Stable Dynamics in Transportation Systems

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Abstract

We present a new class of transportation systems, the stable dynamics models, which provides a natural link between the static and dynamic traffic network models. They can be seen as steady states of dynamic networks (flows are constant in time). These models turn out to be very easy to study analytically for simple networks. Moreover, they can be extended for large networks, for which efficient (and standard) algorithms to solve for equilibrium can be derived. We also present a formulation for the endogenous origin-destination case. Finally, this class of models leads to very natural and simple calibration methods.

Key words: Convex optimization, static models, shortest path, variational inequalities, stable dynamics.

1 Introduction

The static models are still very popular in transportation science as well as for practical applications all around the world [4]. A key element of those models is the specification of the link performance functions (congestion law). Consider a link of a transportation network. The independent variable is the input flow on this link and the dependent variable is the travel time to cross this link (or crossing time). This relation, known as the travel cost function or the impedance function, states that the travel time should be an increasing (or at least non-decreasing) function of the input flow (see [3], [11] and [6]). Many studies have been performed to estimate such functions (e.g. see [2] and [5]). With the inherent problem that (1) in many situations of practical interest, the flows (as well as the travel times) are rarely constant over time, and (2) when the field estimates of the cost functions are used in large transportation software they need some re-calibration. Calibration of those performance functions is a long and expensive task that is often hard to reproduce for many city planners. First, because these specialized skills may not be available (especially in developing countries) and second, because the necessary data (traffic counts, etc.) may neither be available. For these reasons, we propose the exploration of a new avenue of research which we argue is much simpler than current practice from both the computational and the data requirements points of view.

The idea is to base the description of the congestion cost only on a small number of parameters which have an *unambiguous* physical interpretation. We show below that the knowledge of the length of the road, the maximum speed and the capacity of the road (defined as maximal flow) provide enough information to determine the *equilibrium* travel time. This may appear unfamiliar for the reader. Indeed, this information is not enough to compute the travel time out of the equilibrium path. However, it appears that the missing information is provided by a behavioral rule, the Wardrop principle, which states that the users select the minimum generalized cost shortest path.

The models we consider in this paper are based on the mathematical background developed in [9]. However, we present a version adapted to the study of congestion in large congested urban areas. Moreover, on simple examples, we show that the solutions of our models are the only *logical* solutions, which are consistent with some minimal set of behavioral assumptions and physical constraints.

The paper is organized as follows. In Section 2, we introduce the basic notation and present the standard results on static models (along Beckmann's formulation). In Section 3, we study several examples of simple networks and derive their equilibrium solutions. Two of these examples are related to Braess paradox. We show that the negative effect of an additional arc can appear even in the simple triangular network. Moreover, we show that this effect *cannot* be seen in the standard framework for Braess paradox, in which the impedance functions are assumed to be linear in arc flows. In Section 4, we introduce the mathematical formulation of the Stable Dynamic model for general networks and provide it with some intuitive explanation. The social optimum solution is also described. Finally, we propose a formulation where the origin-destination matrix is computed endogenously in a consistent way. Conclusions are provided in Section 5. The proofs of all statements of the paper are provided in Appendix. Note that the existence results can be obtained directly from the general theory for nonatomic games [12]. However, provide the corresponding statements with the proofs based on the technique of Convex Analysis since we need to characterize the solutions in a constructive way.

2 Historical remarks

From the very beginning, the development of the models for congested transportation network was based on the following *Wardrop principle:*

- *At equilibrium, all routes, which are used for traveling between two nodes A and B of the network, have the same travel time.*
- *None of the unused routes between A and B has a smaller travel time.*

This principle perfectly fits our intuition (see [15]). Indeed, if there is a route between A and B, which is faster than the used routes, then the drivers will try to use it. So, any steady state of the transportation flows must ultimately satisfy the above rule.

In order to describe the standard models, we need to introduce a description of the transportation network. Suppose we have a transportation network $\mathcal{R}$ composed by some set of nodes $\mathcal{N}$ and the set of directed arcs $\mathcal{A}$. For this network we define the set of origin-destination pairs (OD-pairs):

$$\mathcal{OD} = \{(i,j),\ i,j \in \mathcal{N},\ i \neq j\}.$$

Each OD-pair (i,j) generates a demand $d_{(i,j)}$. This demand is traditionally considered as an average flow of drivers, which need to travel from some node i to another node j; so the demand is a non-negative real number. In order to describe the travel cost of a trip we introduce some cost functions (*travel time functions*)

$$c_\alpha(f_\alpha), \quad f_\alpha \geq 0, \quad \alpha = 1, \ldots, m \ \equiv \ |\mathcal{A}|,$$

which can be specific for each arc. In order to ensure the existence of the solution, we assume that the cost function is *non-decreasing in the flow f_α on the arc α*. (In a more general case we can assume that the vector function $c(f) = (c_1(f), \ldots, c_m(f))$ with $f \in R^m_+$ depends on all flows in the network and that is a monotone operator in f.)

Further, for each OD-pair (i,j) let us define the set of the routes connecting i with j:

$$\left\{ a^{(r)}_{(i,j)} \in R^m,\ r = 1, \ldots r_{(i,j)} \right\},$$

where the α-component of the vector $a^{(r)}_{(i,j)}$, $\alpha = 1, \ldots, m$, is equal to one if the arc α is included in this route; otherwise this component is zero. For the sake of simplicity, we denote $A_{(i,j)}$ the matrix with the columns $a^{(r)}_{(i,j)}$:[1]

$$A_{(i,j)} = \left(a^{(1)}_{(i,j)}, \ldots, a^{(r_{(i,j)})}_{(i,j)} \right).$$

Finally, for each OD-pair (i,j) we introduce the set of feasible flows

$$\Delta_{(i,j)} = \left\{ F_{(i,j)} \in R^{r_{(i,j)}}_+ : \sum_{r=1}^{r_{(i,j)}} F^{(r)}_{(i,j)} = d_{(i,j)} \right\}.$$

[1]Sometimes this matrix is called the arc-route incidence matrix.

