On Knots and Dynamics in Games*

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Abstract

We extend Kohlberg and Mertens' (1986) structure theorem concerning the Nash equilibrium correspondence to show that its graph is not only homeomorphic to the underlying space of games but that it is also unknotted. This is then shown to have some basic consequences for dynamics whose rest points are Nash equilibria.

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1 Introduction

Although the set of Nash equilibria of a given normal form game may be rather badly behaved, it has been shown by Kohlberg and Mertens (1986) that the graph of the Nash equilibrium correspondence defined over the whole space of games with fixed numbers of players and strategies is topologically speaking a relatively nice object as it is homeomorphic to the space of games, which is itself an Euclidean space. We strengthen their result to show that not only does the graph of the Nash equilibrium correspondence have the intrinsic structure of the space of games, i.e., an Euclidean space, but it can be continuously deformed within its natural ambient space of mixed strategies times games to a graph of a function from games to strategies. In technical terms the graph of the Nash equilibrium correspondence is unknotted.

We use this result to show that any two Nash dynamics (i.e. dynamics whose rest points coincide with Nash equilibria) can be continuously connected to each other via dynamics from within the class of Nash dynamics, i.e., without adding any extra zeros. An immediate implication is that the index of any Nash equilibrium does not depend on the dynamic chosen. In fact the indices are shown to be completely determined by the geometry of the Nash equilibrium correspondence. This allows to make statements concerning the stability properties of Nash equilibria that hold for any dynamics from the class of Nash dynamics. More particularly, we show that any regular equilibrium of degree -1 (e.g. the mixed strategy equilibrium of a 2 player 2×2 coordination game) cannot be stable under any Nash dynamics. Modulo arbitrarily small perturbations, this carries over to the class of aggregate monotonic selection dynamics (see Samuelson and Zhang (1992) or Ritzberger and Weibull (1995) for definitions), and via stochastic approximation (see Fudenberg and Levine (1998), Ch. 4) also to discrete time dynamical systems that approximate Nash dynamics with some noise. Since equilibria of degree -1 are always mixed strategy equilibria and n out of $2n + 1$ equilibria of a given generic normal form game have degree -1, (see Gül, Pearce, and Stacchetti (1993), Ritzberger (1994)), the result complements existing literature on learning mixed strategy equilibria like Fudenberg and Kreps (1993), Kaniovski and Young (1995), Oechssler (1997) and Benaim and Hirsch (1999), by allowing to discriminate against a subset of mixed strategy equilibria for any dynamics from within a large class, while leaving open the possibility that remaining mixed strategy equilibria may be stable under at least some of the dynamics from the class. The result provides also a straightforward proof of a conjecture of Govindan and Wilson (1997a) that the fixed point indices of any map whose fixed points are Nash equilibria do not depend on the particular map chosen. This conjecture was proved by DeMichelis and Germano (1998).[1]

The latter paper made heavy use of homology and cohomology theory, the present unknottedness result allows us to use an elementary argument instead.

The paper is organized as follows. Section 2 introduces basic notation, Section 3 contains the unknottedness result, and Section 4 some consequences.

2 Preliminary Notation

Let $I = \{1, .., n\}$ denote the set of players, let S_i denote player i's space of pure strategies $i \in I$, and $S = \times_{i \in I} S_i$ the space of pure strategy profiles. Let Σ_i denote the set of probability distributions on S_i, $\Sigma = \times_{i \in I} \Sigma_i$, and let $\partial \Sigma = \{\sigma \in \Sigma \; \sigma^i_k = 0 \text{ for some } k, i\}$ denote the boundary of Σ. Let also $S_{-i} = \times_{j \neq i} S_j$, and $\Sigma_{-i} = \times_{j \neq i} \Sigma_j$ and set $K_i = \# S_i$, $K = \sum_{i \in I} K_i$, $\kappa = \Pi_{i \in I} K_i$. In what follows, we consider finite normal form games, i.e., where n and each K_i, $i \in I$ are finite, and fix both the set of players and the set of strategy profiles, so that we can identify a game with a point in Euclidean space $\gamma \in \mathbb{R}^{\kappa n}$ and, accordingly, the space of games with the whole Euclidean space $\mathbb{R}^{\kappa n}$. We also denote by $\gamma^i \in \mathbb{R}^{\kappa}$ the payoff array of player i.

