilde{y} = y'$.
- If $|S| > 1$, then (η, y') is a ComE-ICI of S upon (μ, \tilde{y}) in Γ_y if
 - (η, y') is a ComE-improvement of S upon (μ, \tilde{y}) in Γ_y
 - if $\tilde{S} \subset S$ and $\tilde{S} \neq \emptyset$, then \tilde{S} has no ComE-ICI upon (η, y') in the game $\Gamma_{\mu, \tilde{y}, S}$.

Definition 8' Let Γ_y be an extended Bayesian game.

- A communication equilibrium with transfers (μ, \tilde{y}) is a *collusion-proof communication equilibrium* (cpComE) if no coalition of players has a ComE-ICI upon (μ, \tilde{y}) .
- A communication equilibrium with transfers (μ, \tilde{y}) is a *strong Bayesian equilibrium* if no coalition of players has a ComE-improvement upon (μ, \tilde{y}) .

The strong Bayesian equilibrium and the collusion-proof communication equilibrium cannot fully implement the incentive compatible *interim* efficient allocations. Indeed, when players are allowed to simultaneously deviate in types and actions, they may get a final allocation which is strictly better for them than the allocation they would get under truthful revelation, as Figure 1 shows.

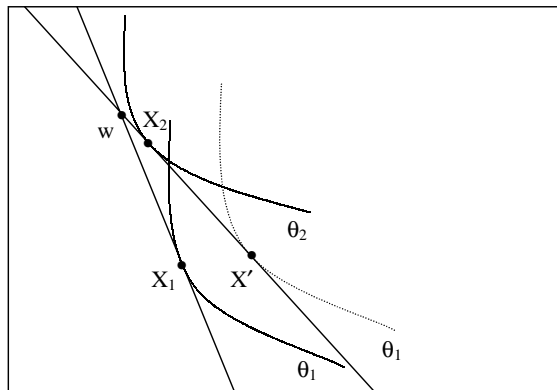


Figure 1

Indeed, assume that a player of type Θ_1 gets the allocation X_1 while when he is of type Θ_2 , he gets X_2 . If, when he is of type Θ_1 , the player announces he is of type Θ_2 but is not allowed to deviate in actions, then he gets X_2 as final allocation, which is worse for him than if he had revealed his true type. However, if he is allowed to also deviate in action, he could ensure himself the final allocation X' , which is strictly better for him.

References

- [1] Abraham R, Robbin J (1967) Transversal mappings and flows. W.A. Benjamin, New York
- [2] Allen B (1991) Market games with asymmetric information and nontransferable utility: representation results and the core. CARESS Working paper 91-09, University of Pennsylvania
- [3] Allen B (1992) Incentives in market games with asymmetric information: the core. CORE Discussion Paper 9221, Université catholique de Louvain
- [4] d'Aspremont C, Gérard-Varet LA (1979) Incentives and incomplete information. *Journal of Public Economics* 11: 25-45
- [5] Aumann R (1959) Acceptable points in general cooperative n-person games. In: Contributions to the Theory of Games IV, Princeton University Press, Princeton
- [6] Aumann R (1974) Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics* 1: 67-96
- [7] Bernheim B, Peleg B, Whinston M (1987) Coalition-proof Nash equilibria I. Concepts. *Journal of Economic Theory* 42: 1-12

- [8] Dubey P, Geanakoplos J, Shubik M (1987) The revelation of information in strategic market games. *Journal of Mathematical Economics* 16: 105-137
- [9] Dubey P, Rogawski JD (1990) Inefficiency of smooth market mechanisms. *Journal of Mathematical Economics* 19: 285-304
- [10] Einy E, Peleg B (1995) Coalition-proof communication equilibria. In: Barnett W, Moulin H, Salles M, Schofield N (eds) *Social choice, welfare and ethics: Proceedings of the 8th international symposium in economic theory and econometric*. Cambridge University Press
- [11] Forges F (1986) An approach to communication equilibria. *Econometrica* 54: 1375-1386
- [12] Forges F, Mertens JF, Vohra R (2000) The incentive compatible core of a quasi-linear economy. mimeo
- [13] Giraud G, Rochon C (2000) Consistent collusion-proofness and correlation in exchange economies. mimeo
- [14] Hirsch MW (1976) *Differential Topology*, Springer Verlag, New York
- [15] Holmström B, Myerson R (1983) Efficient and durable decision rules with incomplete information. *Econometrica* 51: 1799-1819
- [16] Hong L (1995) Nash implementation in production economies. *Economic Theory* 5: 401-417
- [17] Hurwicz L (1979) Outcome functions yielding Walrasian and Lindahl allocations at Nash equilibrium points. *Review of Economic Studies* 46: 217-225
- [18] Jackson M (1999) A crash course in implementation theory. Caltech, mimeo (<http://masada.hss.caltech.edu/jacksonm/Jackson.html>)
- [19] Koutsougeras L, Yannelis N (1993) Incentive compatibility and information superiority of the core of an economy with differential information. *Economic Theory* 3: 195-216
- [20] Laffont JJ, Martimort D (1997) Collusion under asymmetric information. *Econometrica* 65: 875-911
- [21] McLean R, Postlewaite A (1999) Informational size and incentive compatibility. mimeo
- [22] McLean R, Postlewaite A (2000) Informational size, incentive compatibility and the core of a game with incomplete information. mimeo
- [23] Mookherjee D, Reichelstein S (1992) Dominant strategy implementation of Bayesian incentive compatible allocation rules. *Journal of Economic Theory* 56: 378-399

- [24] Myerson R, Satterthwaite M (1983) Efficient mechanisms for bilateral trading. *Journal of Economic Theory* 28: 265-281
- [25] Prescott E, Townsend R (1984) General competitive analysis in an economy with private information. *International Economic Review* 25: 1-20
- [26] Postlewaite A, Wettstein D (1989) Feasible and continuous implementation. *Review of Economic Studies* 56: 603-611
- [27] Radner R (1979) Rational expectations equilibrium: generic existence and the information revealed by prices. *Econometrica* 47: 655-678
- [28] Ray I (1996) Coalition-proof correlated equilibrium: a definition. *Games and Economic Behavior* 17: 56-79
- [29] Vohra R (1999) Incomplete information, incentive compatibility and the core. *Journal of Economic Theory* 86: 123-147