Now, for any choice of the route flows

$$F = \{F_{(i,j)}\}_{(i,j)\in\mathcal{OD}} \quad \in \quad \Delta \equiv \prod_{(i,j)\in\mathcal{OD}} \Delta_{(i,j)},$$

we can compute the vector of arc flows in the network:

$$f = \sum_{(i,j)\in\mathcal{OD}} A_{(i,j)} F_{(i,j)} \equiv AF,$$

where $A = \{A_{(i,j)}\}_{(i,j)\in\mathcal{OD}}$. Therefore, for each OD-pair (i,j) the choice of the vector F results in the following cost of the routes:

$$C_{(i,j)}(F) = A_{(i,j)}^T c(AF).$$

Thus, we can pose the well known *static equilibrium problem*:

Find a flow pattern $F \in \Delta$ *such that:*

$$F_{(i,j)}^{(k)} > 0 \quad \Rightarrow \quad C_{(i,j)}^{(k)}(F) = \min_r C_{(i,j)}^{(r)}(F).$$

Clearly, this problem is just a formal mathematical reformulation of the Wardrop principle.

It is known that this problem can be written in the form of variational inequality [8]:

$$\text{find } F^* \in \Delta : \quad \langle C(F^*), F - F^* \rangle \geq 0 \quad \forall F \in \Delta, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in the corresponding space. In the case when the cost function of the arc depends only on the flow of this arc, the variational inequality (1) is equivalent to a minimization problem, which is known as *Beckmann model*:

$$\min_{f,F} \left\{ \sum_{\alpha\in\mathcal{A}} \int_0^{f_\alpha} c_\alpha(\tau) d\tau \ : \ f = AF, \ F \in \Delta \right\}. \tag{2}$$

If all $c_\alpha(\cdot)$ are non-decreasing functions, the problem (2) is convex, so we can guarantee the existence of the solution.

The approach we have described is completely standard and during many years there have been no major attempts to revise it. It was also shown that the equilibrium flows in congested networks can exhibit very non-trivial behavior (Braess paradox). Namely, it appears that construction of a new road in the network can *increase* the equilibrium travel time. After finding an example of such a situation in a real transportation network (Germany), the static models became one of the main tools for transport engineers. Now almost all existing software packages (see, for example, [7], [4]) are using this theory for analyzing traffic congestion in the real urban networks.

However, in this paper we present an alternative formulation to the original Beckmann's model. The main motivation for our research is that the traditional travel time model describes a situation which hardly happens in situations of practical interest. Indeed, the intuition confirms that a large flow corresponds to a fast movement. Then, the travel time cannot be too large. On the contrary, if a route is congested, then the flow can be very close to zero, but the travel time can be very large. These simple observations show that the assumption that the travel time is an *increasing function* of flow is quite artificial. Moreover, the date we get from the traffic counts very often does not support this assumption as well. On the other hand, as we argued in Introduction, it is clear that the flow on a given arc cannot be used as the main modeling parameter. If we say that the flow is small, we should also explain somehow why it is so: Either the road is congested or nobody is using it.

3 Logical solutions

We start our by discussing several simple examples. We will try to restrict ourselves by the minimal amount of reliable parameters. Clearly, for each arc α of the network we can estimate the minimal free traffic travel time $\bar{t}_\alpha$. Let us introduce also another characteristic of the arc, the maximal output flow $\bar{f}_\alpha$. In urban network the maximal flow depends on the number of lanes of the road, the duration of the green light at the intersection, the weather conditions, etc. Surprisingly enough, even from this restricted (and easily available) information we can retrieve the equilibrium travel time. Formally, we will derive the equilibrium solution from the following behavioral assumption.

Assumption 1 *Given the arc travel time pattern $t = \{t_\alpha\}_{\alpha\in\mathcal{A}}$ in the network $\mathcal{N}$, each driver chooses one of the shortest (with respect to t) paths to travel from his origin to the destination.*

Moreover, we introduce an assumption characterizing the congestion pattern.

Assumption 2 *The flow f_α, observed on arc α, never exceeds $\bar{f}_\alpha$. If $f_\alpha < \bar{f}_\alpha$, then the travel time on this arc is equal to $\bar{t}_\alpha$ (that is the free traffic travel time). If $f_\alpha = \bar{f}_\alpha$, then the travel time t_α can be any value greater or equal to $\bar{t}_\alpha$.*

In the remainder of this section, we examine how these assumption work in some simple cases.

3.1 Two routes in parallel

Let us consider a simple (sub)network consisting of two nodes A and B connected by two directed parallel arcs. Assume that for these two arcs we know the minimal travel time $\bar{t}_1$ and $\bar{t}_2$, $\bar{t}_1 < \bar{t}_2$, and the upper bound on the maximal output flows $\bar{f}_1$ and $\bar{f}_2$. In addition, let us introduce an arc, which goes out of the node B and on which we can measure the flow f. In order to simplify our analysis, let us assume that the flow bounds on this and on the second arcs are big enough to carry out any flow without congestion. It appears, that in the most cases the measurement f gives us a possibility to *reconstruct* the equilibrium flows and travel time on the first two arcs.

Indeed, if $0 \le f < \bar{f}_1$, then the capacity of the shortest arc is enough to carry all flow. Therefore, in this case the travel time t_1 on the first arc is equal to $\bar{t}_1$ (Assumption 2). Thus, in view of Assumption 1, nobody will use the second arc ($f_1 = f$, $f_2 = 0$).

Now, assume that $f > \bar{f}_1$. Such measurement gives us an evidence that the drivers started to use the second arc (Assumption 2). The *only* explanation of this phenomena, *compatible* with Assumption 1, is that the travel time on the first arc, t_1, *was somehow* *increased* up to the level $\bar{t}_2$ (see also the discussion of dynamics - or adjustment processes - in the context of static models in [13]). Thus, in this case we also can write down the equilibrium solution:

$$f_1 = \bar{f}_1, \quad f_2 = f - \bar{f}_1, \quad t_1 = t_2 = \bar{t}_2.$$

Note that we managed to get this solution saying *nothing* about the reason for the delay on the first arc. Nevertheless, we know that the delay is $\bar{t}_2 - \bar{t}_1$. If we are interested in a particular mechanism of arising of congestion, we need to extend our model and introduce some additional assumptions. For example, if we work with urban network, we can assume that the only reason for the arc delay can be a queue, growing in the end of each arc. Then, introducing additional parameters of the model (the length of the arc, the average length of a car, number of lanes on the road, etc.) we can estimate the size of the queue. For inter-urban highways we have to use another model of congestion. However, in any case this particular mechanics *does not* affect the equilibrium travel time and the delay on the arc. Indeed, in our example we can see that the delay on the first arc depends only on the performance of the surrounding network structure.

