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ENDOGENOUS BUSINESS CYCLES AND BUSINESS FORMATION WITH STRATEGIC INVESTMENT

Claude d'ASPREMONT[1] Rodolphe DOS SANTOS FERREIRA[2]
and Louis-André GERARD-VARET[3]

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Abstract

We study an endogenous business cycle model with Cournotian monopolistic competition and an endogenous number of firms in each sector. Our model is a simple general equilibrium macroeconomic model introducing overlapping generations both of consumers *and* firms. Firms strategically decide on investment in the first period of their life, and compete à la Cournot in the second period. Investment is taken to be in human capital or technological know-how, to have spillover effects and to be formed from simple labour supplied by young consumers in anticipation of the profit share they get when old. It is Cournot competition that allows to analyze the variation of monopoly power along the cyle, since the number of firms is endogenized. As this number increases, firms behave more and more competitively. The properties along the cycle are generated by business formation. They will include the counter-cyclicality of markups and prices, the pro-cyclicality of the number of firms and of real wages.

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[1]CORE, Université Catholique de Louvain, Belgium.
E-mail: daspremont@core.ucl.ac.be
[2]BETA-Theme, Université Louis Pasteur, France.
[3]GREQAM, Ecole des Hautes Etudes en Sciences Sociales, France.

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1 Introduction

Although the literature on endogenous fluctuations in overlapping generations models has well integrated imperfect competition, allowing for increasing returns at the firm level, markup variability and endogenous business creation or destruction, it usually reduces the treatment of capital formation to consumer's saving decisions[1]. It is our purpose here to introduce a model with oligopolistic firms taking investment decisions strategically. Modelling firms in the same way as consumers, we assume them to have a two-period life[2]. Firms invest during their first period and compete à la Cournot in their second period of life. In the first period, investment is taken to be in technological know-how, to have spillover effects and to be formed from simple labor, supplied by young consumers in anticipation of the profit share they get when old. This modelling is linked to the industrial organization literature on R&D investment with spillovers[3]. In the second period, Cournot competition is a natural way to relate market power to the number of firms. This number is endogenously determined through capital market clearing. Hence, in contrast with the standard free-entry approach, both markup variability and profit variability are the indicators of market power fluctuations along the cycle.

Fluctuations in market power have already been considered in the pre-war literature[4], but formal models of such dynamic phenomena are more recent. For instance, Rotemberg-Saloner [19] and Rotemberg-Woodford [20] both introduce a repeated-game type of model, leading to the argument that oligopolies, competing repeatedly in prices, are less likely to coordinate on a collusive equilibrium during booms. However, in their model, fluctuations in prices, output and profits are generated by exogenous shocks in aggregate demand and the number of firms is fixed: market power varies with the degree of collusion among this fixed number of firms. Endogenous fluctuations models with free-entry are used by Chatterjee, Cooper and Ravikumar [8], and by Jacobsen [14]. But both models adjust the number of firms imposing a zero-profit condition. Fluctuations in market power are only reflected by markup variability.

In our model, an intertemporal Cournotian equilibrium concept leads to a dynamic system in two variables, namely individual investment and the number of firms. We proceed to a local analysis of this system around its unique steady state. The appearance of endogenous economic fluctuations can be established, through the existence of a Hopf bifurcation, using the degree of internal economies of scale as the bifurcation parameter. A property of this model is to allow to identify the effects on the dynamics of the parameters of internal and external economies of scale[5]. Moreover, some properties along the cycle, gen-

[1] For references see, for instance, Rotemberg and Woodford [21], d'Aspremont, Dos Santos Fereira and Gérard-Varet [3], Cooper and John [9]. The literature on endogenous fluctuations in overlapping generations has, however, been mostly devoted to the competitive case, beginning with Grandmont [12].

[2] As in Hahn and Solow [13].

[3] See, e.g., d'Aspremont and Jacquemin [4] and Suzumura [22].

[4] The main references are Robinson [18], Abramovitz [2] and Kalecki [15].

[5] This is in contrast with the model of Cazzavilan, Lloyd-Braga and Pintus [7].

erated by business formation, can be illustrated. Typically, an increase in the number of active firms, depressing prospective monopoly power, will have the Schumpeterian property of being associated with decreasing investments. This will be followed by increasing costs, and will result eventually in the reduction of the number of firms. But then market power will increase, more investment will take place and eventually more firms will be created.

The paper is organized as follows. In section 2, we give the model and define the intertemporal Cournotian equilibrium. In section 3, the dynamic system is derived and compared to the alternative systems obtained when the number of firms is fixed, or determined by a zero-profit condition. The dynamic properties of the system are then analyzed. Section 4 concludes and refers to the empirical literature.

2 The Model

We consider an overlapping generations model with two types of agents: consumers and firms. Both types live for two periods.

2.1 Consumption

On the consumers side, we have a standard overlapping generations model. Young consumers in period t, assumed to be identical and to form a continuum of mass $\bar{L}$, choose x_t and $\hat{x}_{t+1}$, their current and expected consumptions, evaluated at current prices p_t and expected prices p_{t+1}, as well as their current labor l_t, taking values in $[0,1]$. We assume that the young consumer can allocate his labor either to wage labor at a wage normalized to 1, in order to support current consumption, or to investment z_t in some firm, in order to build up cost-reducing technological know-how. When investing in a firm, a young consumer is anticipating, in addition to the reimbursement of his investment, a share of profit to get when old, at an expected rate of return on capital $r_{t+1}-1$. We assume that factor markets are competitive.

Young consumers, with identical preferences, separable in consumption and labor and homothetic with respect to consumption, face the following program:

$$\max_{(x_t,\hat{x}_{t+1},z_t,l_t)\in\mathbb{R}_+^{2m+1}\times[0,1]} U\left[u\left(x_t\right),u(\hat{x}_{t+1})\right]-V\left(l_t\right)$$

$$\begin{aligned}\text{s.t. } p_t x_t + z_t &\leq l_t,\\ p_{t+1}\hat{x}_{t+1} &\leq r_{t+1}z_t,\end{aligned}$$

where U is a two-dimensional intertemporal utility function (with weights α and $1-\alpha$ for present and future consumption, respectively) and u is an instantaneous m-dimensional symmetric sub-utility function. Both are assumed Cobb-Douglas[6]. The function V is continuously differentiable in $]0,1[$, increasing and convex. As well known, the solution to the young consumer's problem

[6] More explicitly, we put: $U(X,\hat{X})=X^{\alpha}\hat{X}^{1-\alpha}$ and $u(x)=m\prod_{i=1}^{m}x_i^{1/m}=X$.

verifies in this case:

$$x_{it} = \frac{\alpha l_t}{mp_{it}},\ \hat{x}_{it} = \frac{(1-\alpha)\, r_{t+1} l_t}{mp_{it+1}},\ i = 1, ..., m,\ \ z_t = (1-\alpha) l_t,$$

and, for l_t to be positive it must also satisfy, for price indices $P_\tau \equiv \prod_{i=1}^{m} p_{i\tau}^{1/m}$ $(\tau = t, t+1)$,

$$V'(l_t) \leq \alpha^\alpha (1-\alpha)^{1-\alpha} \frac{r_{t+1}^{1-\alpha}}{P_t^\alpha P_{t+1}^{1-\alpha}} \text{ (with an equality if } l_t < 1).$$

