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CONSTRAINED SUBOPTIMALITY AND FINANCIAL INNOVATION IN GEI WITH A SINGLE COMMODITY

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Abstract

In this paper we exploit *global analysis* to explore welfare properties of a standard one-commodity GEI, under different notions of constrained Pareto optimality. In a unifying framework we revise and extend some of the leading results of the literature on incomplete markets and government intervention, including those concerning *financial innovation*.

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1 Introduction

1.1 A general overview and motivation

The literature on government intervention in incomplete market economies has followed two general approaches (See Geanakoplos (1990)). The traditional one, of neoclassical roots, analyses weather it is preferable to restore market completeness, as for instance in the Lindhal model, or assume a position of non interference with private economic activities. The second approach is in general identified in the literature on general equilibrium economies with incomplete markets (hereafter GEI). GEI (on which we mainly concentrate) is characterized by the idea that, even when restoring the missing markets is unfeasible, there exist public policies that may still be desirable. Thus, for a given market structure, certain forms of government intervention may lead the economy toward socially preferred equilibria.

Before starting our discussion we sketch the underlying structure of the economy. We consider a competitive exchange economy of two periods, with uncertainty over a finite number $S \geq 2$ of possible states of nature revealed in the second period (denote $S + 1 = N$). Commodities are perishable, thus consumers can only transfers income across time and states using the available asset markets. Yet financial markets are incomplete; more precisely, asset returns span $\mathcal{L}$, a subspace of $\Re^S$ of dimension $1 < J < S$. If the number of financially constrained consumers is $H - 1 > J$, there does not exist an asset structure that allows consumers to achieve a full diversification of risk. In other words, by trading on financial markets, consumers are at most able to determine J components of theirs future state-income distribution, rather than S. This immediately implies that competitive equilibria are typically Pareto suboptimal. Multiple equilibria with different welfare properties are likely to be observed.

In a second best world, weaker notions of optimality are then needed in order to obtain some form of social ranking of suboptimal market equilibria; criteria that may serve as a guidance to evaluate alternative government policies. The most "natural" bench mark criterion is based on the idea that the planner faces the same financial constraints of the private sector. Thus, a centralized attainable allocation is one that is resource-feasible and that can be realized by implementing a system of transfers that lies on the span of the marketed securities. More precisely, adapting Diamond's criterion to our exchange economy, an allocation is Diamond-Constrained Pareto Optimal (hereafter *D-CPO*) if there does not exists an alternative attainable allocation that is Pareto improving.[1]

[1] Diamond's economy is a one-commodity GEI with production and ray technologies. The central planner is allowed to allocate asset portfolios, production inputs, and date 0-endowments (See Diamond (1967).

The assumption that the planner faces the same constrains of the private sector, gives rise to the (correct) intuition that equilibria are typically *D-CPO*. Most of the subsequent GEI literature on government intervention has then turned the attention to economies with multiple goods. A general belief was that Diamond's constrained efficiency result was entirely driven by the assumption that only one commodity was traded in each state. This was first argued in Stiglitz (1982), for the case of a production economy, and then more rigorously proved in Geanakoplos et Al. (1990). In the context of exchange economies, the seminal contribution is Geanakoplos-Polemarchakis (1986).[2] Yet two common features are central in these analysis, and must be kept distinct: *a*) the notion on constrained Pareto optimality, and *b*) the conditions imposed on the number of consumer types.[3]

To understand why so much importance has been given to the multiple commodity case one should observe that the constrained inefficiency literature relies on the ability of the central planner to induce a change in the *asset span* $\mathcal{L}$, via an (indirect) relative price effect. A change in $\mathcal{L}$, altering the profile of asset payoffs, modifies risk sharing opportunities achievable in the markets. Thus, although the degree of market incompleteness is generically left unchanged, when $\mathcal{L}$ varies individuals face a new set of financial possibilities and constraints. This in general implies that they will then be willing to redefine theirs investment, production, and consumption decisions. If the number of consumer types is not too large, these allocation changes may have positive welfare effects.

Yet relative price effects are not the only source of span effects. There are government policies which may directly affect the asset span. Policies that, for this reason, are effective in a single-commodity economy too. Obviously, different definitions of policy instruments and of the set of centralized attainable allocations give rise to distinct notions of constrained suboptimality. In this paper we refer to the notion of $\mathrm{J}-CPO$ as the one in which the planner acts as in $D-CPO$, but he is also given the extra degree of freedom of designing (or simply controlling) the asset span of the economy. In other words -using the J assets available in

[2] Recall that in Geanakoplos-Polemarchakis (1986) the central government behaves as in Diamond (1967), but an important further assumption is added: after each policy intervention asset markets are kept closed, and consumers are only allowed to trade on spot markets ("weak constrained Pareto criterion"). This assumption is crucial in explaining their constrained inefficiency. While this is not so in the production economy of Geanakoplos et Al. (1990), who however maintain such assumption. The reason is that, in the latter paper, the planner can also directly affect the asset span by choosing production plans. This produces a direct spanning effect of the type examined in Tirelli (2000).

[3] For production economies, *b* should also account for firm types. Moreover, a third feature must be considered: the pricing criterion adopted by the firm to compare and evaluate alternative feasible production plans (or investments).

the economy- the planner can choose the payoff structure to use to implement a system of centralized transfers. Two interesting applications of this general notion of constrained Pareto optimality are monetary policy, and capital income taxation. A third important one is *financial innovation.*

1.2 Direct spanning effects: examples

Consider an economy in which the value of the single good traded is denominated in an abstract unit of account, "money". Let $v_s > 0$ be the purchasing power of money in every state s. Further, assume that financial markets are characterized by the following asset structure: a $(S \times J)-$asset matrix, R, in general position (i.e. with full column rank), and with payoffs denominated in an abstract unit of account (or *numèraire*). Denoting by v_s the purchasing power of the *numèraire* in state s, a *real numèraire* matrix, $\overline{R}$, is

$$\overline{R} = [v]\, R, \text{ where } \ [v] = \begin{pmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_S \end{pmatrix}.$$

Now take two *real numèraire* economies identically defined except than for $[v]$. From Geanakoplos Mas-Colell (1989, Lemma 1), we know that, if $J < S$, and for sufficiently many consumers $(H > J)$, $\langle [v]\, R \rangle \neq \langle [\widetilde{v}]\, R \rangle$ implies that the resulting competitive equilibrium allocations are also different.[4] Moreover, if we let the v vary freely, the degree of real indeterminacy is, typically, $S - 1$. In fact, with only one good, commodity prices can be normalized to one in every state, and no nominal indeterminacy arises. Now, let a fictitious central bank determine the level of money supply, M_s, across states. Further, assume that in each state s, for a given M_s, and total resources, he can determine the purchasing power of money, $v_s > 0$, applying a Fisher-like equation. Then, controlling M_s, independently, across states the central authority can affect the allocation. Yet, this is not enough to say something about the welfare consequences of such a change.[5]

A second example of policies with direct span effects is capital income taxation. Taxes are levied proportionally on security payoffs. In an exchange economy the structure of the after-tax

[4]Indeterminacy with *real numeraire* assets was first shown by David Cass (Cass 1985). See also Balasko and Cass (1989).

[5]On the real effects of monetary policy in GEI see Magill-Quinzii (1992). Further, Polemarchakis (1988) consider the case of an open economy in which assets are denominated in different currencies. In this case, the observable degree of real indereminacies increases in the number of currencies: exchange rate variations may have real effects.

asset matrix is similar to $\overline{R}$. Thus, an effect analogous to the one just considered for the case of monetary policy can be observed.[6]

Turning to *financial innovation*, two are the definitions that can be found in the literature. The first is represented as a problem of security design applied to a given set of assets. This notion is used, for example, in Demange Laroque (1995), and fits our definition of constrained inefficiency ($\mathsf{J} - CPO$). The second, notion focuses on the problem of designing new assets, and has been explored in Cass-Citanna (1998), and Elul (1995, 1999). Observe that the latter departs from GEI's logic of government intervention, when it assumes that the central planner can affect the market structure by effectively reducing its degree of incompleteness.

1.3 The paper contempt

Our analysis is organized as follows. In **section 2** we introduce a standard GEI economy with fixed resources, and two notions of constrained Pareto optimality. The first, substantially corresponds to the one introduced in Diamond (1967), in which the planner can only reallocate portfolios ($D - CPO$). In the second, denoted by J-CPO, we assume that the planner can also affect private investment opportunities by choosing $\mathcal{L}$. To avoid the trivial case of effectively complete markets, we concentrate on the case of $H - 1 > J$.

In **section 3**, we study the structure of the set of constrained Pareto optima.

In **section 4**, we define the concepts of competitive economy and equilibrium. These are the single commodity analogues of those presented in Duffie and Shafer (1985). Then, we study the welfare properties of equilibria. First, we show that, generically in endowments, competitive equilibria are constrained Pareto suboptimal (i.e. the I Welfare Theorem fails under J-CPO). Second, we establish the generic applicability of the II Welfare Theorem. The constrained inefficiency property of incomplete markets equilibria is entirely explained by the general characteristics of the asset markets. If the planner is revoked the ability to design asset payoffs, it is immediate to see that equilibria are, typically, constrained efficient (*D-CPO).*

In **section 5**, we explore *financial innovation.* The generic failure of the I welfare Theorem, directly implies a generalization of Demange-Laroque (1995) to a large class of GEI economies. Moreover, we extend Elul (1999) by relaxing the assumption that preferences are quadratic, and showing that this holds for a strictly generic set of economies parametrized by endowments.

In **section 6**, we briefly analyze how to use our general definition of the planner's maximum problem to study policy optimality. To do so we refer to the two examples of monetary policy

[6] For the case of a single commodity GEI with production, see Tirelli (2000). In that paper securities are equities, claims on future profits acruing from prduction activities.

and capital income taxation given in the introduction.