Let us check what happens with our solutions in dynamics. Assume that the flows in our network change in time τ. It is more convenient to work with the total outgoing flow $f_0(\tau)$ from the node A. Our dynamic characteristics will have the following sense:

- $f_i(\tau)$, $i = 1, 2$ - the input flow on arc i at time τ.
- $t_i(\tau)$, $i = 1, 2$ - the travel time for drivers entering the arc i at time τ.

Clearly, $f_0(\tau) = f_1(\tau) + f_2(\tau)$. We can assume also, without loosing too much in generality, that the travel time functions $t_i(\tau)$ are continuous in τ.

Let us look first at the following dynamics: there is a critical moment $\hat{\tau}$ such that

$$\begin{aligned} f_0(\tau) &\leq \bar{f}_1 && \text{for } \tau \leq \hat{\tau}, \\ \bar{f}_1 < \; f_0(\tau) &< \bar{f}_1 + \bar{f}_2 && \text{for } \tau > \hat{\tau}. \end{aligned}$$

For simplicity, let us assume that $\int\limits_{\hat{\tau}}^{\infty} (f_0(\tau) - \bar{f}_1) d\tau = \infty$.

From Assumption 2, we know that before the moment $\hat{\tau}$ there is no congestion:

$$f_1(\tau) = f_0(\tau), \; f_2(\tau) = 0, \; t_1(\tau) = \bar{t}_1, \; t_2(\tau) = \bar{t}_2 > \bar{t}_1, \quad \tau \leq \hat{\tau}.$$

What happens at $\tau = \hat{\tau}$? Since $t_i(\tau)$ are continuous, we will have $t_2(\tau) = \bar{t}_2 > t_1(\tau)$ for τ from some interval $[\hat{\tau}, \bar{\tau})$. During this period of time, the first arc will stay more attractive for the drivers. So,

$$f_1(\tau) = f_0(\tau), \quad \tau \in [\hat{\tau}, \bar{\tau}).$$

However, during this period of time, the input flow of the first arc will be greater than the maximal output flow. This must result in a queue growing at the end of this arc. This queue will increase the travel time $t_1(\tau)$.

Thus, we have seen that starting from the time $\hat{\tau}$ the function $t_1(\tau)$ strictly increases. How long that can happen? Clearly, up to the moment its reach the level of the travel time on the second arc:

$$t_1(\bar{\tau}) = \bar{t}_2.$$

Starting from time $\bar{\tau}$, the drivers will be shared between the two arcs in such a way that keeps the travel times on both arcs remain the same. Therefore, for any $\tau \geq \bar{\tau}$ we have the following equilibrium solution:

$$t_1(\tau) = t_2(\tau) = \bar{t}_2, \quad f_1(\tau) = \bar{f}_1, \; f_2(\tau) = f_0(\tau) - \bar{f}_1.$$

We see that the above dynamics can be seen as a *transition process* between two *steady equilibrium states* of the network. The dynamics of this process can be quite complicated, but it does not change the final *stable* steady state. This state is stable in the sense that for any flow pattern $f_0(\tau)$ such that

$$\bar{f}_1 < f_0(\tau) < \bar{f}_1 + \bar{f}_2, \quad \text{for } \tau \geq \bar{\tau},$$

the equilibrium arc travel times, flows and queues remain *constant in time*.

Let us look now at the last variant of our static example, in which we have $f = \bar{f}_1$. In this case

$$f_1 = \bar{f}_1, \quad f_2 = 0,$$

but we cannot assign a specific value to the equilibrium travel time. We can only say that $t_1 \in [\bar{t}_1, \bar{t}_2]$.

This uncertainty has an obvious dynamic interpretation. If in our dynamic setting the integral

$$\int_{\hat{\tau}}^{\infty} (f_0(\tau) - \bar{f}_1) d\tau$$

is finite and small enough, the travel time $t_1(\tau)$ can stabilize at some level in the interval $[\bar{t}_1, \bar{t}_2]$. But this level is not stable: any local change of the flow $f_0(\tau)$ will result in the change of the size of the queue and, as a result, in the stabilization of the travel time $t_1(\tau)$ at some new level.

Thus, let us make some intermediate conclusions.

- In our example we managed to find some static equilibrium *without* using any predefined travel time functions. We use only the logical consequences of Assumptions 1 and 2.
- We have seen that the *arc performance model* is not necessarily a functional dependence. In our example it is represented as some bounds on the arc travel time and the arc flow:
$$t_i(\tau) \geq \bar{t}_i, \quad 0 \leq f_i(\tau) \leq \bar{f}_i, \quad i = 1, 2.$$
- Our static equilibria can be seen as *steady equilibrium states* of the dynamic transportation network.

We will focus in this paper on the study of stationary regimes. We refer to these regimes as *Stable Dynamics*. Traffic count data suggest that this approach is likely to be relevant for both the peak and the off-peak hours. What will be ignored, is how the transportation system evolves from one stationary regime to another. Later on, we will prove that the existence of a stationary regime can be guaranteed for general networks under very mild assumptions and that the similar conclusions could then be derived. However, before that we wish to develop our intuition on some other examples.

3.2 Triangle network

Let us consider a simple network with two origin nodes, A and B, and one destination node C. Node A is connected with C by an arc with characteristics $\bar{t}_1$ and $\bar{f}_1$ and node B is connected with C by the second arc with characteristics $\bar{t}_2$ and $\bar{f}_2$. Without loss of generality, we can assume that $\bar{t}_1 < \bar{t}_2$.