Finally, denote by $\eta \subset \Sigma \times \mathbb{R}^{\kappa n}$ the graph of the Nash equilibrium correspondence, i.e.,

$$\eta = \{(\sigma, \gamma) \in \Sigma \times \mathbb{R}^{\kappa n} \; \sigma \text{ is a Nash equilibrium of } \gamma\}.$$

3 The Unknottedness Result

Kohlberg and Mertens (1986), Theorem 1, p. 1021, show that the graph of the Nash equilibrium correspondence η is homeomorphic to the space of

[1]Some particular cases (two players case and the dynamics of Gül, Pearce, and Stacchetti (1993)) were proved in Govindan and Wilson (1997 a and b)

games $I\!R^{\kappa n}$. We extend their result by showing that η is not only homeomorphic to $I\!R^{\kappa n}$, but also that it can be continuously deformed to a trivial copy of it say $\eta_0 = \{\sigma_0\} \times I\!R^{\kappa n}$, for some $\sigma_0 \in \Sigma$. This immediately implies that η is unknotted and, as we shall see, also Propositions 1 and 2 below. Before giving the proofs we introduce some new objects that will simplify the technicalities in dealing with the equilibria on the boundary of Σ. We let $\Sigma_\varepsilon = \{\sigma_k^i | \sum_k \sigma_k^i = 1 \text{ and } \sigma_k^i \geq -\varepsilon \forall k, i\}$ be a copy of Σ slightly enlarged, $\phi(\sigma)$ will be a C^1 function on Σ_ε identically equal to 1 on Σ , ≥ 0 everywhere and identically equal to 0 on a neighborhood of $\partial\Sigma_\varepsilon$, $\pi_\Sigma(\sigma)$ will be the projection from Σ_ε to Σ , a piecewise linear retraction, $V(\sigma)$ will be a differentiable vector field tangent to Σ_ε identically equal to zero on Σ and such that its scalar product with $\pi_\Sigma(\sigma) - \sigma$ is strictly positive elsewhere, i.e. a vector field that points towards Σ outside it. Finally E_ε will be $\Sigma_\varepsilon \times I\!R^{\kappa n}$. We will always identify the original spaces and their subsets with their images in the enlarged spaces.

Unknottedness Theorem *There exists a homeomorphism* $\Phi : (E_\varepsilon, \eta) \rightarrow (E_\varepsilon, \eta_0)$ *which takes the ambient space* $E_\varepsilon = \Sigma_\varepsilon \times I\!R^{\kappa n}$ *homeomorphically onto itself and the graph of the Nash equilibrium correspondence* η *homeomorphically onto a trivial copy of the space of games :* $\eta_0 = \{\sigma_0\} \times I\!R^{\kappa n}$ *with* $\sigma_0 \in \Sigma$ *(not just* Σ_ε*).*

Moreover, the pairs (E_ε, η) *and* (E_ε, η_0) *are (topologically) isotopic, i.e., there exists a family of homeomorphisms* $\Phi_t : (E_\varepsilon, \eta) \rightarrow (E_\varepsilon, \eta_t)$*, continuous in* $t \in [0, 1]$*, such that* $\Phi_0 = \Phi$ *and* Φ_1 *is the identity on* E*, all the* Φ_t *may be taken to be the identity on* $\partial E_\varepsilon = \partial\Sigma_\varepsilon \times I\!R^{\kappa n}$*. Also all* Φ_t *map* E *into itself.*

Proof. We make use of the proof of Kohlberg and Mertens (1986), p. 1021-1022. The only essentially new thing is the observation that their homeomorphism is defined not only on η but also on all E and that there is an obvious extension to E_ε; here are the details.

Consider the reparameterization of player i's payoffs:

$$\gamma_{s,t}^i = A_{s,t}^i + a_s^i, s \in S_i, t \in S_{-i},$$

where $A_{s,t}^i, a_s^i \in I\!R$ are uniquely determined such that $\Sigma_{t \in S_{-i}} A_{s,t}^i = 0$, for all $s \in S_i$. Associating to each game $\gamma \in I\!R^{\kappa n}$ unambiguously the pair $(A, a) \in I\!R^{\kappa n} \times I\!R^K$, consider the following family of maps defined for any $t \in [0, 1]$:

$$\Phi'_t : E_\varepsilon \to E_\varepsilon, (\sigma, (A^i, a^i)_{i\in I}) \mapsto (\sigma, (A^i, a^i + \phi(\sigma)(1-t) \cdot \alpha^i(A^i, \pi_\Sigma(\sigma)))_{i\in I}),$$

where for $i \in I$:

$$\alpha^i : I\!R^\kappa \times \Sigma \to I\!R^{K_i}, (A^i, \sigma) \mapsto ((\sigma^i_s + \Sigma_{t\in S_{-i}} A^i_{s,t} \Pi_{j\neq i} \sigma^j_{t_j})_{s\in S_i}). \quad (1)$$