The old consumers solve, for a pre-decided z_{t-1}, and given r_t, the actual return factor, and p_t, the price vector, the following program:

$$\max_{x'_t \in \mathbb{R}^m_+} u(x'_t),\ \text{s.t. } p_t x'_t \leq r_t z_{t-1},$$

to obtain

$$x'_{it} = \frac{r_t z_{t-1}}{mp_{it}}.$$

We can now derive aggregate demand. Since we are in the Cobb-Douglas case, and by symmetry, demand for good i has the form:

$$d(p_{it}, A_t) = \frac{A_t}{mp_{it}},$$

where A_t is equal to the aggregate expenditure $P_t X_t$, that is, to the sum of consumption expenditures by both the youngs and the olds. This is (denoting by $L_\tau = l_\tau \overline{L}$ aggregate labor in period τ, $\tau = t, t+1$):

$$P_t X_t = \sum_{i=1}^{m} p_{it} d(p_{it}, A_t) = A_t = \alpha L_t + r_t (1-\alpha) L_{t-1}. \qquad (1)$$

2.2 Production

On the producers side, there is a large number, say m, of oligopolistic industries. In each industry i, there is, in each period t, a small (and varying) number n_{it-1} of firms, created in the preceding period, producing good i for immediate consumption and using two kinds of inputs. One kind is simple labor, provided by the young consumers in the production period. The other kind, simply called capital, is the above mentioned cost-reducing technological know-how, formed on a one-to-one ratio from simple labor supplied by young consumers during the investment period. As explained below, the number of active firms is fixed parametrically so as to clear the capital market.

From the point of view of the n_{it-1} firms active in industry i in period t, the situation appears as a two-stage non-cooperative game $\Gamma(n_{it-1}, A_t)$, parametrized in n_{it-1} and A_t. In the first stage, corresponding to the investment period $(t-1)$, each firm j, producing good i, chooses strategically a committing quantity of capital k_{ijt-1}. In the second stage, it chooses strategically, as in the classical Cournot approach, a quantity produced y_{ijt}. All firms within each industry will be symmetric. We will consider subgame perfect equilibria: each game $\Gamma(n_{it-1}, A_t)$ will be solved by backward induction

To define firms profits at each stage, we assume a Cobb-Douglas production function F with increasing returns to scale and linear in labor. Moreover, we suppose that production is subject to spillover effects, coming from capital investments made by the other firms (represented here by the geometric mean of the others' investments in the same sector). Such spillovers may result from the incomplete appropriability of technological know-how as stressed in the literature on strategic R&D investments. We write, for each period t and each firm j of industry i:

$$F\left(l_{ijt}, k_{ijt-1}, \mathbf{k}_{i(-j)t-1}\right) = l_{ijt} k_{ijt-1}^{\gamma} \left[\prod_{j' \neq j} k_{ij't-1}\right]^{\delta/(n_{it-1}-1)},$$

$\gamma > 0$ denoting the degree of (internal) economies of scale, $\delta \in [0, \gamma]$ the spillover coefficient, and with $\mathbf{k}_{i(-j)t-1} = \left(k_{ij't-1}\right)_{j' \neq j}$.

For any given investment vector $\mathbf{k}_{it-1} = (k_{ijt-1})_j$ in period $t-1$, the second stage profit of firm i in sector j and period t is:

$$\left[\frac{A_t/m}{y_{ijt} + \sum_{j' \neq j} y_{ij't}} - c_{ijt}\right] y_{ijt},$$

with

$$c_{ijt} = C(k_{ijt-1}, \mathbf{k}_{i(-j)t-1}) \equiv k_{ijt-1}^{-\gamma} \left[\prod_{j' \neq j} k_{ij't-1}\right]^{-\delta/(n_{it-1}-1)},$$

the constant marginal cost associated with the investment vector $\mathbf{k}_{it-1}$. Each second stage subgame is thus a standard Cournot problem with isoelastic demand and linear cost functions. The solution is easily computed to be:

$$y_{ijt} = \frac{A_t(n_{it-1} - 1)}{m \sum_{j'=1}^{n_{it-1}} c_{ij't}} \left[1 - \frac{(n_{it-1} - 1)c_{ijt}}{\sum_{j'=1}^{n_{it-1}} c_{ij't}}\right],$$

provided $c_{ijt} \leq \sum_{j'=1}^{n_{it-1}} c_{ij't}/(n_{it-1} - 1)$ for any j (otherwise the firm(s) with the highest costs will not be active in equilibrium). Using the equilibrium of each subgame, one for every investment vector $\mathbf{k}_{it-1} = (k_{ijt-1})_j$, we finally compute the first stage profit function, which is net of the investment cost k_{ijt-1}:

$$\Pi\left(k_{ijt-1}, \mathbf{k}_{i(-j)t-1}\right) = \frac{A_t}{m}\left(1 - \frac{(n_{it-1} - 1)c_{ijt}}{\sum_{j'=1}^{n_{it-1}} c_{ij't}}\right)^2 - k_{ijt-1}. \tag{2}$$

2.3 Investment

In equilibrium, aggregate investment K_{t-1} in period $t-1$ must be equal to aggregate saving as determined by consumers: $Z_{t-1} = (1-\alpha)L_{t-1}$. The crucial question is however to determine the number of firms operating in the different sectors, together with the corresponding individual investments into which the aggregate K_{t-1} decomposes. The usual assumptions consist either in taking the number of firms in each sector as exogenously fixed, or in supposing free entry and imposing a zero-profit condition. In the first case, if we assume complete symmetry, both in each industry and across industries, we get the level of individual investment $k_{ijt-1} = k_{t-1} = K_{t-1}/m\overline{n}$, with $\overline{n} = n_{it-1} = n_{t-1}$ denoting the exogenous number of firms. In the second (free-entry zero-profit) case, we see from (2) that, disregarding the integer problem, $n_{t-1} = A_t/K_{t-1}$. Then, using (1), with $r_t = 1$ by the zero profit condition, we obtain:

$$n_{t-1} = 1 + a\frac{K_t}{K_{t-1}},$$

where $a \equiv \alpha/\left(1-\alpha\right)$ is the degree of preference for present consumption. Again, individual investment is then given by $k_{t-1} = K_{t-1}/mn_{t-1}$.