Now, if the demand flows d_a from A to C and d_b from B to C satisfy the relation

$$d_a < \bar{f}_1, \quad d_b < \bar{f}_2,$$

then there is no congestion in the network. In this case

$$t_1 = \bar{t}_1, \quad t_2 = \bar{t}_2,$$

and the social cost in the network is as follows:

$$c_0^* = d_a \bar{t}_1 + d_b \bar{t}_2.$$

In what follows, we assume that

$$d_a + d_b > \bar{f}_1. \tag{3}$$

Let us modify our network. Namely, let us connect the nodes B and A by a new very short and efficient arc with the characteristics $\bar{t}_3$, $\bar{f}_3$. We assume that these characteristics satisfy the following relations:

$$\bar{t}_1 + \bar{t}_3 < \bar{t}_2, \quad \bar{f}_3 > \bar{f}_2.$$

What is the impact of this modification? Clearly, it changes nothing for drivers of OD-pair (A, C). However, for the drivers of OD-pair (B, C) it creates a new shortest path. Therefore these drivers will try to use it. But, since the capacity of the first arc is not enough to carry out the total demand flow (see (3)), the drivers from B will create a queue on the first arc. The size of this queue will be growing up to the moment the travel time along the route $B \to A \to C$ becomes equal to $\bar{t}_2$. Since there is no congestion on the new arc, the travel time on the first arc must be as follows:

$$t_1 = \bar{t}_2 - \bar{t}_3.$$

Thus, the above modification of the network results in the following equilibrium travel times and arc flows:

$$\begin{aligned} t_1 &= \bar{t}_2 - \bar{t}_3, & f_1 &= \bar{f}_1, \\ t_2 &= \bar{t}_2, & f_2 &= d_a + d_b - \bar{f}_1, \\ t_3 &= \bar{t}_3, & f_3 &= \bar{f}_1 - d_a. \end{aligned}$$

The equilibrium travel time for the drivers of OD-pair (B, C) was not improved. But the travel time for traveling from A to C become worse. The new social cost is as follows:

$$c^* = d_a(\bar{t}_2 - \bar{t}_3) + d_b \bar{t}_2 > c_0^*.$$

Thus, nobody is better off after the above modification and some drivers are worse off.

Such a phenomenon is known in transportation science as the *Braess paradox*. However, the classical form of that paradox corresponds to the static model with linear arc performance functions. Moreover, it is derived for a very special network (we consider this network in Section 3.3). Up to now the theoretical conditions for arising such negative effects in general static models are still not well understood (see [14] for details).

The example we have seen in this section seems to be new. Moreover, it corresponds to the simplest network structure and it is much easier for complete theoretical analysis. An interesting question is why this triangular network was never considered as an example for the Braess paradox. The answer is quite intriguing: It appears that for the triangular network the negative impact *cannot* be observed in a static model with strongly increasing linear travel time functions.

Indeed, let us assume that the arc travel time depends linearly on the arc flow:

$$t_i = \alpha_i + \beta_i f_i, \quad i = 1 \ldots 3.$$

Then the equilibrium travel time in our initial network are as follows:

$$t_1 = \alpha_1 + \beta_1 d_a, \quad t_2 = \alpha_2 + \beta_2 d_b.$$

In order to find an equilibrium for the new network we need to solve the following linear system:

$$\begin{aligned} t_2(f_2) &= t_3(f_3) + t_1(d_a + f_3), \\ f_2 + f_3 &= d_b, \\ f_2, f_3 &\geq 0. \end{aligned}$$

Then, if the new arc is used ($f_3 > 0$), the new equilibrium travel time for OD-pair (B, C) must decrease provided that $\beta_2 > 0$:

$$t_2(f_2) = t_2(d_b - f_3) = t_2(d_b) - \beta_2 f_3.$$

However, even in this framework the social impact of the new arc can be negative:

$$\begin{aligned} & [d_a t_1(d_a + f_3) + d_b t_2(f_2)] - [d_a t_1(d_a) + d_b t_2(d_b)] \\ & = f_3[d_a\beta_1 - d_b\beta_2]. \end{aligned}$$

To conclude with this example, note that the above equilibrium solutions can be supported by the same type of dynamic analysis for the stable steady states as it was done in Section 3.1.

3.3 Braess network

Let us check what kind of logical solutions we can have in the classical four-node network of Braess paradox. Consider a network with four nodes A, B_1, B_2 and C. The node A is an origin, the node C is the destination. Let us start with the following arc structure. The node A is connected with B_1 (first arc) and the node B_1 is connected with C (second arc). Similarly, A is connected with B_2 and B_2 is connected with C (third and fourth arcs). For all arcs we have our standard characteristics

$$\bar{t}_i, \quad \bar{f}_i, \quad i = 1, \ldots, 4.$$

In order to simplify our analysis we assume that

$$\bar{f}_2 = \bar{f}_3 = +\infty.$$

Thus, no congestion ever occurs at the second and the third arcs.

Denote by d the demand flow from A to C. In this initial network there are only two routes for travelling from A to C. Hence, the situation is absolutely the same as in the example of Section 3.1. We can represent the route $A \to B_1 \to C$ by a single arc with characteristics

$$\bar{t}_{12} = \bar{t}_1 + \bar{t}_2, \quad \bar{f}_{12} = \bar{f}_1,$$

and the route $A \to B_2 \to C$ by another arc with characteristics

$$\bar{t}_{34} = \bar{t}_3 + \bar{t}_4, \quad \bar{f}_{34} = \bar{f}_4.$$

Without loss of generality, we can assume that $\bar{t}_{12} > \bar{t}_{34}$. Then, there are two cases. First, if $d < \bar{f}_4$ then there is no congestion in the network. If $d \in (\bar{f}_4, \bar{f}_1 + \bar{f}_4)$, then the congestion occurs only on the fourth arc. In this case, the equilibrium travel time on the fourth arc is

$$t_4^* = \bar{t}_{12} - \bar{t}_3 = \bar{t}_1 + \bar{t}_2 - \bar{t}_3 > \bar{t}_4.$$

Thus, the equilibrium travel time tt_0^* in our example is as follows:

$$tt_0^* = \begin{cases} \bar{t}_{34}, & \text{if } d < \bar{f}_4, \\ \bar{t}_{12}, & \text{if } \bar{f}_4 < d < \bar{f}_1 + \bar{f}_4. \end{cases}$$

Let us assume that in our network we have the following relations between the minimal arc travel times:

$$\bar{t}_1 < \bar{t}_3, \quad \bar{t}_2 > \bar{t}_4.$$

Then, we can add to our network a very short and efficient fifth arc, which connects the nodes B_1 and B_2 and which has the following characteristics:

$$\bar{t}_5 < \min\{\bar{t}_3 - \bar{t}_1, \bar{t}_2 - \bar{t}_4\}, \quad \bar{f}_5 = +\infty. \tag{4}$$

What happens with the equilibrium travel time? In what follows we analyze the most interesting case

$$d \in (\max\{\bar{f}_1, \bar{f}_4\}, \bar{f}_1 + \bar{f}_4).$$

First of all, note that we have created a new shortest path

$$A \to B_1 \to B_2 \to C.$$

Therefore the fifth arc will be definitely in use. On the other hand, since d is large enough, all three routes of the network must be used.