Section 7 contains an example. **Section 8** gather most of the proofs of the results presented in the paper.

1.4 About the methodology used in the paper

A fundamental methodological aspect characterizing GEI literature on government intervention is that it is entirely based on genericity analysis. The results are proved to hold in a neighborhood of some equilibrium point, and then they are extended to a larger environment by showing that the set of economies for which they fail to hold is negligible in a topological sense, or in a measure theoretical sense when possible.

In this paper we depart from the latter approach, by analyzing separately the planner's problem from equilibria. In analogy to the case of complete markets, this is possible when the planner's problem is formulated as a programming problem. The advantage of this choice is that it provides a richer set of information about the structure of constrained Pareto optima; information that is then used to derive our main results. Thus, for example, from the analysis of $J-CPO$ we observe that, in every economy, there exists an optimal asset structure. However, knowing the structure of the equilibrium manifold, we can say that if this asset structure is provided to the market economy, new government interventions may still be desirable. In fact, if the number of consumers is not to large, the corresponding set of equilibria is lower dimensional with respect to the whole set of competitive equilibria. Thus, small deviations from the original economy (due, say, to a marginal change of endowments) result in an equilibria for which that asset structure is not optimal (in the sense of $J-CPO$). The information on the structure of the set of constrained Pareto optima is also used to generalize and sharpen the results on financial innovation, without increasing the level of technical difficulty.

2 The Economy

2.1 Commodities and spot markets

There are two dates indexed by 0 and 1, and a finite number S of possible states of nature in date 1. Including date 0 as one of the states, we use the indexing $s = 0, 1, .., S$, and define N $= (S+1)$.

Commodities are represented in a (column) vector $x \in \Re^N$. Further, we denote *date* 0 *present value prices* with the (row) vector $p \in \Re^N$.

2.2 Financial structure

We assume that the financial structure of the economy is represented by the *Grassmanian* of $J-$plains in $\Re^S$, and we denote it by G. In the rest of this subsection we briefly review the notion and the main properties of *Grassmanians*.

(Equivalence classes and quotient spaces) Let G be a collection of $J-$ dimensional linear subspaces in $\Re^S$, with typical element $\mathcal{L} = \{y \in \Re^S : Ry = 0\}$. Associated with every subspace $\mathcal{L}$, there is its unique orthogonal complement $\mathcal{L}^{\perp} = \{y \in \Re^S : y \perp \mathcal{L}\}$; a subspace of $\Re^S$ spanned by $(S-J)$ independent vectors. R is an $(S-J) \times S$ matrix whose independent rows are a *basis* for $\mathcal{L}^{\perp}$. We denote with $\mathsf{M}^{(S-J),S}$ the collection of $(S-J)-$dimensional subspaces of $\Re^S$ of typical element $\mathcal{L}^{\perp}$. Similarly, $\mathsf{M}^{J,S}$ is the collection of orthogonal complements $\mathcal{L}$. Notice that $\mathcal{L}$ can be viewed as the set of solutions to the system $Ry = 0$, with coefficients in R. However, if R *induces* $\mathcal{L}$, it is also true that R' induces $\mathcal{L}$, if there exists a nonsingular $(S-J)$ square matrix B such that $R' = BR$. This is because any linear combination of the rows of R is still a basis for $\mathcal{L}^{\perp}$ (or equivalently induces the same $\mathcal{L}$). If such a matrix B exists we say that R' and R belong to the same equivalence class $\sim$. Analogously, $\mathcal{L} \sim \mathcal{L}'$ if these subspaces are, respectively, induced by R and R', and $R \sim R'$.

Finally we can define the *quotient space* $\mathsf{G} = \mathsf{M}^{J,S} / \sim$; a subspace of $\Re^S$ whose typical element $\mathcal{L} = proj_{\mathsf{G}}(R)$.

(Quotient topology) $\mathcal{L}$ is open in G if and only if $R = proj^{-1}_{\mathsf{M}^{S-J,S}}(\mathcal{L})$ is open in G. The projection defines a *quotient topology* on G.

(Permutation and normalized representation) Let the rows of R be a basis for $\mathcal{L}^{\perp}$. Again, interchanging the rows of R leaves the subspace unchanged. We define a *permutation* σ a mapping from $\{1,..,S\}$ onto itself. Σ denotes the set of all such permutations. Further, let

$$\pi_{\sigma} = \begin{pmatrix} \underset{(S-J)\times(S-J)}{\pi_{\sigma}^{(1)}} & \underset{(S-J)\times J}{\pi_{\sigma}^{(2)}} \\ \underset{J\times(S-J)}{\pi_{\sigma}^{(3)}} & \underset{J\times J}{\pi_{\sigma}^{(4)}} \end{pmatrix}$$

be the partition of an $(S \times S)$ *permutation matrix* associated to the mapping σ. Every matrix R can be represented as

$$R\pi_{\sigma} = \begin{pmatrix} R_{/\overline{\sigma}} \mid & R_{\overline{\sigma}} \end{pmatrix}$$

for some $\overline{\sigma}, \sigma \in \Sigma$ and $\overline{\sigma} = \sigma^{-1}$.[7] Further, if $\mathcal{L} \in \mathsf{G}$, there exists a permutation $\sigma \in \Sigma$ and a $A \in \Re^{(S-J)J}$ such that $rank \begin{pmatrix} I_{S-J} \mid & A \end{pmatrix} = S - J$, and the set of $S-$vectors y such that

[7] $R_{/\overline{\sigma}} \equiv R^{(1)}\pi_{\overline{\sigma}}^{(1)} + R^{(2)}\pi_{\overline{\sigma}}^{(3)}$, $R_{\overline{\sigma}} \equiv R^{(1)}\pi_{\overline{\sigma}}^{(2)} + R^{(2)}\pi_{\overline{\sigma}}^{(4)}$, and $A = \left(R_{/\overline{\sigma}}\right)^{-1}\left(R_{\overline{\sigma}}\right)$. For further details see Duffie and Shafer (1985).

$\begin{pmatrix} I_{S-J} \,|\, & A \end{pmatrix} y = 0$ fully describes $\mathcal{L}$. We shall call the latter *normalized representation.* Obviously, $R \sim \begin{pmatrix} I_{S-J} \,|\, & A \end{pmatrix}$. Further it is easy to check that for every $\mathcal{L}$, A is uniquely determined.

For every permutation σ, define

$$W_\sigma = \left\{ \mathcal{L} \in \mathsf{G} : \exists\, A \subset \Re^{(S-J)J} \text{ such that } \begin{pmatrix} I_{S-J} \,|\, & A \end{pmatrix} \pi_\sigma \in \mathcal{L} \right\}$$

And $\{W_\sigma : \sigma \in \Sigma\}$ is an open cover of G (i.e. $\mathsf{G} \subset \cup_{\sigma\in\Sigma} W_\sigma$). Finally, we define the map $\psi_\sigma : W_\sigma \to \Re^{J(S-J)}$ such that $\psi_\sigma(\mathcal{L}) = A$. This map is an homeomorphism of W_σ onto $\Re^{J(S-J)}$ (i.e. continuous bijection).

(Grassmanian manifold) Being a topological space homeomorphic to $\Re^{(S-J)J}$, G is a manifold of dimension $J(S-J)$, called *Grassmanian* manifold. The *chart* of G is defined by (ψ_σ, W_σ), and its *atlas* is $\{(\psi_\sigma, W_\sigma) : \sigma \in \Sigma\}$. Further, G is compact without boundary. More precisely, every element of its open cover, W_σ contains the limit points of $W_{\sigma'}$, for some $\sigma' \neq \sigma$. In turn, the boundary points of W_σ are contained in $W_{\sigma''}$, for some $\sigma'' \neq \sigma$. The structure of this manifold implies that its charts never overlap if coordinates change.

2.3 Consumers

There are $H \geq 2$ consumers' types indexed by $h = 1, .., H$, each of whom is endowed of a column vector $e^h \in \Re^N$ of contingent commodities. For convenience, we assume that consumption sets coincide with the non negative orthant of the commodity space, $\Re^N_{++}$. Further, we assume that a utility function $u^h : \Re^N_{++} \to \Re$ represents consumer $h's$ preference ordering over his consumption set. Finally, some standard properties on preferences are summarized in the following.

Assumption a (strictly positive endowments)
$e^h \in \Re^N_{++}$

Assumption b (smooth preferences)

1. $\forall h$, u^h is continuously differentiable on $\Re^N_+$, and strictly monotonic (i.e. $Du^h(x) \in \Re^N_{++}$, $\forall x \in \Re^N_{++}$);

2. $\forall h$, u^h is differentiably strictly concave in the neighborhood of each point $x \in \Re^N_{++}$ (i.e. $qD^2u^h(x)q^T < 0$, $\forall x \in \Re^N_{++}, \forall q \in \Re^N$, $q \neq 0$, such that $Du^h(x)\, q^T = 0$);

3. $\forall h, \forall x' \in \Re^N_{++}$, indifference curves are bounded (i.e. if $\mathcal{R}^h(x) = \left\{ x \in \Re^N_{++} : u^h(x') \geq u^h(x) \right\}$ then $cl\mathcal{R}^h(x) \subset \Re^N_{++}$).

We denote with U the space of utility functions satisfying assumption **b**.

Finally, we state two assumptions on the number of consumer types that will be alternatively used later in the paper.

Assumption c

$\min [S-J, J] \leq H-1$

Assumption d

$J < H-1 < S-J$.