Our purpose is to relax these usual assumptions, allowing both for the variability of the number of firms and the variability of profits, and to relate business formation in the economy to strategic investment by the firms. For that purpose, we solve in $\mathbf{k}_{it-1}$ the first stage game as defined by the profit functions $\Pi\left(k_{ijt-1}, \mathbf{k}_{i(-j)t-1}\right)$, given by (2). After some computations (presented in Appendix 1), from the first order condition for a maximum of Π we obtain the symmetric solution k_{it-1} (inside industry i):

$$k_{it-1} = \frac{A_t/m}{n_{it-1}^2} 2\left[\gamma\left(n_{it-1}-1\right)-\delta\right]\left(1-\frac{1}{n_{it-1}}\right) \equiv \frac{A_t/m}{n_{it-1}^2}\frac{1}{\rho\left(n_{it-1}\right)}, \quad (3)$$

introducing the simplifying term $\left[1/\rho\left(n_{it-1}\right)\right]$ and requiring $\gamma > \delta/\left(n_{it-1}-1\right)$ for k_{it-1} to be positive. Also, as shown in Appendix 1, the second order condition gives an additional restriction on the parameters, so that we must assume the two inequalities:

$$0 < \gamma - \frac{\delta}{n_{it-1}-1} < \frac{n_{it-1}}{1+\left(n_{it-1}-1\right)\left(n_{it-1}-2\right)}. \quad (4)$$

The second inequality, limiting the degree γ of internal scale economies, is clearly stronger, the larger the number of firms. It is alleviated whenever the spillover effect, as measured by δ, is stronger.

Assuming symmetry across industries, we get from (2) that the factor of return to capital is equal to the aggregate revenue per unit of capital multiplied by the Lerner's index of degree of monopoly:

$$r_t = \frac{A_t}{K_{t-1}}\frac{1}{n_{t-1}}.$$

This equality enables us to establish, using (1), a general expression for the factor of return on capital as the product of the degree of preference for present consumption, the factor of capital growth and the rate of markup on marginal cost:

$$r_t = a\frac{K_t}{K_{t-1}}\frac{1}{n_{t-1}-1}. \tag{5}$$

Using (3), we also obtain an expression for the same factor that is specific to the strategic investment approach:

$$r_t = \rho\left(n_{it-1}\right). \tag{6}$$

Notice that non-negativity of profits imposes $\rho\left(n_{t-1}\right) \geq 1$, and hence (by (3)):

$$0 < \gamma - \frac{\delta}{n_{t-1}-1} \leq \frac{n_{t-1}}{2\left(n_{t-1}-1\right)^2}, \tag{7}$$

a stronger condition than the second order condition (4).

It seems useful at this point to contrast the different approaches to individual investment, materialized in the different expressions for r_t, by resorting to the graphical representation of the space (n_{t-1}, r_t) in Figure 1. Imbedded in equation (5), there is a family of decreasing curves in this space (two of which appear in the figure), resulting from equilibrium conditions, independently of any assumption on investment decisions. Assuming an exogenous number of firms would lead to a vertical line $n_{t-1} = \overline{n}$ in the same space (not represented in the figure). Imposing the zero profit condition results in the horizontal line $r_t = 1$. Finally, the strategic approach, as rendered by condition (6), gives a steep decreasing curve in the same space[7].

The lower flat curve is the one selected in the family (5) by taking a stationary value of capital, so that its intersection with the steeper decreasing curve, corresponding to the strategic investment approach, occurs at the stationary point $(n^*, r^*) = (2.86, 2.15)$. The former curve shifts upwards as capital increases more and more quickly. Thus, if n_{t-1} is *smaller* than n^*, we must have $K_t/K_{t-1} > 1$, leading to a higher flat curve in the same family. More generally, a smaller number of new firms, leading to a higher degree of monopoly in the next period, induces a larger rate of growth. This is to be contrasted with the exogenous number of firms approach in which n_{t-1} remains fixed at $\overline{n}$, and with the zero profit approach, in which r_t remains fixed at 1. In the latter case in particular, $K_t/K_{t-1} > 1$ if n_{t-1} is *larger* than its stationary value. This switch in the relation between n_{t-1} and K_t/K_{t-1}, when switching from the zero profit approach to our strategic investment approach, will appear to play a crucial role in the shift of the dynamic properties of the two models.

2.4 Cournotian equilibrium

Using the solutions to the family of two-stage games $\{\Gamma(n_{it-1}, A_t)\}$, we can now define our concept of equilibrium for the whole economy. In particular, this will

[7] The curves have been calculated with the parameter values $a = 4$, $\gamma = .3$ and $\delta = .2$.

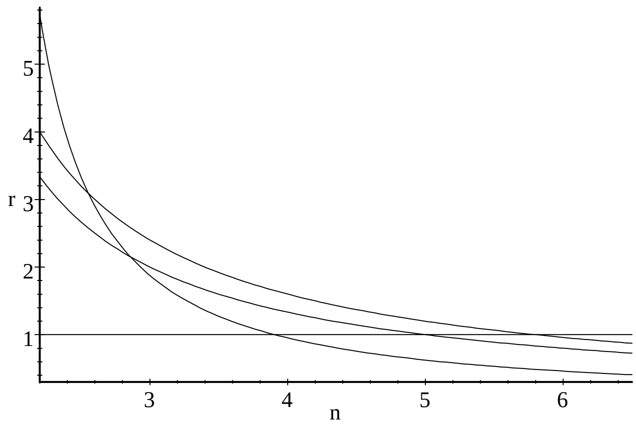

Figure 1:

fix parametrically the number of firms in each industry at each period. Because of the symmetry of the model, we shall only define an equilibrium that assigns identical choices *inside* each industry.

Given $\left(\bar{l}_{t-1}, \left(\overline{n}_{it-1}, \overline{k}_{it-1}\right)_i\right)$, a (non-trivial) symmetric *temporary Cournotian equilibrium* at time t, with anticipated aggregate expenditure $\hat{A}_{t+1}$, future prices $\hat{p}_{t+1}$, and return factor $\hat{r}_{t+1}$, is defined as a solution $\left(l_t, \left(n_{it}, k_{it}\right)_i, A_t, p_t, r_t\right)$ to the following system:

(i) output market clearing at Cournot equilibrium prices in each industry i

$$A_t = \left[\alpha l_t + r_t \left(1-\alpha\right) \bar{l}_{t-1}\right] \overline{L}, \tag{8}$$

with prices

$$p_{it} = \frac{\overline{n}_{it-1}}{\overline{n}_{it-1} - 1} \left(\overline{k}_{it-1}\right)^{-(\gamma+\delta)}; \tag{9}$$

(ii) labor market clearing

$$\sum_{i=1}^{m} \frac{A_t}{m} \left(1 - \frac{1}{\overline{n}_{it-1}}\right) = \alpha l_t \overline{L}, \tag{10}$$

with labor supply l_t such that (for $P_\tau \equiv \prod_{i=1}^{m} p_{i\tau}^{1/m}$, $\tau = t, t+1$)

$$V'\left(l_t\right) \leq \alpha^\alpha \left(1-\alpha\right)^{1-\alpha} \frac{\widehat{r}_{t+1}^{1-\alpha}}{P_t^\alpha \widehat{P}_{t+1}^{1-\alpha}} \text{ (with an equality if } l_t < 1\text{);} \tag{11}$$

(iii) capital market clearing

$$\sum_{i=1}^{m} n_{it} k_{it} = (1-\alpha) l_t \overline{L}, \tag{12}$$

with investment strategically determined by firms

$$k_{it} = \frac{\hat{A}_{t+1}/m}{n_{it}^2} \frac{1}{\rho(n_{it})} \text{ (with } \rho(n_{it}) \geq 1). \tag{13}$$

Determination of the temporary equilibrium can be described as follows. Current prices are entirely determined by past decisions of business and capital formation (condition (9)). Labor (and capital) supply is then determined by current prices and expectations of future prices and returns on capital (condition (11)). Labor market clearing determines employment and aggregate expenditure (condition (10)) and, given past employment, output market clearing determines the equilibrium return factor on capital (condition (8)). Capital market clearing determines aggregate investment (condition (12)), which, given the anticipation of future expenditure, decomposes into business and capital formation in the different industries, according to strategic investment decisions (condition (13)) and a symmetry condition.