Observe that on E, where ϕ is 1 and π_Σ is the identity, (Φ'_t) extend the map of Kohlberg and Mertens. First we show that the maps $(\Phi'_t)_{t\in[0,1]}$ are homeomorphisms from E_ε onto itself. Consider the family of maps:

$$\Psi_t : E \to E, (\sigma, (A^i, a^i)_{i\in I}) \mapsto (\sigma, (A^i, a^i - \phi(\sigma)(1-t) \cdot \alpha^i(A^i, \pi_\Sigma(\sigma)))_{i\in I}), t \in [0, 1],$$

Clearly Φ'_t, Ψ_t are all continuous, and it is easily verified that, for any given $t \in [0, 1]$, they are inverses of each other. Next, notice that Φ'_1 is the identity on E and that it is the identity on the boundary of E_ε.

Now $\Phi'_0(\eta)$ projects onto $I\!R^{\kappa n}$ in a one to one fashion (in fact the projection is the parametrization given by Kohlberg and Mertens); in other words, it is the graph of a function from the space of games $I\!R^{\kappa n}$ to the interior of Σ_ϵ. We use this fact to construct a further isotopy Φ'' that flattens it to $\{\sigma_0\} \times I\!R^{\kappa n}$ and is the identity on ∂E_ε. Note that $\Phi'_0(\eta)$ touches the boundary of Σ in the not completely mixed equilibria and so such a map cannot be the identity on ∂E; that is why we introduced E_ε. At this point we can define Φ by glueing together Φ' and Φ''.

△

Observe also that the proof of the unknottedness theorem shows that the graph of the Nash equilibrium correspondence $\eta \subset E$ is deformed by Φ' within the same ambient space E into the graph of a one valued function, i.e. $R : I\!R^{\kappa n} \to \Sigma, (A^i, a^i) \mapsto \pi_\Sigma(a^i)$.

4 Some Basic Consequences

The rest of the paper is devoted to the derivation of some basic consequences of the previous theorem. The consequences we draw concern possible dynamic behavior of Nash equilibria for essentially arbitrary dynamics whose zeros coincide with Nash equilibria. In order to state the results, we formally introduce the relevant class of dynamics, which will be generalized later.

Definition 1 *A* **Nash dynamics** *is a map $F : E \to I\!R^{K-n}$, the tangent space to Σ, satisfying:*
(i) F is continuous,
(ii) $F^{-1}(0) = \eta$,
and
(iii) $F(\cdot, \gamma)$ points weakly to the interior of $\Sigma \times \{\gamma\}$ along the boundary $\partial\Sigma \times \{\gamma\}$ for any $\gamma \in I\!R^{\kappa n}$.

The map F defines a vector field on the strategy space for any game, condition (i) and (iii) are standard: (iii) insures that the vectors never point outside the space of mixed strategies. Observe that condition (i) is already sufficient to insure the existence of a solution of the associated differential equation on Σ, $\dot{\sigma} = F_\gamma(\sigma) = F(\sigma, \gamma)$. [2] Condition (ii) is more demanding: it says that the zeros of the Nash dynamics actually coincide with the Nash equilibria. Because this excludes many interesting selection dynamics used in evolutionary game theory, at the end of the paper we will briefly show how to include these dynamics too.

The following dynamics derived from Gül, Pearce, and Stacchetti (1993) (henceforth GPS) is an example of a Nash dynamics:

$$F_{GPS} : E \to I\!R^{K-n}, (\sigma, (A^i, a^i)_{i\in I}) \mapsto (\pi_{\Sigma_i}(a^i + \alpha^i(A^i, \sigma)) - \sigma^i)_{i\in I}, \qquad (2)$$

where α^i is defined in (1) and $\pi_{\Sigma_i} : I\!R^{K_i} \to \Sigma_i$ is the projection from $I\!R^{K_i}$ to player i's mixed strategy space Σ_i. Rewriting it as $F_{GPS}(\sigma, \gamma) = (\pi_{\Sigma_i}(\sigma^i + \nu^i(\gamma^i, \sigma)) - \sigma^i)_{i\in I}$, where $\nu^i(\gamma^i, \sigma^{-i}) = (\Sigma_{t\in S_{-i}} \gamma^i_{s,t} \Pi_{j\neq i} \sigma^j_{t_j})_{s\in S_i}$ is the vector of expected payoffs against strategy profile σ^{-i}, one notices that it points in direction of the better-responses to σ.