Further, the free traffic travel time for going from A to B_2 directly (using the third arc) is greater then the travel time of the trip $A \to B_1 \to B_2$. Therefore, the queue will appear at the first arc (the only possible place for congestion). On this arc the queue will establish the following travel time:

$$t_1^* = \bar{t}_3 - \bar{t}_5 > \bar{t}_1$$

(see (4)). Similarly, the drivers which travel to C through node B_1 can see that the free traffic travel time for going to C directly (using the second arc) is worse then the travel time of the trip $B_1 \to B_2 \to C$. Therefore, the queue will appear on the fourth arc. The travel time on this arc will be adjusted up to the level

$$t_4^* = \bar{t}_2 - \bar{t}_5 > \bar{t}_4$$

(see (4)). On all other arcs the travel time will be equal to the free traffic travel time.

Now we can easily check, that the travel time along all three routes from A to C is the same. Thus, we have found the new equilibrium travel time

$$tt^* = \bar{t}_2 + \bar{t}_3 - \bar{t}_5. \tag{5}$$

The equilibrium flows in our network are as follows:

$$f_1^* = \bar{f}_1, \quad f_2^* = d - \bar{f}_4, \quad f_3^* = d - \bar{f}_1, \quad f_4^* = \bar{f}_4, \quad f_5^* = \bar{f}_4 + \bar{f}_1 - d.$$

Note that in our example we again have $tt^* > tt_0^*$ (in view of (4)). That is not surprising, since the Braess network can be seen as a combination of two triangular networks, on which we have already observed such a phenomena (see Section 3.2).

We have seen that the equilibrium travel time in our network is a decreasing function of $\bar{t}_5$ (see (5)). Thus, the largest value of tt^* corresponds to $\bar{t}_5 = 0$. Topologically, this means that in the worst case the nodes B_1 and B_2 coincide and become a single node B. Then we have two consecutive two-route-in-parallel network structures (see Section 3.1): $A \to B$ and $B \to C$. Since these networks are independent, the equilibrium travel time for going from A to C is the sum of equilibrium travel times for going from A to B and from B to C; that is $\bar{t}_2 + \bar{t}_3$. The above analysis shows that this equilibrium travel time decreases if we do not allow drivers, coming at B by the first arc, enter the fourth arc.

4 Stable Dynamics

In the previous section we have shown that it is possible to find equilibrium solutions for some simple networks. We managed to do that only by using the logical consequences of Assumptions 1 and 2, without any use of the standard concept of arc performance function. In this section, we show that we can find such solutions in general networks. Of course, in the most cases that cannot be solved in a closed form. However, we show that these solutions can be found numerically using the standard tools of convex optimization. In this section we use the formal network description introduced in Section 2. The proofs of all statements below can be easily derived from the theory developed in [9]. For completeness of the paper we present all necessary proofs in Appendix.

4.1 Structure of equilibrium flows

Consider now a general transportation network. Let us assume that the arc travel time pattern $t = \{t_\alpha\}_{\alpha \in \mathcal{A}}$ is given. Which route will be used by an individual to travel between his origin-destination pair? Of course, that must be a shortest path defined by t. A single individual cannot change significantly the travel time for other drivers (recall that we have a continuum of drivers). However, if we introduce in the network an additional finite *flow* of drivers, we can expect some changes. We have discussed the possible dynamics of that change in Section 3.1. Initially, the additional drivers will create queues (or, increase the size of existing queues) somewhere along their shortest paths. When this queue becomes sufficiently large, we will see a birth of a *new* shortest path. Thus, at this moment the set of the shortest paths is increased and all such paths are in use to carry out the demand flow.

From our considerations, we can make two important observations.

- In congested networks we can expect that the shortest path, connecting an origin-destination pair, is not unique.
- The equilibrium travel time on the used arcs is established in such a way that it creates a variety of shortest paths large enough to carry out the demand flow.[2]

In other words, in congested networks the equilibrium arc travel time becomes a function of demand flows. Let us try to describe this dependence.

Let in our network $\mathcal{R}$ the arc travel time pattern t is fixed. Then, for each OD-pair (i, j) we can compute the shortest-path travel time. This value is a function of t and it has the following analytical form

$$T_{(i,j)}(t) = \min\{\langle a^{(r)}_{(i,j)}, t\rangle, \; r = 1, \dots, r_{(i,j)}\}.$$

Thus, $T_{(i,j)}(t)$ is a *concave* piece-wise linear function of t. It is defined for any $t \in R^m$.

Recall, that for any function $f(x)$, which is concave on R^m, at each point we can define a *superdifferential* $\partial f(x)$. This is a closed convex set such that for any $g \in \partial f(x)$ we have

$$f(y) \le f(x) + \langle g, y - x\rangle, \quad \forall y \in R^m.$$

Let us look at the superdifferential of the function $T_{(i,j)}(t)$.

Define

$$I_{(i,j)}(t) = \{r \in [1 \dots r_{(i,j)}] : \; \langle a^{(r)}_{(i,j)}, t\rangle = T_{(i,j)}(t)\}.$$

In other words, $I_{(i,j)}(t)$ is the set of all shortest paths (with respect to t), which connect the origin i with the destination j. Then

$$\partial T_{(i,j)}(t) = \mathrm{Conv}\,\{a^{(r)}_{(i,j)}, \; r \in I_{(i,j)}(t)\}.$$

This set allows us to characterize the equilibrium flows induced by the OD-pair (i, j) in network $\mathcal{R}$ in a very compact form.

[2] More precisely, on the *minimal level.* That is again a direct consequence of Assumptions 1 and 2. We will discuss a mathematical justification for this conclusion in Section 4.3.

Lemma 1 *The flow vector* $f_{(i,j)}$ *is compatible with Assumption 1 if and only if there exists some* $g \in \partial T_{(i,j)}(t)$ *such that*

$$f_{(i,j)} = d_{(i,j)} g.$$

In the sequel, we call such a vector $f_{(i,j)}$ the *equilibrium flow of the OD-pair* (i,j). Note that the cumulative arc flow vector f is just a summation of all OD-flows:

$$f = \sum_{(i,j)\in\mathcal{OD}} f_{(i,j)}. \tag{6}$$

We call f the *equilibrium flow* if it can be represented as a sum of equilibrium flows of all OD-pairs. Note that the equilibrium flows are defined *with respect to* the arc travel time pattern t.