We have defined here a concept of temporary equilibrium which relies on an equilibrium condition on strategic investment decisions. Alternative approaches to individual investment lead to different concepts of temporary Cournotian equilibria, with either an exogenous number of firms, or a zero profit condition. The definition is easily accomodated to incorporate these cases. The three concepts will be contrasted in the sequel, when studying the dynamic system obtained under the requirement of an equilibrium satisfying all conditions in each period, and such that all anticipations are correct.

More precisely, a symmetric *intertemporal Cournotian equilibrium* is a sequence of symmetric temporary Cournotian equilibria $(l_t, (n_{it}, k_{it})_i, A_t, p_t, r_t)_{t \geq 1}$, given $(l_{t-1}, (n_{it-1}, k_{it-1})_i)$, and with anticipations $(A_{t+1}, p_{t+1}, r_{t+1})$.

3 The dynamic system

Although we have imposed symmetry inside each industry, symmetry *across* industries cannot be satisfied in general with integer values of n_{it-1}. However, by taking a large enough number of industries, and varying the n_{it-1} appropriately from one industry to the other, symmetry across industries can be preserved approximately. There are multiple ways in which this can be done, of which we choose one as close to perfect symmetry as possible. This will allow us, in the sequel, to treat (abusively) the average number of firms n_{t-1} as a continuous variable and to make computations as if we had symmetric outcomes not alone inside each industry, but also across industries. We thus look for a simple characterization of a "completely" symmetric intertemporal Cournotian equilibrium, leading to a system of two dynamic equations.

Using conditions (8), (10) and (12) and imposing symmetry, we get the first dynamic equation:

$$an_t k_t = r_t (n_{t-1} - 1) n_{t-1} k_{t-1}. \tag{14}$$

Then, using (9), (11) and, again, (12), together with symmetry and fulfilment of expectations, we obtain the second dynamic equation:

$$\begin{aligned} & V'\left(mn_t k_t/\left(1-\alpha\right)\overline{L}\right)\left[\frac{n_t}{n_t-1}k_t^{-(\gamma+\delta)}/r_{t+1}\right]^{1-\alpha} \\ = \; & \alpha^\alpha\left(1-\alpha\right)^{1-\alpha}\left[\frac{n_{t-1}}{n_{t-1}-1}k_{t-1}^{-(\gamma+\delta)}\right]^{-\alpha}. \end{aligned} \tag{15}$$

For simplicity and clarity of the results, we will limit our analysis to the limit case of a linear function V, with positive derivative v. Also, we will admit that the degree of consumers' preference for present consumption $a \equiv \alpha/\left(1-\alpha\right)$ is larger than 1. Finally, we will denote $\beta \equiv \gamma + \delta$ the overall degree of scale economies (both internal and external).

3.1 Closing the system: a general approach

Consider the dynamic system given by equations (14) and (15). One may close it by assuming that one of the variables follows an exogenous trajectory ($n_t = \overline{n}$ or $r_t = 1$ for any t are the usual candidates), or by imposing an additional relation between two variables (and making the system autonomous), as in our strategic investment approach, where $r_t = \rho\left(n_{t-1}\right)$. By making r_t be determined by n_{t-1}, or by making it constant, we obtain a system with two pre-determined variables: In period t, the number n_{t-1} of active firms and their working capital k_{t-1} entirely depend upon past decisions. On the contrary, when fixing the number of firms exogenously, we are left with only one pre-determined variable, k_{t-1}, the factor of return on capital r_t being partially dependent upon the (correct) expectation r_{t+1}.

It is easy to check that the dynamic system has, in any case, a unique steady state, with n^* (or r^*) determined by the first equation, and k^* determined by the second equation. We may then proceed to a local analysis of the dynamic system by log-linearization around the steady state. This analysis will be first conducted in general terms, independently of the particular way of closing the system, by introducing the elasticity $\varepsilon\rho\left(n^*\right)$ of r_t with respect to n_{t-1} at the steady state. This elasticity is calculated from (3) in the strategic investment approach. It is nil under the zero profit condition (imposing $r_t = 1$), and infinite when n_{t-1} is restricted to the exogenous value $\overline{n}$.

By differentiating with respect to the logarithms of n and k both sides of the logarithmic transforms of equations (14) and (15), evaluated at the steady state, we obtain:

$$\begin{aligned} & \begin{bmatrix} 1 & 1 \\ -1/\left(n^*-1\right)-\varepsilon\rho\left(n^*\right) & -\beta \end{bmatrix}\begin{bmatrix} d\ln n_t \\ d\ln k_t \end{bmatrix} \\ & \quad = \begin{bmatrix} 1+n^*/\left(n^*-1\right)+\varepsilon\rho\left(n^*\right) & 1 \\ a/\left(n^*-1\right) & a\beta \end{bmatrix}\begin{bmatrix} d\ln n_{t-1} \\ d\ln k_{t-1} \end{bmatrix}, \end{aligned}$$

leading to the Jacobian matrix

$$J = \frac{1}{1/(n^* - 1) - \beta + \varepsilon\rho(n^*)} \times$$
$$\begin{bmatrix} -\beta & -1 \\ 1/(n^* - 1) + \varepsilon\rho(n^*) & 1 \end{bmatrix} \begin{bmatrix} 1 + n^*/(n^* - 1) + \varepsilon\rho(n^*) & 1 \\ a/(n^* - 1) & a\beta \end{bmatrix}.$$

The trace T and the determinant D of the Jacobian matrix J are:

$$T = 1 - a - \frac{\beta n^*/(n^* - 1) - (a - \beta)\varepsilon\rho(n^*)}{1/(n^* - 1) - \beta + \varepsilon\rho(n^*)},$$

$$D = -a\left(a + T - \frac{(1 + a)\varepsilon\rho(n^*)}{1/(n^* - 1) - \beta + \varepsilon\rho(n^*)}\right).$$

As a first application, consider the free-entry zero-profit case where $r_t = 1$, for any t. We replace $\varepsilon\rho(n^*)$ by 0, and obtain in particular $D = -a(a + T)$. Consequently, the characteristic polynomial $\lambda^2 - T\lambda + D = (\lambda + a)(\lambda - a - T)$ has roots $\lambda_1 = -a < -1$ and $\lambda_2 = a + T$, so that the steady state can only be either a saddle (if $|\lambda_2| < 1$) or a source (if $|\lambda_2| > 1$). Persistent fluctuations are only possible when the parameter β is close to its flip bifurcation value $2/(3n^* - 2) = 2/(1 + 3a)$. Also, notice that this case does not discriminate between internal and external economies of scale, γ and δ appearing exclusively in the sum $\gamma + \delta = \beta$.