3 Command economy and Constrained Pareto Optimality

Consider an economy with fixed aggregate resources $r \in \Re^N_{++}$. The pace of endowments is, $\Omega(r) = \left\{ e \in \Re^{NH}_{++} : \sum_h e^h = r \right\}$. Assume that there is a benevolent central Planner whose action set is characterized by two fundamental aspects: (a) wealth transfers are restricted by the financial structure of the economy; (b) the planner can affect consumers' risk sharing opportunities by controlling the asset span. The first aspect implicitly exclude lump sum transfers as feasible instruments. Point (b) is equivalent to impose that the Planner is enabled to optimally choose an element $\mathcal{L}$ on G. Notice that this is equivalent to assume that the Planner can pick feasible income-transfers which can at most be of dimension equal to the minimum between the number of assets, J, and the number of financially constrained consumers, $H-1$. In particular, we restrict our attention to the case of $J < H-1$. The opposite case $(H-1 \leq J)$ is non generic since, given J asset, it is always possible to design their payoff structure such that asset j has a return profile that exactly match the needs of income redistribution of consumer j, and thus for $j = 1, .., H$, allows to reach an equilibrium with full insurance (i.e. a *first best*).

typically, leads to an allocation which can be decentralized as a contingent market equilibrium, and therefore as a first best (see section 4 on the non genericity of first best allocations). Further, observe that, assuming that G is given to the Planner, implies that the existent market structure (fully represented by G) is also given.

We are now ready to formalize the set of the Planner's attainable transfers z^h at $\mathcal{L}$,

$$\mathbb{N}(\mathcal{L}) = \left\{ z \in \Re^{NH} : \begin{array}{c} \sum_h z^h = 0 \\ z^h_1 \in \mathcal{L}, \forall h \geq 2 \end{array} \right\}$$

<u>**Fact 1**</u> : $\mathbb{N}(\mathcal{L})$ is nonempty, convex, and compact for every $\mathcal{L} \in \mathsf{G}$.

$\mathbb{N}$ differs with respect to the standard feasible set for the addition of the financial constrains, $z_1^h \in \mathcal{L}$, for $h = 2, .., H$.. These are a total of $(S-J)(H-1)$ linear constrains[8]. The set is bounded, otherwise some resource constraints would be violated. Finally, it is closed since we can think at $\mathbb{N}$ as the intersection of itself with the closed set

$$\mathbb{M} = \left\{ z \in \Re^{NH} : \begin{array}{c} z_s^h \leq r_s, \forall s \\ \sum_h z_s^h = 0, \forall s \end{array} \right\}$$

Define the (Lagrangian) objective function faced by the planner as

$$\Psi(z, \mathcal{L}, \upsilon, \gamma, e, \lambda) = \sum_h \lambda^h u^h(z^h + e^h) - \upsilon \sum_h z^h - \sum_{h \geq 2} \gamma^h \left(\begin{array}{cc} I_{S-J} \mid & A \end{array} \right) \pi_\sigma z_1^h$$

For given $(\lambda, e) \in \Re^H \times \Omega(r)$, the typical Planner's maximum problem can be represented as,

$$\max_{(z,\mathcal{L}) \in \Re^{HN} \times \mathsf{G}} \Psi(z, \mathcal{L}, \upsilon, \gamma, e, \lambda)$$

with $(\upsilon, \gamma) \in \Re^N \times \Re^{(H-1)(S-J)}$ denoting the multipliers of the programming problem.

Definition 1 *($\mathsf{J}-$Constrained Pareto optimum) A local maximum for $\Psi(z, \mathcal{L}, \upsilon, \gamma, e, \lambda)$ at $(\lambda, e) \in \Lambda^{H-1} \times \Omega(r)$ is a constrained Pareto optimum $\mathsf{J}-CPO$. The set of $\mathsf{J}-CPO$ allocations is*

$$\mathcal{P} = \left\{ z + e : (z, \mathcal{L}) \in \arg\max \Psi(z, \mathcal{L}, \upsilon, \gamma, e, \lambda), (\lambda, e) \in \Lambda^{H-1} \times \Omega(r) \right\}$$

Proposition 2 *(CPO existence) Under assumptions* **a** *and* **b***, $\mathcal{P}$ is nonempty.*

This follows from the fact that the Planner's problem consists in maximizing a continuous real valued function on a compact set.

3.1 The structure of $\mathcal{M}$

A problem that immediately arises from our formulation consists in establishing the topological properties of the set of solutions of the planner's maximum problem. This is fundamentally due to the existence of two source of non-convexity: the first is the direct consequence of the intrinsic non-convexity of *Grassmanians*; the second depend on the fact that two choice variables $(z, \mathcal{L})$ enter in the problem multiplicatively. Yet, for any given $\mathcal{L} \in \mathsf{G}$, assumption **b** implies that Ψ_λ is a differentiably strictly concave function on the commodity space.

Let Λ^{H-1} denote the interior of the unit $(H-1)-$ simplex. Using the concept of *strict genericity* to indicate the properties of openness and full Lebesgue measure of a set, we state the following.

[8] After each permutation, σ, we have the following $(S-J)-$system of linear constraints, $z_{\backslash\sigma}^h = Az_\sigma^h$, for $h = 2, .., H$.

Proposition 3 *Let assumptions* $\mathbf{a},\mathbf{b},\mathbf{c}$ *hold. There exists a strictly generic set of endowments and welfare weights* $\widehat{\Pi}^{*}=\Omega^{*}\times\Lambda^{H-1}\subset\Omega\left(r\right)\times\Re^{H}$*, such that for every* $(\lambda,e)\in\widehat{\Pi}^{*}$*,* $\mathcal{M}$ *is a smooth manifold of dimension* $(H-1)+(H-1)N$*.* $\mathcal{M}$ *is diffeomorphic to* $\widehat{\Pi}^{*}$*.*

Theorem 4 *Let assumptions* $\mathbf{a},\mathbf{b},\mathbf{c}$ *hold. There exists a strictly generic set of endowments and welfare weights* $\widehat{\Pi}^{**}=\Omega^{*}\times\Lambda^{*}\subset\Omega\left(r\right)\times\Re^{H}$*, such that for every given* $e\in\Omega^{*}$*,* $\mathcal{M}_{e}$ *is a smooth manifold of dimension* $(H-1)$*.*

In order not to break the flow of exposition we defer the proofs of these results to Section 8. Here we only wish to provide to the reader the underlined idea of the proof. The non-convexity of the problem suggested us the following strategy. First, we show that there exists a strictly generic set of endowments Ω^{*} such that for every $(e,\lambda)\in\Omega^{*}\times\Re^{H}$ the set of *quasi-saddle* point solutions (denoted by $\mathcal{G}$) is a manifold. Second, we prove that the set of local maxima, $\mathcal{M}$, is a manifold (strictly) contained in the set of regular points of $\mathcal{G}$. Then, we apply transversality arguments to infer on the structure of $\mathcal{M}_{e}$.

Remark 5 *The last proposition formalizes the idea that* $\mathcal{M}_{e}$ *is characterized by a finer structure than* $\mathcal{M}$*. Moreover, we can now apply the Implicit Function Theorem to express the policy instruments* $(z,\mathcal{L})$ *as smooth functions of the welfare weights* $\lambda\in\Lambda^{*}$ *(and this holds for every initial distribution of endowments in* Ω^{*}*).*

3.2 The structure of $\mathcal{P}$, and of the Pareto frontier

Recall that Λ^{*} denotes an open subset of welfare weights contained in the interior of the $(H-1)$ unit simplex. And fix the endowments such on Ω^{*}, so that $\mathcal{M}_{e}$ is a smooth manifold (Theorem 4). We are going to show that both $\mathcal{P}$, and the corresponding Pareto frontier, are intrinsically equivalent to the generic space of welfare weights. However, since we restrict the planner's action on economies in Ω^{*}, and since we also restrict the space of attainable transfers, the graph of the Pareto frontier lies strictly below the graph of the *first best* (or complete markets) frontier. One may expect the existence of paths connecting the two frontiers, yet these will correspond to a negligible subset of endowments that is a complement of Ω^{*} (i.e. a set that is closed and of null Lebesgue measure in $\Omega(r)$).

Observe that, by assumption b, each consumption set coincides with $\Re^{N}_{++}$, and $x\in\mathcal{P}$ implies $u^{h}(x)\gg 0$, for all h. We can thus abstract from boundary problems, and apply standard arguments.[9]

[9] See, for example, Mas-Colell (1985), chapter 4.

Theorem 6 *In the context of Theorem* 4, $\mathcal{P}$ *is a smooth manifold diffeomorphic to the* $(H-1)$ *dimensional space* Λ^*.

Proof: Consider an allocation x corresponding to a regular point in $\mathcal{M}_e$. Define the mapping $\phi : \Lambda^* \rightarrow \mathcal{P}$, such that $\phi(\lambda) = x$. ϕ is one to one (see Remark 5). To prove that ϕ is a diffeomorphism, we also need to show that the inverse of ϕ is smooth. This follows immediately if we observe that ϕ^{-1} is a map $x \rightarrow \left(.., \frac{D_{x_0}u^1(x)}{D_{x_0}u^h(x)}, ..\right)$.□

We can also represent constrained efficient allocations in the space of consumer utilities. Let $U : \mathcal{P} \rightarrow \Re^H_+$, such that $U(x) = \left(u^1(x^1), .., u^H(x^H)\right)$. Again, by assumption b, $x \in \mathcal{P}$ implies that $x \in \Re^{HN}_{++}$, and $U(x) \gg 0$. For fixed endowments $e \in \Omega^*$, define

$$\begin{aligned} \mathcal{U} &= \left\{U \in \Re^H_+ : U(x) = \left(u^1(x^1), .., u^H(x^H)\right), \, x = z + e, z \in \mathbb{N}\right\} \\ \widehat{\mathcal{U}} &= \left\{U \in \Re^H_+ : U(x) = \left(u^1(x^1), .., u^H(x^H)\right), \, x \in \mathcal{P}\right\} \end{aligned}$$

The following can be stated.