Given a Nash dynamics $F(\sigma; \gamma)$ we define its extension to E_ϵ as follows:

$$F_\epsilon(\sigma; \gamma) = \phi(\sigma) F(t(\sigma); \gamma) + V(\sigma).$$

Note that:
(i) $F_\epsilon(\sigma; \gamma) = F(\sigma; \gamma)$ on Σ
(ii) F_ϵ is continuous on Σ_ϵ
(iii)$F_\epsilon^{-1}(0) = \eta$
(iv) F_ϵ is strongly interior pointing on $\Sigma_\epsilon - \Sigma$.

[2] If in (i) we required F Lipschitz we would also have uniqueness of solutions, one can check that if the F are such to begin with, all our deformations would stay in this category.

We will denote the family of all F_ϵ satisfying (ii), (iii) and (iv) by $\mathcal{F}$ and call it extended Nash dynamics, also we will drop the subscript ϵ when no confusion is likely.

An important property of the family of extended Nash dynamics is the following.

Proposition 1 *The family of extended Nash dynamics $\mathcal{F}$ is arcwise connected, i.e., any two dynamics $F, F' : E \to I\!R^{K-n}$ in $\mathcal{F}$ are homotopic within $\mathcal{F}$.*

Proof. Recall that an homotopy is a continuous one parameter family of maps. Given two extended dynamics F and F' in $\mathcal{F}$ let $G' = F' \circ \Phi^{-1}$ and $G = F \circ \Phi^{-1}$, where Φ is the homeomorphism in the previous theorem, for these new functions we have $G^{-1}(0) = G'^{-1}(0) = \{\sigma_0\} \times I\!R^{\kappa n}$. If H_t is a homotopy from G to G' such that $H_t^{-1}(0) = \{\sigma_0\} \times I\!R^{\kappa n}$, $H_t \circ \Phi$ will be the required homotopy from F to F'. We define such a H_t as the composition of the homotopies defined below, all of them will be denoted by H for notational simplicity. Given a $\sigma \neq \sigma_0$ in Σ_ϵ let $\overline{\sigma}$ be the unique point in $\partial\Sigma_\epsilon$ lying on the ray from σ_0 to σ. For $\sigma \in \Sigma_\epsilon - \sigma_0$ and $t \in [0, 1]$ let :

$$h(\sigma, t) = \begin{cases} \sigma_0 + \frac{(\sigma-\sigma_0)}{(1-t)} \text{ if this point is in } \Sigma_\epsilon \text{ and } t < 1 \\ \\ \overline{\sigma} \text{ else} \end{cases} .$$

It is an easy (but tedious!) exercise to check the continuity of this and of all the maps defined in the following. If $\|G\|$ is the length of G and $d(\cdot, \cdot)$ denotes the distance of two points in Σ_ϵ, we perform first the homotopy $H_t(\sigma, \gamma) = (1 - t)G(\sigma, \gamma) + t\frac{G(\sigma,\gamma)}{\|G(\sigma,\gamma)\|}d(\sigma, \sigma_0)$ so that we can assume that $\|G(\sigma, \gamma)\| = d(\sigma, \sigma_0)$, (check the continuity also in σ_0!). The next homotopy is defined as:

$H_t(\sigma_0, \gamma) = 0$ and $H_t(\sigma, \gamma) = G(h(\sigma, t), \gamma)\frac{d(\sigma,\sigma_0)}{d(h(\sigma,t),\sigma_0)}$ when $\sigma \neq \sigma_0$.

For t=1 $H_1(\sigma_0, \gamma) = G(\overline{\sigma}, \gamma)\frac{d(\sigma,\sigma_0)}{d(\overline{\sigma},\sigma_0)} = G'(\overline{\sigma}, \gamma)\frac{d(\sigma,\sigma_0)}{d(\overline{\sigma},\sigma_0)}$ because G and G' agree on $\partial\Sigma_\epsilon$. We now perform the last two homotopies in reverse order with G' instead of G and we get the result.

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This proposition shows that all dynamics in $\mathcal{F}$ are related to each other via a homotopy of elements of $\mathcal{F}$. This means that dynamics on E_ϵ , and hence on

E, cannot differ too much from each other if they have to satisfy properties (i) to (iii). In particular they can all be deformed to the GPS dynamics defined above. Notice, however, that while any two dynamics in $\mathcal{F}$ are homotopic on the whole of E_ϵ within $\mathcal{F}$, any two arbitrarily given vector fields defined on say $\Sigma_\epsilon \times \{\gamma\}$, for a fixed γ , that are inward pointing along $\partial\Sigma_\epsilon \times \{\gamma\}$ and whose zeros coincide with the Nash equilibria of the underlying game need not be homotopic on $\Sigma_\epsilon \times \{\gamma\}$ within this class. So, in particular, a necessary condition to have a homotopy is that the dynamics be defined for all games $\gamma \in I\!R^{\kappa n}$.