If we take the fixed firm number case, *i.e.* $n_t = \overline{n}$ for any t, we must replace $\varepsilon\rho(n^*)$ by ∞, thus obtaining $T = 1 - \beta$ and $D = a\beta > 0$. Thus, the roots of the characteristic polynomial $\lambda^2 - T\lambda + D$ are either non-real or real of the same sign. Since

$$1 - T + D = (1 + a)\beta > 0 \text{ and } 1 + T + D = 2 + (a - 1)\beta > 0,$$

the characteristic roots are both either inside or outside the unit circle, so that the steady state is either a sink or a source. These are essentially the properties that we are going to find in the dynamic system with strategic investment, when log-linearized in the neighborhood of the steady state. A first difference is however that an exogenous number of firms leads, in our specification, to a log-linear dynamic system, whereas by switching to strategic investment we gain non-linearity, hence robustness of persistently fluctuating trajectories. A second difference is that, by construction, this way of closing the system results in a constant number of firms and a constant markup factor on marginal cost, making it impossible to reproduce the characteristic cyclical features we are looking for. Finally, as in the previous case, discrimination between internal and external economies of scale is excluded.

3.2 The strategic investment approach

We come to our strategic investment approach, with $r_t = \rho(n_{t-1})$, as given by (3). As already indicated, there is a unique steady state. From (14), we get

$$n^* = \frac{2a\beta}{2a\gamma - 1}. \tag{16}$$

The scale factor γ appears here independently of the sum $\gamma + \delta = \beta$, a characteristic which opens the possibility of discriminating between internal and external economies of scale. Notice that increasing the former relatively to the latter, while keeping constant the overall degree of economies of scale β, leads to a decrease in the number of firms that are active at the steady state, and hence to a strengthening of market power.

Since n^* cannot be smaller than 2 and must satisfy the non-negative profit condition (7), there is a restriction on the admissible values of parameters a, β and γ, additional to $a > 1$, $\beta > 0$ and $\gamma > 0$ (and taking into account the condition $0 \leq \delta \equiv \beta - \gamma \leq \gamma$):

$$\max\left\{\frac{\beta}{2}, \frac{1}{2a} + \frac{\beta}{1+a}\right\} \leq \gamma \leq \min\left\{\beta, \frac{1}{2a} + \frac{\beta}{2}\right\}. \tag{17}$$

This condition imposes a lower bound on β: $\beta \geq (1+a)/2a^2$.

As to steady individual investment, we obtain, using (15):

$$k^* = \left[\frac{v}{\alpha}\frac{n^*}{(n^*-1)^\alpha}\right]^{1/\beta}.$$

Notice that, assuming a large enough population $\overline{L}$ is enough to ensure attainability of aggregate steady investment $K^* = mn^*k^*$.

In order to analyze the properties of the log-linearized system in the neighbourhood of the steady state, we must replace $\varepsilon\rho(n^*)$, in the Jacobian matrix J, by its value, calculated from (3):

$$\varepsilon\rho(n^*) = -\frac{1}{n^*-1} - \frac{\gamma n^*}{\gamma n^* - \beta} = -\frac{1}{n^*-1} - 2a\gamma = -\frac{2a\gamma - 1}{2a\beta - (2a\gamma - 1)} - 2a\gamma.$$

Clearly, this elasticity increases in absolute value as internal economies of scale, given by γ, become more important relatively to external economies. We shall take advantage of this property, by assuming that the degree of overall scale economies, as measured by β, is given, while varying γ and correspondingly adjusting δ. We thus obtain the following values of the trace and determinant of J:

$$T = \frac{1}{2a\gamma + \beta}\left[a\frac{2a\gamma - 1}{2a\beta - (2a\gamma - 1)} + 2(1 - a\gamma)\beta + 2a\left(\gamma - \frac{\beta}{2}\right)\right],$$

$$D = \frac{a}{2a\gamma + \beta} \left[\frac{2a\gamma - 1}{2a\beta - (2a\gamma - 1)} - 2(1 - a\gamma)\beta \right].$$

Consider again the characteristic polynomial $\lambda^2 - T\lambda + D$ of the Jacobian matrix J. We see that, for $\lambda = 1$ and $\lambda = -1$, the polynomial takes the values:

$$\begin{aligned} &1 - T + D \\ &= \frac{(1+a)\beta}{2a\gamma + \beta}(2a\gamma - 1) > 0 \text{ and} \\ &1 + T + D \\ &= 1 + \frac{2}{2a\gamma + \beta}\left[a\frac{2a\gamma - 1}{2a\beta - (2a\gamma - 1)} - (a-1)(1 - a\gamma)\beta + a\left(\gamma - \frac{\beta}{2}\right)\right] > 0, \end{aligned}$$

respectively. Indeed, the last expression is either larger than 1 (if $a\gamma \geq 1$) or larger than $1 - T + D > 0$ (if $a\gamma < 1$, entailing $T > 0$). We conclude that the characteristic roots must either be non-real or belong both to one of the intervals $]-\infty, -1[$, $]-1, 1[$ or $]1, \infty[$, so that the steady state is either a sink (if $D < 1$) or a source (if $D > 1$). Also if, by fixing the parameters a and β, and by varying γ along the admissible interval given by (17), we make the value of D go through 1, we can establish existence of a Hopf bifurcation, and hence of periodic or quasi-periodic trajectories along a closed orbit in the neighbourhood of the steady state, for some values of γ close to its bifurcation value. Clearly, the determinant D becomes negative as γ approaches $2a$, and tends to infinity as γ tends to $\beta + 1/2a$. Hence, the difficulty in establishing conditions for occurrence of a Hopf bifurcation lies in the restrictions on the admissible values of γ. The purpose of the following proposition is precisely to formulate such conditions.