Proposition 7 *In the context of Theorem* 4, $\widehat{\mathcal{U}}$ *is diffeomorphic to the* $(H-1)$ *dimensional space* Λ^*.

Proof: First, it straightforward to see that $\mathcal{U}$ is homeomorphic to the unit simplex. In fact, $i)$ preferences are monotonic, $ii)$ the space of attainable utility levels $\mathcal{U}$ is compact. Thus, $\varsigma : \Lambda^* \rightarrow \mathcal{U}$ such that $\lambda = \alpha U(x(\lambda))$, $\alpha \in \Re_{++}$, is an homeomorphism. To prove that it is also a diffeomorphism it is sufficient to observe that $\widehat{\mathcal{U}}$ is a manifold (when $e \in \Omega^*$).□

4 Welfare Theorems

In this section we are going to introduce a standard notion of competitive economy. This is instrumental to define the concept of competitive equilibrium and discuss its welfare properties.

4.1 Competitive economy and equilibria

The structure given to our economy is the one-commodity version of Duffie and Shafer (1985). Suppose that the financial structure of our economy is represented by a fixed matrix Y in general position (i.e. of full column rank J). The requirement $\langle Y \rangle \subseteq \mathcal{L}$ is always trivially satisfied with equality. Therefore, the financial opportunities of private agents can be equivalently represented with $\mathcal{L}$ (or A); and the economy can be directly parametrized by $\mathcal{L}$. This is not true when multiple goods are considered since in that case the asset payoff structure depends also on commodity prices and thus it is (at least partially) determined endogenously (i.e. the *asset span* depends on Y and on commodity prices). For fixed preference $u \in \mathrm{U}$, and e satisfying assumption **a**, denote such economy as $\mathbb{E}(e, \mathcal{L})$.

Without loss of generality, assume that the first consumer ($h = 1$) is financially unconstrained.[10]. Let w^h define the present value wealth of consume h, evaluated at $s = 0$. This results in the following budget representation,

$$\beta\left(p, w^1\right) = \left\{x^1 \in \Re^N_{++} : px^1 - w^1 = 0\right\}.$$

The budget set of every other consumer h, for $h = 2, .., H$, is

$$\mathbb{B}\left(p, \mathcal{L}, e^h\right) = \left\{ x^h \in \Re^N_{++} : \begin{array}{c} p\left(x^h - e^h\right) = 0 \\ z^h_{\mathbf{1}} \in \mathcal{L} \end{array} \right\}$$

Next, define the domain of prices and financially feasible reallocation by $D = \Delta^S \times \mathsf{G}$.[11] This, simplifying the consumer problem, will reduce redundancies in the equilibrium implicit asset prices and allocations.[12]

Definition 8 *($\sigma-$CE) A $\sigma-$competitive-equilibrium in $\mathbb{E}(e, \mathcal{L})$, for fixed $u^h \in \mathrm{U}$ and every $(\mathcal{L}, e) \in \mathsf{G} \times \Re^{NH}_{++}$, is a vector $(\overline{x}, \overline{p}) \in \Re^{NH}_{+} \times \Delta^S$ such that,*

(i) $\overline{x}^1 = \arg\max\left\{u^1(x^1) : x^1 \in \beta\left(\overline{p}, w^1\right)\right\}$,

(ii) $\overline{x}^h = \arg\max\left\{u^h(x^h) : x^h \in \mathbb{B}\left(\overline{p}, \mathcal{L}, e^h\right)\right\}$, $h = 2, .., H$,

(iii) $\sum_h \left(\overline{x}^h - e^h\right) = 0$.

Under assumptions **a**, **b**, and for every $\mathcal{L} \in \mathsf{G}$, one can use a standard fixed point theorem to show that a $\sigma-CE$ exists, and is locally unique. The following system of first order conditions is necessary and sufficient to establish existence. $(\eta, p, \delta) \in \Re^{(S-J)(H-1)}_{++} \times \Re^N_{++} \times \Re^H_{++}$ are Kuhn-Tucker multipliers of the consumers' problems, and we denote trade by $z^h = x^h - e^h$,

[10] This is the so called "Cass trick", introduced to prove existence of equilibria in Cass (1984). This condition is sufficient to ensure that the aggregate demand is well behaved at the boundary. Generic existence for the case of symmetrically defined consumers was proved in Polemarchakis and Siconolfi (1995).

[11] Δ^S denotes the interior of the $S-$dimensional simplex. The normalization of p follows from the linear homogeneity of the budget constraint.

[12] We follow the convention to denote by p^h the vector of state prices. Since p^h is restricted to lie in the set of no-arbitrage prices, we have that $p^h \in \mathcal{L}^{\perp}$, i.e. $p^h_{\mathbf{1}}$ is a vector that contained in an $(S-J)$-dimensional subspace in $\Re^S$. Moreover, in equilibrium, no-trade requires that $p^h - p^1 \in \mathcal{L}^{\perp}$, for all h. This implies that, without loss of generality, one can restrict the set of personalized prices to $D \in \left(S^S_{++} \times \mathsf{G}\right)/\tilde{}$, of typical element $(p, \mathcal{L})$, with the equivalence class $\sim$ on $S^S_{++} \times \mathsf{G}$ defined by

$$(p, \mathcal{L}) \sim (p', \mathcal{L}') \;\Leftrightarrow\; \mathcal{L} = \mathcal{L}' \text{ and } (p - p') \perp \mathcal{L}$$

For more on this see Polemarchakis and Siconolfi (1995).

[**Foc CE**]

$$\begin{array}{ll} & Du^1\left(z^1+e^1\right)-\delta^1 p=0, \\ I) & D_{z_0^h}u^h\left(z^h+e^h\right)-\delta^h p_0=0, \forall h\geq 2, \\ & D_{z_1^h}u^h\left(z^h+e^h\right)-\delta^h p_1+\eta^h\left(\begin{array}{cc} I_{S-J} \mid & A \end{array}\right)=0, \forall h\geq 2, \\ II) & pz^h=0, \forall h \\ III) & \sum_h z^h=0, \\ IV) & \left(\begin{array}{cc} I_{S-J} \mid & A \end{array}\right)\pi_\sigma z_1^h=0, \forall h\geq 2. \end{array}$$

Observe that for $h \geq 2$, we can rewrite $I)$ as

$$Du^h\left(z^h+e^h\right)=\rho(p,\mu^h,\delta^h)$$

where

$$\mu^h \equiv \mu(\eta^h,\mathcal{L};\sigma)=\eta^h\left(\begin{array}{cc} I_{S-J} \mid & \psi_\sigma(\mathcal{L}) \end{array}\right)\pi_\sigma \in \Re^S$$

and ρ is a *personalized price functional*

$$\rho(\delta^h,p,\mu^h)=\left(\delta^h p+\left(0,\mu^h\right)\right)\in \Re^N$$

The set of pseudo-equilibria is

$$\mathcal{E}=\{z,p,\eta,\delta,\mathcal{L},e: \text{ [\textbf{Foc CE}] hold}\}$$

4.1.1 The structure of $\mathcal{E}$

Proposition 9 *$\mathcal{E}$ is a smooth manifold diffeomorphic to $\mathsf{G}\times\Re^{HN}_{++}$.*

We omit a formal existence proof since this result is well known (see Duffie and Shafer (1985)).

For fixed resources $r \in \Re^{S+1}_{++}$, we have the following.

Corollary 10 *For fixed total resources, $\mathcal{E}(r)$ is a manifold diffeomorphic to $\mathsf{G}\times\Re^{(H-1)N}_{++}$.*

4.2 Welfare properties of equilibria

We now turn to examine the welfare properties of equilibria under the $\mathsf{J}-CPO$ criterion.

Theorem 11 *(II Welfare Theorem): Every constrained Pareto optimum allocation can be decentralized as a competitive equilibrium for an appropriate choice of endowments.*

Proof: First, notice that if $\left(z', \upsilon', \gamma', \mathcal{L}', e', \lambda'\right) \in \mathcal{M}$, then - for given $\mathcal{L}'$ - $z' + e'$ is a $D - CPO$. We can find market prices needed to decentralize the optimal allocation at $\mathcal{L}'$. To this end, just pick $\delta^{h'} = \upsilon_0'/\lambda^h$, $p_s' = \upsilon_s'/\upsilon_0'$ for all $s \geq 1$, $\eta^h = \gamma^{h'}/\lambda^h$, for all $h \geq 2$. Then, $\left(z', p', \eta', \delta', \mathcal{L}', e'\right)$ satisfies $[\mathbf{Foc\,A - CE}]$. Therefore, $\left(z', p', \eta', \delta', \mathcal{L}', e'\right) \in \mathcal{E}(r)$. □

Remark 12 *Since a* $\mathsf{J} - CPO$ *exists and is regular, the II Welfare Theorem implies that the equilibrium set* $\mathcal{E}(r)$ *is nonempty (i.e. equilibria exist). Moreover, for every* $e \in \Omega^*$ *equilibria are regular (i.e. they are smooth functions of endowments) and unique.*

However, the most interesting result explored in the literature is the (generic) failure of the I Welfare Theorem. An intuitive way to present this result is to compare the set of equilibria, $\mathcal{E}(r)$, with the set of constrained Pareto optimal equilibria. Formally, the latter set is given by $\mathcal{E}(r) \cap \mathcal{M} = \mathcal{E}_{\mathcal{M}}(r)$. Thus, it is easy to see that an element of $\mathcal{M}$ does also belong to $\mathcal{E}_{\mathcal{M}}(r)$, if the planner is also restrict to satisfy $(H-1)$ additional constraints: the individual budget constraint $px^h - pe^h$, for all $h = 2, .., H$. Then the following can be stated.