Consider now the cost function

$$C(t) = \sum_{(i,j)\in\mathcal{OD}} d_{(i,j)} T_{(i,j)}(t).$$

Theorem 1 *The arc flow vector* $f \in R^m$ *is an equilibrium flow with respect to the arc travel time pattern* t *if and only if*

$$f \in \partial C(t). \tag{7}$$

This theorem allows us to answer some interesting questions. Let the demand flow $\{d_{(i,j)}\}_{(i,j)\in\mathcal{OD}}$ be known.

1. Given the arc travel time vector t, we can describe all possible equilibrium flows, which can arise in the network. That is

$$f_{(i,j)} \in d_{(i,j)} \partial T_{(i,j)}(t), \ (i,j) \in \mathcal{OD},$$

$$f = \sum_{(i,j)\in\mathcal{OD}} f_{(i,j)}.$$

2. Given the arc flow vector f, we can check whether it is possible to find an arc travel time vector t, with respect to which f is an equilibrium flow.

Lemma 2 *The arc flow* f *is an equilibrium flow in the network* $\mathcal{R}$ *if and only if the following optimization problem*

$$\max_t \, [\, C(t) - \langle f, t \rangle \,] \tag{8}$$

admits a non-negative solution t^*.

This statement shows that the information from traffic counters can help to reconstruct the equilibrium travel time in the network. If all arc flows are known, this data is enough in order to find the equilibrium travel time. However, even a partial knowledge can help (see Section 4.4).

Note that the results presented in this section are based *only* on Assumptions 1 and 2. We did not use anyhow a particular model of the arc performance.

4.2 Max-flow model

Let us show now how we can get the equilibrium solutions discussed in Section 3. We use the following *max-flow* arc performance model:

$$t_\alpha \geq \bar{t}_\alpha, \quad 0 \leq f_\alpha \leq \bar{f}_\alpha, \quad \alpha \in \mathcal{A}. \tag{9}$$

Theorem 2 *The arc travel time t^* and the arc flow vector f^* is an equilibrium solution of the model (9) if an only if t^* is a solution to the problem*

$$\max_t [C(t) - \langle \bar{f}, t \rangle : \; t \geq \bar{t}], \tag{10}$$

and $f^* = \bar{f} - s^*$, where s^* is a vector of optimal dual multipliers for the inequality constraints in (10).

Let us show on a simple example how this theorem works. Consider the Braess network, described in Section 3.3. In this case the objective function of the problem (10) is as follows:

$$d \min\{t_1 + t_2, t_1 + t_5 + t_4, t_3 + t_4\} - \sum_{i=1}^{5} \bar{f}_i t_i.$$

We have assumed that $\bar{f}_2$, $\bar{f}_3$ and $\bar{f}_5$ are very large. Therefore, in the solution of the problem (10) we necessarily will have

$$t_2 = \bar{t}_2, \quad t_3 = \bar{t}_3, \quad t_5 = \bar{t}_5.$$

Thus, our problem becomes a problem of two variables:

$$\max_{t_1, t_4} [d \min\{t_1 + \bar{t}_2, t_1 + t_4 + \bar{t}_5, t_4 + \bar{t}_3\} - \bar{f}_1 t_1 - \bar{f}_4 t_4 : \; t_1 \geq \bar{t}_1, \; t_4 \geq \bar{t}_4].$$

The solution discussed in Section 3.3 is an interior solution of this problem (inequality constraints are not active). Note that a candidate solution can be found from the system of linear equations:

$$t_1^* + \bar{t}_2 = t_1^* + t_4^* + \bar{t}_5 = t_4^* + \bar{t}_3.$$

That is $t_4^* = \bar{t}_2 - \bar{t}_5$, $t_1^* = \bar{t}_3 - \bar{t}_5$. The necessary and sufficient conditions for that point to be optimal are as follows. Firstly, we must have

$$t_1^* > \bar{t}_1, \quad t_4^* > \bar{t}_4.$$

Secondly, there must exist the multipliers $\lambda_i \geq 0$, $i = 1 \dots 3$, such that

$$d \left[\lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} \bar{f}_1 \\ \bar{f}_4 \end{pmatrix},$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1.$$

That is

$$d \geq \bar{f}_1, \quad d \geq \bar{f}_4, \quad d \leq \bar{f}_1 + \bar{f}_4.$$

4.3 Dual max-flow model

Note that (10) is a convex optimization problem. Therefore it can be reformulated in an equivalent dual form. Let us look at this reformulation.

In order to do that, it is convenient to introduce for each origin $i \in \mathcal{O}$ a demand flow vector $d_i \in R^n$, where $n = |\mathcal{N}|$. Each component of this vector is the demand flow from i to the corresponding node. Denote by $E \in R^{n\times m}$ the incidence matrix of the network $\mathcal{R}$:

$$E_{ij} = \begin{cases} 1 & \text{if arc } j \text{ enters node } i, \\ -1 & \text{if arc } j \text{ goes out from node } i, \\ 0 & \text{otherwise.} \end{cases}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m.$$

The problem dual to (10) can be written in terms of arc flow vectors $f_i \in R^m$, generated by the origins $i \in \mathcal{O}$.

Lemma 3 *The problem*

$$\begin{aligned} &\min_{f, f_i} \quad \langle f - \bar{f}, \bar{t} \rangle, \\ s.t. \quad & f = \sum_{i\in\mathcal{O}} f_i \le \bar{f}, \\ & Ef_i = d_i, \ i \in \mathcal{O}, \\ & f_i \ge 0, \ i \in \mathcal{O}, \end{aligned} \tag{11}$$

is dual to (10).

Thus, the problem dual to (10) is the *minimal cost multi-commodity transportation problem* with bounded arc capacities [10]. In our knowledge, this problem was never considered for finding a user equilibrium in transportation systems. Traditionally, the problems of that type are used for finding a system optimum. However, note that in our framework the solution of this problem gives us only the flow pattern in the congested network. And intuitively it seems reasonable that, when congestion occurs, the drivers increase the set of the used paths in a monotone way, starting from the free traffic shortest path. The equilibrium arc travel times arise in this problem as the optimal dual multipliers for the inequalities $f \le \bar{f}$.