Proposition 1 *Consider the dynamic system given by equations (14) and (15), with strategic investment ($r_t = \rho(n_{it}), \forall t$) and linear labor disutility. Take the degree γ of internal scale economies as a bifurcation parameter, belonging (by (17)) to the interval $[\max\{\beta/2, 1/2a + \beta/(1+a)\}, \min\{\beta, 1/2a + \beta/2\}]$, with $a > 1$ and $\beta \geq (1+a)/2a^2$. Take $a_1 \simeq 1.9032$, $a_2 \simeq 2.4142$ and $a_3 \simeq 2.4498$.*

Then, the (unique) steady state of this dynamic system can never be a saddle, and: (i) the steady state is a sink *whatever the admissible value of γ for economies of scale weak enough, namely for values of β smaller than a critical value $\widetilde{\beta}_2(a)$ if $a \leq a_2$, and smaller than $\widehat{\beta}(a)$ if $a > a_2$ (with $\widetilde{\beta}_2(a_2) = \widehat{\beta}(a_2)$); also, for β in the interval $\left]\widetilde{\beta}_1(a), \widetilde{\beta}_2(a)\right[$ if a lies in the interval $]a_2, a_3[$ (with $\widehat{\beta}(a) < \widetilde{\beta}_1(a)$ in this interval); (ii) the steady state is a* source *whatever the admissible value of γ for economies of scale strong enough, namely for values of β larger than a critical value $\underline{\beta}(a)$ if $a \leq a_1$, and larger than $\overline{\beta}(a)$ if $a > a_1$ (with $\underline{\beta}(a_1) = \overline{\beta}(a_1)$); (iii) the steady state switches from a sink to a source, as the parameter γ of internal scale economies is increased through the Hopf bifurcation value $\gamma_H(\beta, a)$, for intermediate economies of scale (for β in the interval $\left]\widetilde{\beta}_2(a), \underline{\beta}(a)\right[$ if $a \leq a_1$, for β in the interval $\left]\widetilde{\beta}_2(a), \overline{\beta}(a)\right[$ if $a_1 < a \leq a_2$, for*

β in the union of intervals $\left]\widehat{\beta}(a), \widetilde{\beta}_1(a)\right[\cup \left]\widetilde{\beta}_2(a), \overline{\beta}(a)\right[$ *if* $a_2 < a < a_3$*, and for* β *in the interval* $\left]\widehat{\beta}(a), \overline{\beta}(a)\right[$ *if* $a \geq a_3$*).*

Proof. See Appendix 2, where the functions referred to in the statements of the Proposition are specified. ■

3.3 Interpretation and simulation of persistent fluctuations

To conclude this section, we consider a periodic or quasi-periodic trajectory along a closed orbit in the neighborhood of the steady state, close to a Hopf bifurcation. To understand the source of the fluctuating pattern, assume an increase, in period $t-1$, of individual investment k_{t-1} relatively to its stationary value k^*. If $n_{t-1} = n^*$, this increase results in identical proportional increases in present aggregate investment K_{t-1} and, in the next period, in aggregate profits $r_t K_{t-1}$ (recall that r_t is entirely determined by $n_{t-1} = n^*$), in aggregate expenditure by the old, in employment L_t, and finally in future capital formation $K_t = (1-\alpha) L_t$. The increased aggregate investment in period t induces net business formation ($n_t > n_{t-1} = n^*$), which depresses the expected markup on marginal cost, and hence the expected return on investment, so that individual capital formation regresses towards its stationary value: $k_{t-1} > k_t \geq k^*$, anyway in accordance with the fact that an increasing number of firms is accumulating the same aggregate capital: $K_t = K_{t-1}$. Now, this higher number of firms created in period t, and active in period $t+1$, results in a lower degree of monopoly, in weaker returns on capital, and possibly in less employment and less aggregate investment. This Schumpeterian feature of our model, that monopoly power enhances investment, so that competition has adverse effects on capital formation, is the main factor of the fluctuating pattern of non-stationary trajectories. The dampened, explosive or persistent character of such fluctuations is then partly dependent on the degree of scale economies, and in particular on the relative importance of their internal component, as compared to the external one associated with spillover effects[8].

The dynamics involved here are rather complex. Yet, the model is simple enough, so that most relations between the relevant variables are straightforward. Aggregate investment and employment are the same variable, up to a linear transformation. The markup factor and the factor of return on capital depend exclusively, according to decreasing functional relations, upon the

[8] The mechanism built in the model has some flavour of the predator-prey relationship in simple ecological models. In fact, a discrete time version of the Lotka-Volterra system happens to be a borderline case of our dynamic system. It is indeed easy to check that, if we take investment to have identical internal and external effects, that is $\gamma = \delta = \beta/2$, the second element in the first diagonal of the Jacobian matrix J_{kk} is equal to zero. Assuming in addition $a = 2\beta = \sqrt{6}$ leads to the required values $J_{nn} = 0$, $J_{nk} = 1$ and $J_{kn} = -1$. As well known, this Jacobian matrix characterizes a four-period cycle, along which the channels of dynamic influence are reduced to the driving effect of an increase in k_{t-1} upon n_t, and the depressing Schumpeterian effect of an increase in n_t (with the associated erosion of monopoly power) upon k_{t+1}.

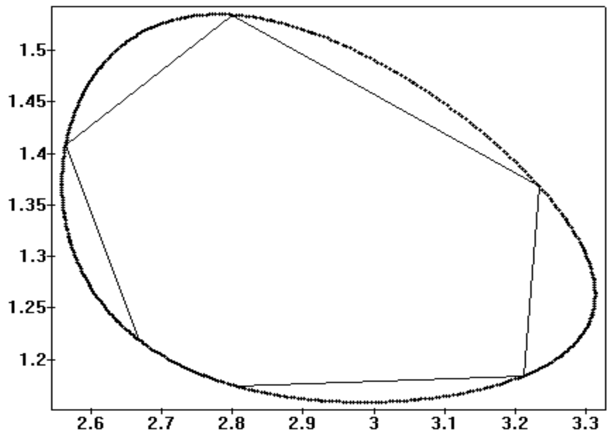

Figure 2:

number of firms created in the previous period. The price level is a decreasing function of that number and of previous individual investment. The real wage is just the reciprocal of the price level. The two main difficulties in assessing co-variations of the relevant variables lie, first in the existence of lagged effects, second in the fact that aggregate investment may be quite differently decomposed into business and individual capital formation. The number of new firms and each firm's investment move sometimes in the same sense, sometimes in opposite senses, entailing effects on the price level and correlated variables that are reinforcing in the former case, counteracting in the latter.

In spite of somewhat complex interactions, simulations display some typical co-movements. Differences in simulations concern mainly periodicity, which is quite sensitive to parameter perturbations. Thus, for illustration, we content ourselves with a single simulation, obtained with a configuration of parameter values close to a Hopf bifurcation ($a = 4$, $\gamma = 0.3$ and $\delta = 0.2$, values which have already been used in Figure 1), and giving a cycle with high periodicity along a closed orbit in the (n, k)-space (see Figure 2). The five lines in the figure link up the six first points in the simulated trajectory (with $(n_1, k_1) = (2.67, 1.22)$.

Figure 3 exhibits the following general co-movements of some of the relevant variables: (i) business formation (i.e. the number of firms created) is pro-cyclical, lagged with respect to the output; (ii) net profits (or dividends) are also pro-cyclical, but with a lead with respect to the output; (iii) the price level is mainly counter-cyclical, so that the real wage is again pro-cyclical.