Lemma 13 *The set of constrained Pareto optimal equilibria* $\mathcal{O}(r)$ *is a lower dimensional manifold contained in* $\mathcal{E}(r)$*, its codimension* $J(S-J)$*.*

The latter lemma immediately implies.

Theorem 14 *(Constrained Pareto Suboptimality) In an economy with (at least) one asset, competitive equilibria are, typically,* $J-$*constrained Pareto suboptimal.*

The following corollary is a refinement of our constrained inefficiency result.

Corollary 15 *Suppose that the planner can independently control the columns of a given asset payoff matrix* $\overline{R}$*. Then, competitive equilibria are, typically,* $J-$*constrained Pareto suboptimal. And the codimension of* $\mathcal{O}$ *in* $\mathcal{E}(r)$ *is* S*.*

Proof: Take any $\overline{R} = [v]\,R$, and v such that $\left\langle \overline{R} \right\rangle = \mathcal{L}$. This implies that we require $(I_{S-J} \mid A)\,\pi_\sigma \overline{R} = 0$ for $\sigma \in \Sigma$ such that $A = \varphi_\sigma(\mathcal{L})$ and $\mathcal{L} \in W_\sigma$. Just rewriting the latter, $\overline{R}_{/\sigma} = -A\overline{R}_\sigma$, or $A = -\overline{R}_{/\sigma}\overline{R}_\sigma^{-1}$. Thus A has elements $a_k^j = \left(\frac{v_{/\sigma,k}}{v_{/\sigma,j+S-J}}\right) r_k^j$, where r_k^j is the (given) typical element of $-R_{/\sigma}R_\sigma^{-1}$, with $k = 1, .., S-J$, $j = 1, .., J$. Define $\tau : \Delta^{S-1} \to W_\sigma$, such that $\tau(v) = \mathcal{L}$. Moreover, the composition $\varphi_\sigma \circ \tau$ is such that $(\varphi_\sigma \circ \tau)(v) = A$, with each element of A equal to $a_k^j = \left(\frac{v_{/\sigma,k}}{v_{/\sigma,j+S-J}}\right) r_k^j$, for all k and j. Using the fact that the mapping $A = \varphi_\sigma(\mathcal{L})$ is a smooth bijection, it is immediate to see that $(\varphi_\sigma \circ \tau)$ is a smooth injection. □

5 Financial Innovation

5.1 Designing the asset structure with J available assets

Consider a notion of *financial innovation* that consists in giving a fictitious centralized planner the ability to optimally define the payoff structure of J existent assets. Then, Theorem 14, and Corollary 15, above directly implies the following generalization of Demange-Laroque (1995), obtained for every parametrization of preferences $u \in \mathrm{U}$.

Corollary 16 *In the context of Theorem 14, assume that* $(H-1) < \min[S, J(S-J)]$ *and* $u \in \mathrm{U}$. *There exists a strictly generic set of economies, parametrized by endowments, in which financial innovation is welfare improving.*

5.2 Designing "new" assets

Cass-Citanna (1998), and Elul (1995), explored this notion in a multiple goods GEI. In those papers it is shown that, when economies have a sufficiently large degree of market incompleteness with respect to the number of agent types, it is possible to introduce new assets in the economy such that the resulting equilibria are generically Pareto superior. To avoid discontinuity problems, it is assumed that the new assets are initially redundant (or non-redundant but untraded or unwanted). Then, loosely speaking, it is sufficient to show that, generically, the resulting equilibrium has as an asset matrix of full column rank. Therefore, the policy intervention has the final effect of both changing the degree of market incompleteness (i.e. to push the economy toward market completeness), and to modify the position of the asset span. Elul (1999) extends this analysis to a single good economy. Elul's idea is based on the observation that, typically, financial innovation changes the equilibrium asset prices, and may cause welfare effects of opposite sign. However, he noticed that it also has indirect income effects in the form of gains from trading the new assets (i.e. pure *span effects*). The latter are never socially undesirable, since they cannot hart anyone. But, most importantly, they are generically beneficial, since equilibria are (typically) characterized by the fact that consumers fully exploit financial trade opportunities. Elul shows that some forms of financial innovation do not have price effects, and thus, by a *revealed preference* argument, concludes that (under certain conditions) they are generically constrained Pareto improving.

The results obtained in the first part of this paper allow to re-examine Elul (1999). Thus, imagine that the planner can modify the asset structure introducing a new asset. More precisely assume that he can now choose an asset span on $\mathsf{G}^{J+1,S}$ rather than on $\mathsf{G}^{J,S}$; call the new problem $(\mathsf{J}+1)-CPO$. Once a *Grassmanian* is in place, the planner problem is specified as above.

Corollary 17 *Let assumptions* $\mathbf{a}, \mathbf{b}, \mathbf{d}$ *hold. For every economy in a strictly generic subset of economies in* Ω^* *there exists a direction in which financial innovation is welfare improving.*

Proof: see Section 8.

Our proof applies the logic used in Elul (1999) to the planner's problem. Consider the action set of the planner's problem in $\mathsf{G}^{J+1,S}$; this trivially contains the one obtained in $\mathsf{G}^{J,S}$ (just observe that any $\mathcal{L}^J \in \mathsf{G}^{J,S}$, is contained in all $\mathcal{L}^{J+1} \in \mathsf{G}^{J+1,S}$). Starting from a $\mathsf{J}-CPO$, it is possible to introduce an asset in a way that -at the new optimum- consumers' evaluations of the pre-existent asset structure is unchanged, and the new asset is valued. To do so, first, it is necessary that, after the introduction of the new asset, consumers' gradients still belong to the subspace induced by the original asset span, $\left(\mathcal{L}^J\right)^{\perp}$. In fact, the condition of no-arbitrage and the one ensuring that consumers' agree on the evaluation of the existent assets imply that $(\varsigma^h_{J+1} - \varsigma^H_{J+1}) \in \left(\mathcal{L}^J\right)^{\perp}$, for all $h \neq H$, with $\varsigma^h_J \equiv \gamma^h \left(I_{S-J} \mid \psi_\sigma \left(\mathcal{L}^J\right)\right)$. Since the dimension of $\left(\mathcal{L}^{J+1}\right)^{\perp}$ is $S-J-1$, this is possible if $H-1 \leq S-J-1$, a condition that we also find in Elul's. Finally, if the new asset is actually used by the planner to achieve the new optimal allocation, one can apply a revealed (social) preference argument and conclude the proof.

Although more abstract, this argument provides a good geometric intuition of the above argument. As the number of assets increases, the dimension of $\mathcal{L}$ increases, while that of its normal, $\mathcal{L}^{\perp}$, decreases. This implies that consumers, progressively, face weaker financial constraints, and their evaluation of assets tend to converge to unanimity. Yet, this also implies that it becomes, progressively, more difficult to ensure that consumers evaluation of pre-existing assets remains unchanged as $\mathcal{L}^{\perp}$ shrinks.

Since we look directly at the planner's problem some differences with respect to Elul's arise. The most relevant one concerns the assumption on preferences. In fact, our result is proved to hold for any class of preferences satisfying assumption **b**, rather that only for quadratic preferences. Moreover, under Elul's assumptions, we show that some form of social desirable financial innovation can be found for every generic economy. Where now "generic" refers to a set of endowments that is open and of full Lebesgue measure in Ω^*, and for fixed preferences.

A final qualification concerns the possibility that financial innovation be socially undesirable, either because it induces a strictly Pareto suboptimal allocation or just because it harts someone. In general, when a central planner intervention affects consumers evaluations of the existent assets than it may not even be a weak Pareto optimum: some consumers, and eventually all of them, may be made worse off. In a single commodity world, asset price variation results in a change of real allocation.

6 More on Policy and Spanning

Our notion of constrained Pareto optimality can now be used to study specific policies operating through asset span effects. Take the extreme case of monetary policy, as it is described in the introduction. Here, the planner controls a minimal set of instruments, lowering its number by just one would not allow him to control the asset span. Recall that the payoff matrix in real terms is now,

$$\overline{R} = [v]\, R,\ where\ [v] = \begin{pmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_S \end{pmatrix}$$

where v_s is the purchasing power of the unit of account, M, in state s.

Corollary 18 *Let assumptions* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *hold. For every solution to the planner's maximum problem, there exists a unique optimal monetary policy.*

Proof: Take $v^* = \tau^{-1}(\mathcal{L})$, where τ has been defined in the proof of Corollary 15. Next, apply a Fisher like equation: an optimal money supply rule is a function $M_s\left(v_s^*, e_s\right) = \frac{1}{v_s^*} \sum_h e_s^h$, for $s = 1, .., S$. $\square$

Next, consider the case of capital income taxes. First, assume that the planner can tax asset payoff *ad valorem*, contingently on states, t_s. Further, assume that he faces a budgetary rule of the type $\sum_{s=1}^{S} t_s r_s^j = 0$ for $j = 1, .., J$. This implies that the payoff matrix has the same structure presented above, $\overline{R} = [t]\, R$. The above results imply.

Corollary 19 *Let assumptions* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *hold. For every solution to the planner's maximum problem, there exists a unique optimal tax policy.*

The tax policy rule is obtained as the preimage of the function $\tau : \Delta^{S-1} \to W_\sigma$, evaluated at the optimal asset span $\mathcal{L}$.