It is interesting that the problem (11) provides us also with a social optimum solution. Indeed, if f^* is a solution to this problem, then $\langle f^*, \bar{t} \rangle$ is the optimal social cost. Thus, the equilibrium solution differs from the social optimum solution only by the queues created by the drivers along the most attractive routes. These queues are clearly inefficient from the social point of view and they are null at the social optimum. However, they increase the travel time along the best routes up to an equilibrium level. As we have seen, the flow pattern for the user equilibrium and the social optimum is the same.

4.4 Mixed max-flow model

In the end of Section 4.1, it was mentioned that the complete knowledge of arc flows in the network gives us a possibility to find the equilibrium travel time and the equilibrium OD-flows without any additional model for the arc performance. However, in real networks the measurements of traffic counters are available only for a small number of arcs. In this situation we can combine the available information with the max-flow arc performance model. Let us consider the following model:

$$\begin{array}{ll} f_\alpha = \hat{f}_\alpha, & \alpha \in \mathcal{C}, \\ 0 \leq f_\alpha \leq \bar{f}_\alpha, \; t_\alpha \geq \bar{t}_\alpha, & \alpha \in \mathcal{A} \setminus \mathcal{C}. \end{array} \tag{12}$$

In the above model the set $\mathcal{C}$ corresponds to a subset of arcs, for which we know the measurement of traffic counters.

Theorem 3 *The arc travel time t^* and the arc flow vector f^* is an equilibrium solution to the model (12) if an only if t^* is a non-negative solution to the problem*

$$\max_t \left[C(t) - \sum_{\alpha \in \mathcal{C}} \hat{f}_\alpha t_\alpha - \sum_{\alpha \in \mathcal{A} \setminus \mathcal{C}} \bar{f}_\alpha t_\alpha : \; t_\alpha \geq \bar{t}_\alpha, \; \alpha \in \mathcal{A} \setminus \mathcal{C} \right], \tag{13}$$

and $f^*_\alpha = \bar{f}_\alpha - s^*_\alpha$, $\alpha \in \mathcal{A} \setminus \mathcal{C}$, where s^*_α are the optimal dual multipliers for the inequality constraints in (13).

Note that the problem (13) may be unsolvable. In this case the observed traffic counts contradict to our choice of the demand flows.

4.5 Computing OD-matrix

Usually, in practical applications the exact origin-destination matrix is never known. Instead, for each origin we may know the average number of drivers, which go outside this node, and the average number of drivers, which come to this node from all other origins. Using this information, we can estimate the origin-destination matrix, assuming that the flow of drivers of an origin-destination pair is some function of the number of working places at the destination, the size of population at the origin and the distance between the origin and destination points. It is natural to assume that this function increases in two first parameters and decreases in the last one.

The most popular model of that type is the *gravity-type model*, which states that

$$d_{(i,j)} \approx \frac{\alpha_{i,j}}{(\beta_{i,j} + t_{i,j})^\theta},$$

where θ, $\alpha_{i,j}$ and $\beta_{i,j}$ are some parameters and $t_{i,j}$ is some distance between the origin i and the destination j. Usually $\alpha_{i,j}$ depends on the size of the population and economic activity at i and j. The parameters $\beta_{i,j}$ are chosen to satisfy some balance conditions. Since in the existing models the OD-matrix must be computed *before* the computation of the equilibrium travel time, the natural candidate for $t_{i,j}$ is the free traffic shortest path distance. At the same time, it is clear that in congested networks the free traffic travel time can differ from the equilibrium travel time significantly. In this section we show that in the framework of Stable Dynamics models, we can compute the OD-matrix and the equilibrium travel time *simultaneously*, in the same computational process. Intuitively, it is clear that such an estimation of the OD-matrix will be more reasonable as compared with current practice.

Let $\mathcal{O}$, $|\mathcal{O}| = n_o$, be the set of origins and $\mathcal{D}$, $|\mathcal{D}| = n_d$, be the set of destination. Denote

$$D \in R^{n_o \times n_d}$$

the origin-destination matrix. The feasible set for the matrix D is as follows:

$$D \in \Delta = \{D \geq 0 : \; De_{n_d} = p, \; D^T e_{n_o} = q\},$$

where e_k is the vector of all ones from R^k, p is the demand vector of destinations and q is the output flow vector of origins.

Let us introduce a family of attraction functions

$$\phi_{i,j}(d), \; d > 0, \quad i = 1 \dots n_o, \; j = 1 \dots n_d,$$

which are convex and decreasing functions in d. We assume that $\phi'_{i,j}(d) \to -\infty$ as $d \to 0$, where $\phi'_{i,j}(\cdot)$ stands for the derivative of function $\phi_{i,j)(\cdot)}$.

In the framework of max-flow model (Section 4.2), we can consider the following minimax problem:

$$\max_{t \geq \bar{t}} \min_{D \in \Delta} \left\{ \sum_{i \in \mathcal{O}} \sum_{j \in \mathcal{D}} \left[D_{(i,j)} T_{(i,j)}(t) + \phi_{i,j}(D_{(i,j)}) \right] - \langle \bar{f}, t \rangle \right\}. \tag{14}$$

Note that this is a saddle point convex-concave problem. So, the existence of its solution, (t^*, D^*), follow from the standard results of convex optimization.

Using the first-order optimality conditions for (14), we conclude that there exist the dual multipliers $\lambda \in R^{n_o}$ and $\mu \in R^{n_d}$ such that

$$T_{(i,j)}(t^*) + \phi'_{i,j}(D^*_{(i,j)}) = \lambda_i - \mu_j, \quad i = 1 \dots n_o, \; j = 1 \dots n_d.$$

Solving this equation for $D^*_{(i,j)}$, we can find its dependence in $T_{(i,j)}(t^*)$.