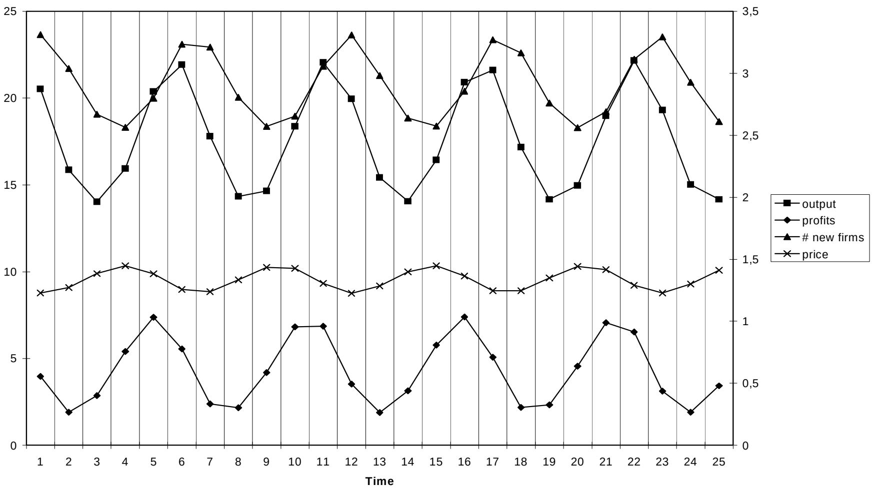

Figure 3:

4 Conclusion

These dynamic properties, which we have brought to the fore on the basis of our Cournotian model, seem to be in accordance with some stylized facts that have been identified in the empirical literature and tend to be widely accepted.

As far as the pro-cyclicality of the real wage is concerned, it goes back to the discussion by Dunlop [11], Tarshis [23] and Keynes [16], leading to the defense of a (weak) positive co-movement of real wages and output largely attributed to the varying "degree of the imperfection of competition". The issue has been the subject of quite a number of controversies, but recent evidence is mainly in favor of such weak pro-cyclicality (surveys are given by Abraham and Haltiwanger [1] and Brandolini [6]). Market power fluctuations are linked to the same co-movements. Rotemberg-Saloner [19] and Rotemberg-Woodford [20] develop both theoretical arguments (as mentioned in the introduction) and empirical ones in favor of more competition during booms. In the same direction, Bils [5] shows, for U.S. manufacturing, that marginal cost relative to price tends to be high in periods of high demand. Finally the pro-cyclicality of business formation was already an argument in the pre-war literature. Robinson [18], Abramovitz [2] and Kalecki [15] insisted on the increasing number of producers and their tendency towards less collusive behavior in a boom. More recent empirical investigations have gone in the same direction, as in Davis and Haltiwanger [10] and Portier [17].

These empirical conclusions may be used to argue that a Cournotian approach to macro-dynamics can be fruitful and relevant to study business cycles and business formation. In our model, endogeneity of business formation is simply linked to strategic investment decisions in cost-reducing technological know-how. With less firms (having more market power), profit prospects per firm tend to be higher, so that investment increases, reducing costs (and the more so, the higher the spillovers), and eventually more firms are created. We thus get a dynamic behavior with some Schumpeterian flavor. However, this is only one facet of the Schumpeterian dynamics, the one linked to business formation, varying market power and cost fluctuations. No accumulation of knowledge, leading to new marketable technology or new products, has been introduced here. To incorporate, in such an oligopolistic model of the business cycle, endogenous growth associated to vertical innovations is an important topic for further research.

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A 1. The investment game

To compute the subgame perfect equilibrium value of the first stage strategic variables, we simply take the first order conditions for maximization of Π (as given by (2)) with respect to k_{ijt-1}, to obtain:

$$\begin{aligned}\frac{\partial \Pi}{\partial k_{ijt-1}} &= \frac{2A_t}{m}\left[1-\frac{(n_{it-1}-1)c_{ijt}}{\sum_{j'} c_{ij't}}\right]\frac{(n_{it-1}-1)c_{ijt}}{k_{ijt-1}\left(\sum_{j'} c_{ij't}\right)^2} \\ &\quad \times \left[\sum_{j'\neq j} c_{ij't}\left(\frac{\partial c_{ij't}}{\partial k_{ijt-1}}\frac{k_{ijt-1}}{c_{ij't}} - \frac{\partial c_{ijt}}{\partial k_{ijt-1}}\frac{k_{ijt-1}}{c_{ijt}}\right)\right] - 1 \\ &= \frac{2A_t\left(n_{it-1}-1\right)}{m}\left(\gamma - \frac{\delta}{n_{it-1}-1}\right)\frac{1}{k_{ijt-1}}\left[1-\frac{(n_{it-1}-1)c_{ijt}}{\sum_{j'} c_{ij't}}\right] \\ &\quad \times \frac{c_{ijt}}{\sum_{j'} c_{ij't}}\left[1-\frac{c_{ijt}}{\sum_{j'} c_{ij't}}\right] - 1 = 0.\end{aligned}$$

Because of the symmetry assumption, we are interested in the symmetric solution, namely:

$$k_{it-1} = 2\frac{A_t}{m}\frac{\gamma\left(n_{it-1}-1\right)-\delta}{n_{it-1}^2}\left(1-\frac{1}{n_{it-1}}\right),$$

requiring $\gamma > \delta/\left(n_{it-1}-1\right)$ for this solution to be positive.

Next, we must check that the second order condition (that the derivative of the profit function be decreasing or, equivalently, that the elasticity of its variable term be negative) is satisfied at the symmetric solution. At the symmetric solution, with $k_{ij't-1} = k_{it-1}$ for any $j' \neq j$, denoting

$$s_{it} \equiv c_{ijt}/\textstyle\sum_{j'} c_{ij't} = \dfrac{1}{1+(n_{it-1}-1)(k_{ijt-1}/k_{it-1})^{\gamma-\delta/(n_{it-1}-1)}},$$

we see that the elasticity of s_{it} with respect to k_{ijt-1} is given by

$$-(1-s_{it})\left[\gamma-\delta/(n_{it-1}-1)\right].$$

Consequently, the elasticity of the variable term of the derivative of the profit function is negative at the symmetric solution if and only if

$$\left[\gamma-\delta/(n_{it-1}-1)\right](1-s_{it})\left[1-\left(\frac{(n_{it-1}-1)\,s_{it}}{1-(n_{it-1}-1)\,s_{it}}+\frac{s_{it}}{1-s_{it}}\right)\right]>-1.$$

Replacing s_{it} by $1/n_{it-1}$ in this expression leads to the restriction given in the text, namely

$$\gamma-\frac{\delta}{n_{it-1}-1}<\frac{n_{it-1}}{1+(n_{it-1}-1)\,(n_{it-1}-2)}.$$

It is easy to check that the left hand side of the former inequality is decreasing in s_{it}. As s_{it} is itself decreasing in k_{ijt-1}, the derivative of the profit function must be negative whenever $k_{ijt-1}>k_{it-1}$. Hence, the profit function has at most another critical value besides the symmetric solution, and that critical value must be a minimum, in the interval $]0,k_{it-1}[$. Since profit is nil at zero, the unique solution is $k_{ijt-1}=k_{it-1}$, if it entails a positive profit.