Remark 20 *(optimal policy rules) Observe that our argument for optimal policies uses the definition of mappings τ. In particular theirs inverses are smooth mappings between an optimal* *choice of $\mathcal{L}$ and a set of policy instruments. Yet, we know that, under appropriate conditions, $\mathcal{L}$ can be expressed as a smooth function of both endowments and welfare weights, for all $(\lambda, e) \in \Lambda^{H-1} \times \Omega^*$. This means that we can establish how optimal policies vary for different distributions of initial resources and (or) for alternative profiles of welfare weights.*

7 Example

Consider an economy with four states of nature in the second period, and one asset ($S = 4$, $J = 1$); we are in the case in which $J < H < S - J$. There are two consumers, characterized by Von Neuman-Morgenstern utility functions, $u(x^h) = \ln x_0^h + \sum_{s=1}^{3} \rho_s^h \ln x_s^h$, $h = a, b$. Endowments are, $e^a = (2, e+h, 0.5, e+h, 0.5)$, $e^b = (2, e, 0.5, e, 0.5)$, with h to be read as a measure of aggregate risk. Further, assume, for simplicity, that welfare weights are uniform, and $\rho_1^a = \rho_2^b = 0.5 - \varepsilon$, $\rho_2^a = \rho_1^b = \varepsilon$, $\rho_s^a = \rho_s^b = 1/4$ for $s = 3, 4$. A $J - CPO$ problem has a maximum with the optimal asset span defined by the 3 - dimensional plane $(1, A^*)$, of typical coordinate $a_k^* = a_k(e, h, \varepsilon)$, $k = 1, 2, 3$,

$$
\begin{aligned}
a_1^*(e, h, \varepsilon) &= 4.0 \frac{(h+e)(1-2\varepsilon) + \varepsilon}{2e - 1} \\
a_2^*(e, h, \varepsilon) &= -4.0 \frac{e(1-2\varepsilon) + \varepsilon}{2e - 1} \\
a_3^*(e, h, \varepsilon) &= \frac{2(e+h) - 1}{2e - 1}
\end{aligned}
$$

Thus, for example, if $e = 2h = 2$, and $\varepsilon = 0.1$,

$$(a_1^*, a_2^*, a_3^*) = (2, -2, 1).$$

If we provide the market economy with a payoff structure compatible with this one, the corresponding GEI equilibrium is $J - CPO$. Moreover, because of the particular distribution chosen for the endowments, the allocation is a first best: both agents attain the same utility level, and are able to cross insurance themselves. Given endowments, the entity of income transfers is also affected by consumers' beliefs about the future. For example, consumer b, who foresees that will be relatively richer in state 2, a state to which he only attach probability 0.1, promises to make a net payment of 1.5 to a, if state 2 occurs; the complete table of net transfers is,

s	$z^a = 1 - z^b$	ρ^a	ρ^b	e^a	e^b
1	-1.5	0.4	0.1	2	0.5
2	1.5	0.1	0.4	0.5	2
3	-0.75	0.25	0.25	2	0.5
4	0.75	0.25	0.25	0.5	2

However, our assumptions on how the economy is structured imply that the set of $J - CPO$ is lower dimensional with respect to the set of equilibria. This implies, that starting from a $J - CPO$ equilibrium, a marginal change in the fundamentals (say, in the endowments), typically, leads the economy to an equilibrium that is not $J - CPO$. For example, suppose that in the above economy the endowments of consumer a in state 1 and 2 are increased by 0.5 units of consumption. The new equilibrium is not even $J - CPO$. The new $J - CPO$ space of feasible transfers has coordinates,

$$\left(a_1^*, a_2^*, a_3^*\right) = (2.53, -2, 1.33) .$$

8 Proofs

8.1 The command economy

Saddle point characterization We start by deriving the structure of the set of *quasi-saddle* point solutions to the Planner's problem.

Lemma 21 *The first order (necessary) conditions of the Planner's maximum problem hold at* $\left(\widehat{z}, \widehat{\mathcal{L}}, \widehat{\upsilon}, \widehat{\gamma}\right)$ *if and only if* $\left(\widehat{z}, \widehat{\mathcal{L}}, \widehat{\upsilon}, \widehat{\gamma}\right)$ *is a quasi-saddle point.*

Proof: Just observe the first order conditions of the maximum problem can be written as,

[**Foc PP**]

$$\begin{array}{ll} & \lambda^1 Du^1\left(z^1+e^1\right) - \upsilon = 0 \\ 1) & \left.\begin{array}{l} \lambda^h D_{z_0^h} u^h\left(z^h+e^h\right) - \upsilon_0 = 0 \\ \lambda^h D_{z_1^h} u^h\left(z^h+e^h\right) - \upsilon_1 - \gamma^h \left(\begin{array}{cc} I_{S-J} \mid & A \end{array}\right) = 0 \end{array}\right. ,\forall h \geq 2, \\ 2) & \sum_h z^h = 0, \\ 3) & \left(\begin{array}{cc} I_{S-J} \mid & A \end{array}\right) \pi_\sigma z_1^h = 0, \forall h \geq 2, \\ 4) & \sum_{h\geq 2}\left(\gamma_1^h z_\sigma^{hT}, ..\gamma_{S-J}^h z_\sigma^{hT}\right) = 0. \end{array}$$

Let $z_\sigma = \left(z_\sigma^2, .., z_\sigma^H\right)$ be a $J \times (H-1)$ matrix, and $\Gamma = \left(\gamma^{2^T}, .., \gamma^{H^T}\right)$ is a $(S-J) \times (H-1)$ matrix. Condition 4) is the one obtained maximizing $\widehat{\Psi}$ with respect to A (or $\mathcal{L}$), and can now be rewritten as $\Gamma z_\sigma^T = 0$. Thus, for 4) to hold it is sufficient either one of the following: (a) $rank(z) = J$; (b) $rank\,(\Gamma) = S - J$. Loosely speaking, an asset span is optimal when is selected in a way that either when allows the planner to perform the maximum number of possible income redistributions, J, (a); or when it allows him to make better off the maximum number of consumer (types), $S - J$, by independently controlling their gradients (i.e. their ex-ante evaluations of future consumption profiles), (b). Conditions (a) and (b) should be considered one the dual of the other.[13]

Denote $\xi = (z, \mathcal{L}, \upsilon, \gamma)$, and $\Xi = \Re^{HN} \times \mathsf{G} \times \Re^N \times \Re^{(H-1)(S-J)}$. [**Foc PP**] are equivalent to the *quasi-saddle point conditions*,

$$\begin{aligned} D_\xi \widehat{\Psi}_\lambda &= 0 \\ (\widehat{\upsilon}, \widehat{\gamma})\, D_{\upsilon,\gamma} \widehat{\Psi}_\lambda^T &= 0, \end{aligned}$$

for $\widehat{\Psi}_\lambda \equiv \Psi_\lambda \left(\widehat{\xi}, e\right)$.$\square$

Let the system **Foc PP** be represented by the mapping $F_\sigma\,(\xi, \lambda, e) = 0$. Define the set of *quasi-saddle* points as

$$\mathcal{G} = \{\xi, \lambda, e \in \Xi \times \Pi\}$$

where

$$\Pi = \left\{ (\lambda, e) \in \Re^H \times \Omega(r) : \begin{array}{l} F_\sigma\,(\xi, \lambda, e) = 0, \xi \in \Xi \\ \text{and } \sigma \; s.t. \; \mathcal{L} \in W_\sigma \end{array} \right\}$$

Proposition 22 *Under assumptions* $\mathbf{a}, \mathbf{b}, \mathbf{c}$, *there exists a strictly generic set* $\Pi^* \subset \Pi$, *such that for every* $(\lambda, e) \in \Pi^*$ *the set of quasi-saddle points,* $\mathcal{G}$, *is a smooth manifold diffeomorphic to* Π^*, *of dimension* $H + (H-1)N$.

Proof :

For clarity of exposition, we shall start by establishing a few preliminary results.

Since F_σ is a continuous real valued function defined on a compact set, it follows that,

<u>Fact 2</u> : For every $\sigma \in \Sigma$, and for every $(e, \lambda) \in \Re_{++}^{HN} \times \Re^H$, $F_\sigma\,(\xi, \lambda, e) = 0$ has a solution. Therefore, $\Pi = \Re_{++}^{HN} \times \Re^H$.

[13] In fact, using (column) subspaces notation, 4) can also be written as $\langle\Gamma\rangle \subset \langle z\rangle^\perp = 0$, and $\langle z\rangle \subset \langle\Gamma\rangle^\perp = 0$.

Let $z_\sigma = \left(z_\sigma^2, .., z_\sigma^H\right)$ be a $J \times (H-1)$ matrix, and $\Gamma = \left(\gamma^{2^T}, .., \gamma^{H^T}\right)$ be a $(S-J) \times (H-1)$ matrix. Further, define the set

$$\widetilde{\Pi} = \left\{ (\lambda, e) \in \Pi : \begin{array}{l} \det(z_\sigma) = 0 \\ \dim \Gamma < S - J \end{array} \right\}$$

and set $\xi = (z, \gamma, \mathcal{L})$; where in the latter we are omitting υ from the previous definition of the endogenous variable vector of the problem.[14] Consider the sub-Jacobian $D_\xi F_\sigma$,