For example, for $\phi_{i,j}(d) = -2\sqrt{d}$ we get

$$D^*_{(i,j)} = \frac{1}{(T_{(i,j)}(t^*) + \mu_j - \lambda_i)^2}.$$

Another reasonable variants for the function $\phi(d)$ can be as follows:

$$\phi(d) = d \ln d, \quad \phi(d) = -\ln d, \quad \text{etc.}$$

The first variant leads to the well known *logit model.*

5 Concluding remarks

In this paper we have presented several simple static models, that we call *Stable Dynamic model* since they can be interpreted as a stationary regime of a dynamic transportation network model. Our approach relies only on a directly observable parameters of the transportation system: minimal travel time to cross a link and maximum flow on this link. Together with the behavioral assumption that users select the shortest path, we were able to solve our models for the user equilibrium and social optimum. Moreover, the formulation can be extended to the case where the origin-destination matrix is endogenous. In this case, standard socio-economic data (population and jobs) are also required.

One of the main result of this paper is that the equilibrium solutions of our models can be found from a multi-commodity flow problem, for which various efficient algorithms have been developed. In a companion paper, we plan to present simulation results that will be compared with actual data (traffic counts).

Since the proposed approach can be easily linked to well established mathematical theories, it is tempting to wonder how it could be extended to fully dynamic models in the vein of William Vickrey (see in particular Part IV in [1]). This (difficult) topic will be addressed, we hope, by some researchers in the near future.

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Appendix

1. Proof of Lemma 1. Let $f_{(i,j)}$ be a feasible flow pattern for the OD-pair (i,j). Consider the sub-network $\mathcal{N}_{(i,j)}$ defined by the set of arcs

$$\mathcal{A}_{(i,j)} = \{\alpha \in \mathcal{A} : \; f^{(\alpha)}_{(i,j)} > 0\}.$$

The flow vector $f_{(i,j)}$ satisfies Assumption 1 if and only if any route from i to j in $\mathcal{N}_{(i,j)}$ has the same travel time and this travel time is equal to $T_{(i,j)}(t)$.

Since

$$\partial T_{(i,j)}(t) = \mathrm{Conv}\,\{a^{(r)}_{(i,j)}, \; r \in I_{(i,j)}(t)\},$$

inclusion $f_{(i,j)} \in d_{(i,j)} \cdot \partial T_{(i,j)}(t)$ is equivalent to the possibility to represent the flow vector as follows:

$$f_{i,j} = d_{(i,j)} \cdot \sum_{l=1}^{N} \lambda_{r_l} a^{(r_l)}_{(i,j)} \tag{15}$$

with some $\lambda_{r_l} \geq 0$ such that $\sum\limits_{l=1}^{N} \lambda_{r_l} = 1$. Consider now an arbitrary route a form i to j, which belongs to sub-network $\mathcal{N}_{(i,j)}$. In view of (15), an arc α can be included in the route a only if there is a shortest path from i to j which goes through this arc. From this observation we can prove that a is also the shortest path using a simple induction in travel time along a. □

2. Proof of Theorem 1. The proof follows from the definition (6), Lemma 1 and the standard relation for superdifferentials:

$$\partial C(t) = \sum_{(i,j)\in \mathcal{OD}} d_{(i,j)} \partial T_{(i,j)}(t).$$

□

3. Proof of Lemma 2. Note that (7) is a first order optimality condition for the problem (8). Since $C(t)$ is a concave function, this condition is necessary and sufficient characterization of the optimal solution t^*. □

4. Proof of Theorem 2. Let us denote by s the dual multipliers for inequality constraints $t \geq \bar{t}$ in problem (10). Then the Lagrangean for this problem has the following form:

$$\mathcal{L}(t,s) = C(t) - \langle \bar{f}, t\rangle + \langle s, t - \bar{t}\rangle \quad \rightarrow \quad \min_{s \geq 0} \max_{t}.$$

The problem (10) is solvable if and only if there exists a saddle point (t^*, s^*) of $\mathcal{L}(t,s)$. This means that

$$\begin{aligned} f^* &= \bar{f} - s^*, \qquad\qquad s^* \geq 0, \\ s^*_i (t_i - t^*_i) &= 0, \; i = 1, \dots, m, \quad t^* \geq \bar{t}, \end{aligned} \tag{16}$$

for some $f^* \in \partial C(t^*)$. In view of the last inclusion and Theorem 1, vector f^* is an equilibrium flow with respect to the arc travel time t^*. $\Box$

5. Proof of Lemma 3. It is enough to show that the problem dual to (11) is (10). Denote by τ the dual multipliers for the inequalities $f \leq \bar{f}$. Then the primal-dual form of problem (11) is as follows:

$$
\begin{aligned}
& \min_{f,f_i} \max_{\tau \geq 0} \left[\langle f - \bar{f}, \bar{t} \rangle + \langle f - \bar{f}, \tau \rangle : \ f = \sum_{i \in \mathcal{O}} f_i, \ E f_i = d_i, \ f_i \geq 0, \ i \in \mathcal{O} \right] \\
= \ & \max_{\tau \geq 0} \min_{f,f_i} \left[\langle f - \bar{f}, \bar{t} + \tau \rangle : \ f = \sum_{i \in \mathcal{O}} f_i, \ E f_i = d_i, \ f_i \geq 0, \ i \in \mathcal{O} \right] \\
= \ & \max_{\tau \geq 0} \min_{f_i} \left[\langle \sum_{i \in \mathcal{O}} f_i - \bar{f}, \bar{t} + \tau \rangle : \ E f_i = d_i, \ f_i \geq 0, \ i \in \mathcal{O} \right] \\
= \ & \max_{\tau \geq 0} \left\{ -\langle \bar{f}, \bar{t} + \tau \rangle + \sum_{i \in \mathcal{O}} \min_{f_i} \left[\langle f_i, \bar{t} + \tau \rangle : \ E f_i = d_i, \ f_i \geq 0 \right] \right\}.
\end{aligned}
$$

It remains to note that

$$
\min_{f_i} \left[\langle f_i, \bar{t} + \tau \rangle : \ E f_i = d_i, \ f_i \geq 0 \right] = \sum_{l=1}^{n} d_i^{(l)} T_{(i,l)}(\bar{t} + \tau)
$$

since the optimal flow f_i^* in this problem corresponds to the shortest path tree (with respect to $\bar{t} + \tau$) with the root at node i. $\Box$

6. Proof of Theorem 3. The proof of this theorem is very similar to the proof of Theorem 2. The only difference is that for arcs with known flows we do not have the inequality constraints. Therefore the first order optimality conditions ensure

$$
f^*_\alpha = \hat{f}_\alpha, \quad \alpha \in \mathcal{C},
$$

(compare with (16)). $\Box$