A 2. Proof of the Proposition

It is easy to establish the equivalence of the following inequalities

$$D=\frac{a}{(2a\gamma-1)+(1+\beta)}\left[\frac{2a\gamma-1}{2a\beta-(2a\gamma-1)}+\beta\,(2a\gamma-1)-\beta\right]\gtrless 1$$

and

$$G\,(2a\gamma-1;\beta,a)\equiv(2a\gamma-1)^2\,(1-a\beta)+(2a\gamma-1)\,(a\beta\,(2a\beta-1)+\beta+1+a)\\ -2a\beta\,(a\beta+\beta+1)\gtrless 0.$$

Of course, we are only interested in admissible values of $2a\gamma-1$, such that:

$$\max\left\{a\beta-1,\frac{2a\beta}{1+a}\right\}\leq 2a\gamma-1\leq\min\left\{2a\beta-1,a\beta\right\}.$$

We must distinguish two cases. If $a\beta\leq 1$, $G\,(\cdot;\beta,a)$ is clearly increasing in the admissible interval, and has a single positive root, denoted $2a\gamma_H\,(\beta,a)-1$. The same is true in the case $a\beta>1$, since $G\,(\cdot;\beta,a)$ is strictly concave, and $G'\,(a\beta;\beta,a)=(1+a)\,(1+\beta)>0$, whereas $a\beta$ is the upper end of the admissible interval ($a\beta<2a\beta-1$, in this case). The root we are interested in is now the smallest one (if real), and it has the same expression as $\gamma_H\,(\beta,a)$ in the former case (so that we keep the same notation). Thus, $G\,(\cdot;\beta,a)$ increases with γ and eventually becomes positive (D larger than 1). The steady state will accordingly become unstable, the dynamic system undergoing a Hopf bifurcation at $\gamma=\gamma_H\,(\beta,a)$, provided this root belongs to the interior of the admissible interval. Otherwise, the steady state is always a sink, if $\gamma_H\,(\beta,a)>\min\{2a\beta-1,a\beta\}$, or a source, if $\gamma_H\,(\beta,a)<\max\{a\beta-1,2a\beta/(1+a)\}$.

(i) The conditions for the steady state to be always a sink, whatever the (admissible) value of γ, are that $G(2a\beta - 1; \beta, a) < 0$, if $a\beta \leq 1$, and that $G(a\beta; \beta, a) < 0$, if $a\beta > 1$. In the first case, we have:

$$G(2a\beta - 1; \beta, a) = 2a^2\beta^2 + \left(2a^2 - 4a - 1\right)\beta - a \equiv \widehat{B}(\beta; a) < 0.$$

This condition is satisfied if and only if $(1+a)/2a^2 < \beta < \widehat{\beta}(a)$, where $\widehat{\beta}(a)$ is the positive root of $\widehat{B}(\cdot; a)$. Notice that $\widehat{B}\left((1+a)/2a^2; a\right) = -(3+a)/2a < 0$, so that $\widehat{\beta}(a) > (1+a)/2a^2$. In the second case, we have:

$$G(a\beta; \beta, a) = a\beta\left[a^2\beta^2 - (2a+1)\beta + a - 1\right] \equiv \widetilde{B}(\beta; a) < 0.$$

This condition is verified if and only if $\widetilde{\beta}_1(a) < \beta < \widetilde{\beta}_2(a)$, where $\widetilde{\beta}_1(a)$ and $\widetilde{\beta}_2(a)$ are the smaller and larger roots of $\widetilde{B}(\cdot; a)$, respectively. Obviously, it can only be satisfied if this interval is not empty, that is, if $-4a^3 + 8a^2 + 4a + 1 > 0$ ($a < a_3 \simeq 2.4498$). Notice that $\widehat{B}(1/a; a) = \widetilde{B}(1/a; a) = \left(a^2 - 2a - 1\right)/a \gtreqless 0$ if and only if $a \gtreqless a_2 = 1 + \sqrt{2} \simeq 2.4142$. As a consequence, if $a \leq a_2$, $\widetilde{\beta}_1(a) \leq 1/a \leq \widehat{\beta}(a)$, and the two conditions can be contracted into the single inequality $\beta < \widetilde{\beta}_2(a)$. If $a > a_2$, $\widehat{\beta}(a) < 1/a < \widetilde{\beta}_1(a)$, and β may belong to either one of two disconnected intervals, for the steady state to be a sink independently of the value of γ: $\left](1+a)/2a^2, \widehat{\beta}(a)\right[$ and $\left]\widetilde{\beta}_1(a), \widetilde{\beta}_2(a)\right[$. The second interval is empty if $a \geq a_3$.

(ii) The conditions for the steady state to be always a source, whatever the (admissible) value of γ, are that

$$G(2a\beta/(1+a); \beta, a) > 0, \text{ if } \beta \leq (1/a)(a+1)/(a-1) \equiv \underline{\beta}(a),$$

and that $G(a\beta - 1; \beta, a) > 0$, if $\beta > \underline{\beta}(a)$. This corresponds in the first case to

$$G(2a\beta/(1+a); \beta, a) \equiv 2a^2\beta^2\left[\frac{2a(a\beta - 1)}{(1+a)^2} - 1\right] > 0,$$

which is verified if and only if $(1/a)\left[1 + (a+1)^2/2a\right] \equiv \underline{\beta}(a) < \beta \leq \underline{\underline{\beta}}(a)$, which is possible only if $a^3 + a^2 - 5a - 1 < 0$ ($a < a_1 \simeq 1.9032$). In the second case, we have

$$G(a\beta - 1; \beta, a) = a^3\beta^3 - a(2a+1)\beta^2 + \left(a^2 - 3a - 1\right)\beta - a \equiv \overline{B}(\beta; a) > 0.$$

As $\overline{B}(1/a; a) = -4 - 2/a < 0$ and $\overline{B}''(1/a; a) = 2a(a-1) > 0$, $\overline{B}(\cdot; a)$ has at most one root, say $\overline{\beta}(a)$, larger than $1/a$. As $\beta > 1/a$ in this case, the condition $\overline{B}(\beta; a) > 0$ is equivalent to $\beta > \max\left\{\underline{\underline{\beta}}(a), \overline{\beta}(a)\right\}$. Notice that $\underline{\underline{\beta}}(a) < \overline{\beta}(a)$ if and only if

$$\overline{B}((1/a)(a+1)/(a-1); a) = -\frac{2}{(a-1)^3}\left[a^3 + a^2 - 5a - 1\right] < 0,$$

or $a > a_1$. Thus, the two conditions can be contracted into $\beta > \underline{\beta}(a)$ if $a \leq a_1$, into $\beta > \overline{\beta}(a)$ if $a > a_1$, with $\underline{\beta}(a_1) = \underline{\underline{\beta}}(a_1) = \overline{\beta}(a_1)$.

(iii) Conditions on β and a such that $\gamma_H(\beta, a)$ belongs to the interior of the admissible interval, so that a Hopf bifurcation will occur, are just established by complementarity with respect to the preceding conditions.