$var.$	z^1	z^2		z^H	γ^2		γ^H	$\left(\left(a_j^k\right)_{k=1}^{S-J}\right)_{j=1}^{J}$
$equ.$	$---$	$---$		$---$	$---$		$--$	$-------$
1	$-\lambda^1 D^2 u^1$	$\lambda^2 D^2 u^2$		0	B		0	N_2
	$\vdots$		$\ddots$			$\ddots$		$\vdots$
	$-\lambda^1 D^2 u^1$	0		$\lambda^H D^2 u^H$	0		B	N_H
2	I_N	I_N	$\cdots$	I_N	0	$\cdots$	0	0
3	0	B^T		0	0	$\cdots$	0	K_2
	$\vdots$		$\ddots$		$\vdots$		$\vdots$	$\vdots$
	0	0		B^T	0	$\cdots$	0	K_H
4	0	N_2^T	$\cdots$	N_H^T	K_2^T	$\cdots$	K_H^T	0

where $B^T = (0, (I \mid A))$. Since column operations do not affect the rank, perform the following transformations. Subtract the first column block from the h^{th}, for $h = 2, .., H$. Then, move the resulting H^{th} row block (a block–row vector of typical element I_N in the first H blocks) to the top row block. The following (symmetric) matrix representation is obtained,

$$D_{\xi,e} F_\sigma(\lambda) = \begin{pmatrix} I_N & 0 & 0 & 0 \\ * & G & C & N \\ 0 & C^T & 0 & K \\ 0 & N^T & K^T & 0 \end{pmatrix} \equiv \begin{pmatrix} I_N & 0 \\ * & Q \end{pmatrix}$$

where

$$G = \begin{pmatrix} \lambda^2 D^2 u^2 + \lambda^1 D^2 u^1 & \lambda^1 D^2 u^1 & & \lambda^1 D^2 u^1 \\ \lambda^1 D^2 u^1 & \lambda^3 D^2 u^3 + \lambda^1 D^2 u^1 & & \lambda^1 D^2 u^1 \\ & & \ddots & \\ \lambda^1 D^2 u^1 & D^2 u^1 & & \lambda^H D^2 u^H + \lambda^1 D^2 u^1 \end{pmatrix}$$

[14] Use the first order conditions of consumer $h = 1$ in [**Foc PP**] to substitute for υ.

and G has dimension $(H-1)N \times (H-1)N$.

$$\underset{(S-J)J\times(S-J)(H-1)}{K^T} = \left(K_2^T..K_h^T..K_H^T\right), K_h^T = \begin{pmatrix} z_{\sigma,1}^h I_{S-J} \\ \vdots \\ z_{\sigma,J}^h I_{S-J} \end{pmatrix}$$

$$\underset{J(S-J)\times N(H-1)}{N^T} = \left(N_2^T..N_h^T..N_H^T\right), N_h^T = \begin{pmatrix} 0_{J\times S-J+1} & \gamma^{h^T} & \\ \vdots & & \ddots & \\ 0_{J\times S-J+1} & & & \gamma^{h^T} \end{pmatrix}.$$

We shell proceed by showing that the rows of DF_σ are linearly independent. Equivalently, we can show that there does not exists non trivial (row) vectors $q \in \Re^n$, and $r \in \Re^{n-N(H-1)}$ such that

$$\begin{pmatrix} qDF_\sigma(\lambda) \\ qq^T - 1 \end{pmatrix} = 0, \quad \begin{pmatrix} rQ \\ rr^T - 1 \end{pmatrix} = 0$$

Consider the following partition $r = \left(r^1, r^2, r^3\right) \in \Re^{(H-1)N} \times \Re^{(H-1)(S-J)} \times \Re^{(S-J)J}$, such that

$$rQ = \begin{pmatrix} r^1G + r^2C^T + r^3N^T \\ r^1C + r^3K^T \\ r^1N + r^2K \end{pmatrix}$$

The following fact proves that G has linearly independent rows.

Fact 3 : If $r^1G = 0$, then $r^1 = 0$.

Proof of Fact 3 : $r^1 \neq 0$ contradicts the assumption **b.2**. In fact if $r^1 \neq 0$, (i) implies $r^1Gr^{1^T} = 0$. For $r^1 = (r_l^1)_{l=2,..,H}$, $\iota_{H-1} = (1,..,1) \in \Re^{(H-1)N}$, the following transformations can be used.

$$r^1G = \left(\lambda^2 r_2^1 D^2u^2, .., \lambda^H r_H^1 D^2u^H\right) + \lambda^1 \left(\left(\sum_{h=2}^{H} r_h^1\right) D^2u^1\right) \Box \iota_{H-1}$$

of typical element,

$$\lambda^h r_h^1 D^2u^h + \lambda^1 \left(\sum_{h=2}^{H} r_h^1\right) D^2u^1$$

Consider the latter; post multiply by $r_h^{1^T}$, and sum over $h = 2, .., H$. This yields

$$\sum_{h=2}^{H} \left(r_h^1 \left(\lambda^h D^2u^h\right) r_h^{1^T}\right) + \lambda^1 \left(\sum_{h=2}^{H} r_h^1\right) D^2u^1 \left(\sum_{h=2}^{H} r_h^1\right)^T$$

the latter being null if $r^1_h = 0$, for $h = 2, .., H$.□

<u>Fact 4</u> : If $r^1_{h,1} \in \mathcal{L}$, $\forall h \geq 2$, then $r^1_1 = 0$.

Proof of Fact 4 : operate substitutions in 1) above (using 2), 3)), one can write $r^1 G r^{1^T} + r^2 C^T r^{1^T} + r^1 C r^{2^T} = 0$. The middle term in the sum has typical argument $r^2_h (I \mid A) r^{1^T}_{h,1}$, with $(I \mid A) r^{1^T}_{h,1} = 0$, $\forall h \geq 2$. The last term, is just its transposed, and thus is null too. By Fact 3, $r^1 G r^{1^T} = 0$ implies $r^1 = 0$.□

<u>Fact 5</u> : If $r^1_1 = 0$, then $r^1_0 = 0$.

Proof of Fact 5 : This directly follows from the fact that 1) would involve terms like $r^1_{0,h} D^2_{z^h_0} u^h r^{1^T}_{0,h}$, which cannot be zero for $r^1_0 \neq 0$ without violating assumption **b.2** of strict convexity of preferences.□

The next two lemmas give decisive insights on the properties of F_σ at 0.

Lemma 23 *For each $\sigma \in \Sigma$, and for every $(\lambda, e) \in \Pi \backslash \widetilde{\Pi}$, $F_\sigma \pitchfork 0$.*

Proof of the Lemma: Observe that $(\lambda, e) \in \Pi \backslash \widetilde{\Pi}$ implies either one (or both) of the following facts:

a) rank $(z_\sigma) = J$

b) rank $(\Gamma) = S - J$.

- <u>if *a*), but not *b*)</u>: Observe that for the allocation to an optimum, $z^h_1 \in \mathcal{L}$ for $h = 2, .., H$. This implies that if we look at equation 2), one of the following must hold: (i) $r^1_h = 0$, and $r^3 = 0$ or (ii) $r^1_{h,1} \in \mathcal{L}$, $\forall h \geq 2$, and $r^3 = 0$. Yet, by Fact 4, 5, the two conditions imply $\left(r^1, r^3\right) = 0$. Therefore, it must be that $r = 0$.

- <u>if *b*), but not *a*)</u>: Consider equation 2). Because of *b*), it reduces to $(0, (I \mid A)) r^{1^T}_h = 0$ for $h = 2, .., H$, and (using Fact 2, 3) holds only if $r^1 = 0$. Rewriting 1) for a typical h, $r^2_h (0, (I \mid A)) + \left(0, r^3_1 \gamma^{h^T}, .., r^3_J \gamma^{h^T}\right) = 0$, we immediately see that $\left(r^2, r^3\right) = 0$ is the only vector satisfying this condition. Therefore, $r = 0$.

Therefore $r = 0$, and DF_σ has linearly independent rows. □

Lemma 24 *Let assumption* **c** *hold. For each $\sigma \in \Sigma$, the set of endowments for which $(\lambda, e) \in \widetilde{\Pi}$, and a quasi-saddle point exists is negligible (closed and of zero Lebesgue measure).*

Proof of the Lemma: For greater clarity, we brake our proof in two steps. In the first, we show that there exists a local perturbation of the endowments such that it is not the case that both $a)$ and $b)$ fail to hold at a quasi-saddle point. In the second step, we show that this result is strictly generic.

Step 1: It is sufficient to show that for every quasi saddle point (ξ, e, λ) such that $(\lambda, e) \in \widetilde{\Pi}$ there exists a (local) perturbation of the endowments, Δe, such that $(\lambda, e + \Delta e) \notin \widetilde{\Pi}$. The perturbation is constructed as follows. $(\lambda, e) \in \widetilde{\Pi}$ implies that $\dim \Gamma < S - J$. Then, since $\min [S - J, J] \leq H - 1$, we can define a set of consumers, J, of cardinality J. For $h \in J$ we can construct an endowment perturbation on the plane of financially feasible transfers: $\Delta e^h \in \mathcal{L} \quad \forall h \in J$. Next, define $\widehat{z}^h = z^h - \Delta e^h$, such that the consumption allocation is left unchanged, and thus the first order conditions $1), 3)$ of the Planner's problem hold $(\widehat{x}^h = \widehat{z}^h + e^h + \Delta e^h = z^h + e^h = x^h)$. The financial constraint is satisfied by construction. Obviously, we can use Δe^1 to offset the effects of our perturbation on the resource constraint, 2). To complete our argument we must ensure that also 4) $\Gamma \widehat{z}_\sigma^T = 0$ holds. But this becomes clear if $H > S - J$, since then $\langle \Gamma \rangle \subset \langle \widehat{z}_\sigma \rangle^\perp \subseteq \langle \widehat{z} \rangle^\perp$. We can conclude that $(\lambda, e + \Delta e) \notin \widetilde{\Pi}$, and therefore that DF_σ is surjective. Observe that what we have show is equivalent to prove that there exists a perturbation of the endowments such that the following mapping, $H_{\sigma,\lambda}$, (locally) does not intersect zero:

$$H_{\sigma,\lambda} (z, \gamma, \mathcal{L}, c_0, c_1, e) = \begin{bmatrix} F_{\sigma,\lambda} (z, \gamma, \mathcal{L}, e) \\ c_0 \Gamma(\gamma) \\ c_1 z_\sigma \end{bmatrix} = 0$$

(i.e. locally, there is no $(z, \gamma, \mathcal{L}, c, e)$ satisfying $H_{\sigma,\lambda} (z, \gamma, \mathcal{L}, c, e) = 0$). The latter implies that $DF_{\sigma,\lambda}$ has linearly independent rows. However, if $\min [S - J, J] \leq H - 1$, the number of unknowns, (c_0, c_1), added to the problem is less than the number of equations.

Next, we can show that this property is generic, holding for every initial $(\lambda, e) \in \Re^H \times \Omega^*$, where Ω^* is relatively open and of full Lebesgue measure.

Step 2.1.: [Ω^* **is open**] For every $\lambda \in \Re^H$, define

$$\begin{aligned} \mathcal{H}^R_{\sigma,\lambda} &= \left\{\xi, e \in \Xi \times \Re^{HN}_{++} : H_{\sigma,\lambda} = 0, rDH_{\sigma,\lambda} = 0 \text{ if } r = 0 \text{ in } \Re^n\right\} \\ \mathcal{H}^C_{\sigma,\lambda} &= \left\{\xi, e \in \Xi \times \Re^{HN}_{++} : H_{\sigma,\lambda} = 0, rDH_{\sigma,\lambda} = 0,\ r \in \Re^n \backslash \{0\}\right\} \end{aligned}$$

$\mathcal{H}^C_\sigma$ is closed in $\Xi \times \Re^{HN}_{++}$. Therefore, the set of critical points of the mapping H_σ, is also relatively closed. Next, define the natural projection $\phi_\sigma : \mathcal{H}^R_{\sigma,\lambda} \to \Re^{HN}_{++}$, a mapping from the set of regular points of $H_{\sigma,\lambda}$ into the endowment space. One can use standard arguments to show that ϕ_σ is proper. This, in turn, is sufficient to ensure that $\phi\left(\mathcal{H}^C_{\sigma,\lambda}\right)$ is closed in $\Re^{HN}_{++}$, and hence that its complement $\Omega^* = \widetilde{\Omega} \backslash \phi\left(\mathcal{H}^C_{\sigma,\lambda}\right)$ is relatively open.

Step 2.2. [Ω^* **is of full measure**] Endow the space of endowments with the natural topology. Since Ω^* is a manifold, by continuity of the map $\phi_\sigma : \mathcal{H}^R_{\sigma,\lambda} \to \Omega^*$, and Sard's Theorem we can conclude that Ω^* has full Lebesgue measure[15]. Regularity of the solution set implies that such set is nonempty and finite. □

The structure of $\mathcal{M}$ The manifold structure of the set of solutions of the Planner's maximum problem can now be easily derived.

Let Λ^{H-1} denote the interior of the set of normalized welfare weights, of dimension $(H-1)$.

$$\widehat{\Pi}^* \supset \Pi^* \equiv \Lambda^{H-1} \times \Omega^*$$

Further, define the set of regular quasi saddle points by

$$\mathcal{G}^R = \left\{\xi, \lambda, e \in \Xi \times \widehat{\Pi}^*\right\}$$

Lemma 25 *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *hold, and restrict parameters on* $\widehat{\Pi}^*$*. The set of local maxima, endowed with a manifold structure, is contained in the set of regular quasi-saddle points,*

$$\mathcal{M} \subset \mathcal{G}^R$$

Proof: First, let (λ, e) be given and suppose that $\left(\widehat{\xi}, \lambda, e\right) \in \mathcal{G}^R$. Then, for $\widehat{\Psi}_{\lambda,e} \equiv \Psi_{\lambda,e}\left(\widehat{\xi}\right)$, the following holds,

$$\begin{aligned} D_\xi \widehat{\Psi}_{\lambda,e} &= 0 \\ (\gamma, \upsilon) D_{\gamma,\upsilon} \widehat{\Psi}_{\lambda,e} &= 0 \\ \det\left(D^2_\xi \widehat{\Psi}_{\lambda,e}\right) &\neq 0 \end{aligned}$$

Next, observe that

$$\begin{aligned} F_\sigma\left(\widehat{\xi}, \lambda, e\right) &= 0 \Leftrightarrow D_\xi \widehat{\Psi}_{\lambda,e} = 0 \\ D_\xi F_\sigma\left(\widehat{\xi}, \lambda, e\right) &= D^2_\xi \widehat{\Psi}_{\lambda,e} \end{aligned}$$

In words, $\mathcal{G}^R$ may contain regular maxima (i.e. solutions of the Planner's problem), but also regular minima. The existence of maxima is ensured by *Proposition* 2 above. Obviously, if $\widehat{\xi}$, and thus $\left(\widehat{z}, \widehat{\mathcal{L}}\right)$, is a (local) regular maximum (i.e. a *CPO*), $\widehat{\xi}$ does also satisfy the second order conditions, i.e. for every (row) vector $y \in \Re^l$ $(l \equiv HN + N + (S-J)(H-1) + (S-J)J)$, $y \neq 0$, $yD^2\Psi_{\lambda,e}y^T < 0$ if $D\Psi_{\lambda,e}y^T = 0$. To understand why $\mathcal{M}$ is strictly included in $\mathcal{G}$, observe

[15] See, for example, Guillemin and Pollack (1974), p.62.

the following. The set $\widehat{\Pi}^*$ is (by construction) a strictly generic subset of $\Re^H \times \Omega(r)$. Based on the latter *Lemma*, we can observe that for every pair $(\lambda, e) \in \widehat{\Pi}^*$, $\left(\widehat{\xi}, \lambda, e\right) \in \mathcal{M}$, and $\widehat{\xi}$ is a solution to the Kuhn-Tucker maximum problem faced by the Planner. Given assumption **b** of strict concavity of utility functions, the planner's problem solved for given welfare weights is equivalent to the one solved for given $(H-1)$ utility levels. In the latter, λ can be seen as strictly positive Kuhn-Tucker multipliers. hence we can take we can normalize them, for example, by taking $\lambda^1 = 1$ (it is important to notice that at the maximum λ belongs to an $(H-1)$-dimensional space). $\square$

Proof of Proposition 3: This substantially follows from Lemma 23 and 24. In fact, the Jacobian $D_\xi F_\sigma$ is of full rank whenever we restrict e on Ω^*, for all $\sigma \in \Sigma$. Applying the Preimage Theorem, we obtain that $F_\sigma^{-1}(0)$ is a smooth submanifold of $\Xi \times \Omega(r) \times \Re^H$. Then $\mathcal{G}^R = \cup_\sigma F_\sigma^{-1}(0)$, has also a manifold structure, and $\dim \mathcal{G}^R = \dim \Lambda + \dim \Omega^*(r) = H + (H-1)N$.$\square$

The structure of $\mathcal{M}_e$ *Proof of Theorem* 4*:* this substantially follows from *Proposition* 3 above, and by Transversality Theorem. In fact, we have shown that for every $\lambda \in \Lambda$ and every $e \in \Omega^*$ $F_\sigma \pitchfork 0$, provided that F_σ is restricted to $\mathcal{M}$. By the Transversality Theorem, for fixed $e \in \Omega^*$, $F_{\sigma,e} \pitchfork 0$ for every $\lambda \in \Lambda^*$ in an open set of full Lebesgue measure $\Lambda^* \subset \Lambda$. Therefore, the standard results in the literature apply: $\mathcal{M}_e$ is diffeomorphic Λ^*, an $(H-1)$ manifold. $\square$

The above analysis implies the following (for details, see the proof of Proposition 3).

The above analysis implies the following (for details, see the proof of Proposition 3).

Corollary 26 *In the context of Proposition* 3*, if we restrict on the generic set* $\widehat{\Pi}^*$*, every quasi-saddle point is such that either one of the following two conditions hold: a)* $z^h \in \mathcal{L}$*, for all* $h \geq 2$*, b)* $rank\,(\Gamma) = S - J$*.*

8.2 Financial innovation

Proof of Corollary 17:

Consider the action set of the planner's problem in $\mathsf{G}^{J+1,S}$; this trivially contains the one obtained in $\mathsf{G}^{J,S}$ (just observe that any $\mathcal{L}^J \in \mathsf{G}^{J,S}$, is contained in all $\mathcal{L}^{J+1} \in \mathsf{G}^{J+1,S}$). Starting from a $\mathsf{J} - CPO$, we are going to show that an asset can be introduced such that a change of consumers' evaluation of the existent transfers would not hart anyone. Take $\mathcal{L}^{J+1} \in \mathsf{G}^{J+1,S}$ such that its orthogonal complement, $\left(\mathcal{L}^{J+1}\right)^\perp$, contains the original consumer's gradients.

Since $\left(\mathcal{L}^{J+1}\right)^{\perp} \subset \left(\mathcal{L}^{J}\right)^{\perp}$, a sufficient condition for this to be possible is that $H - 1 < S - J$, so that $rank(\Gamma) < S - J$. Without loss of generality, let $H = S - J - 1$. We know by

Refer to Corollary 26, and redefine the mapping $H_{\sigma,\lambda}$. Now, $H_{\sigma,\lambda} \pitchfork 0$ implies that if the new asset is (initially) redundant (i.e. $rank(z_{\sigma}) = J$), $rank(\Gamma) = S - J - 1$. Next, recalling the proof of Lemma 24, we can always construct a perturbation of endowments of $\mathcal{L}^{J+1}$, that, leaving unchanged allocations and (thus) consumer gradients, will imply that $\left\langle z_{\sigma}^{T}\right\rangle$ span a $(J+1)-$dimensional space. Once again, the fact that consumer gradients are unchanged implies that their evaluations of the J pre-existing assets is unchanged. Moreover, since now $rank(z_{\sigma}) = J + 1$, the new asset is used by the planner. The second part (step 2.2) of the proof of Lemma 24, can be reiterated to show that our results is strictly generic. Denote the strictly generic set of endowments Ω^{**}. Finally, we can decentralize this new $(J+1) - CPO$ as a competitive equilibrium with $J + 1$ assets using the II Welfare Theorem (Theorem 11).□

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