An immediate consequence of Proposition 1 concerns the indices of zeros of dynamics. Let us recall the definitions of index and degree.

Definition 2 *Let $e = (\sigma, \gamma) \in E$ be such that σ is a regular Nash equilibrium* [3] *of the game γ, let $\pi : \eta \to I\!R^{\kappa n}, (\sigma, \gamma) \mapsto \gamma$, denote the projection map. The* **(local) degree** *of π at e is defined as the integer,*

$$d(e, \pi) = \text{ sign } |D_{(\sigma,\gamma)}\pi(e)|.$$

Similarly, if there is an extension $F_\epsilon \in \mathcal{F}$ of F such that e is a regular zero of it we define the **index** *of F at e as the integer,*

$$i(e, F) = \text{ sign } | - D_\sigma F_\epsilon(e)|,$$

where $|\cdot|$ denotes the determinant (it can be proved that this definition does not depend on the choice of F_ϵ provided it belongs to $\mathcal{F}$).

It is well known that regular equilibria are isolated points, non-regular equilibria (or zeroes) may cluster in larger connected components and the index (or the degree) of a connected components of them is defined using a perturbation of the relevant maps π and F_ϵ (see appendix A below); again it can be proved that the definition does not depend on the choices made provided F is weakly interior pointing. The present definitions are standard and can be proved to coincide with the ones that use homology theory (see DeMichelis and Germano (1998) where the references to the litterature in algebraic topology are given). Notice that, despite the similarity of their definitions,

[3] In this context, we say $\sigma \in \Sigma$ is a regular Nash equilibrium of $\gamma \in I\!R^{\kappa n}$ if the matrix $D_{(\sigma,\gamma)}\pi(e)$, computed with an appropriate parameterization of η, has full rank. Similarly, we say σ is a regular zero of F_ϵ at γ if $D_\sigma F_\epsilon(e)$ has full rank.

index and degree are quite different objects: the index a priori depends (and gives information) on the behaviour of the dynamics F around its zero (independently of equilibria of games close to the given one), on the other side the degree is defined without reference to a dynamics but depends on how equilibria of games that are perturbations of the given one look like, it is e.g. an essential ingredient in the definition of stable sets of Mertens (see Mertens (1989) and Mertens (1991)).

Also an essential feature of the index and the degree is that they are homotopy invariants i.e. they are constant for deformations that don't change the set of zeroes. So , since dynamics in $\mathcal{F}$ can be continuously connected within $\mathcal{F}$, and indices are deformation invariant we have immediately that they are the same for all dynamics in $\mathcal{F}$ and hence for Nash dynamics on E. Explicitly:

Corollary 1 *Let F, F' be two Nash dynamics , then for any C a component of Nash equilibria*

$$i(C, F) = i(C, F').$$

In particular this is true if C is a regular and hence isolated Nash equilibrium.

To identify the indices to the degrees of the Nash equilibria as determined from the graph of the Nash equilibrium correspondence it is enough to check it for a particular Nash dynamics, this has been done for the dynamics of Gül, Pearce, and Stacchetti (1993) in Govindan and Wilson (1997a) (the interested reader can also work out an elementary proof by himself using the parametrization of Kohlberg and Mertens' (1986) along the lines of the unknottedness theorem).

Corollary 2 *Let C be a component of Nash equilibria of the same game, and let F be a Nash dynamics, then we have,*

$$i(C, F) = d(C, \pi).$$

Corollary 2 asserts that the index of a given regular Nash equilibrium $e \in E$, computed with *any* dynamics in $\mathcal{F}$, coincides with the degree of the projection map π evaluated at e. This implies that the indices for arbitrary dynamics in $\mathcal{F}$ are completely determined by the geometry of the Nash equilibrium correspondence. Given the more local nature of the index as compared to the (local) degree, it is somewhat surprising that the two notions agree for such a large class of dynamics as the Nash dynamics.[4] In particular, Corollary 2 implies that the index of components defined in Ritzberger (1994) using perturbations of the replicator dynamics is the same as the (local) degree defined in Govindan and Wilson (1995) directly from the graph of the Nash correspondence. More generally, this allows to compute the (local) degrees of components of Nash equilibria by computing indices of suitably chosen dynamics, which comes down to evaluating determinants of Jacobian matrices and in particular avoids having to compute parameterizations of the graph η. This can be useful when analyzing normal form games. We next state another important consequence, that answers a conjecture made by Hauk and Hurkens (1999)

Corollary 3 *In every game, if a component of Nash equilibria has nonzero index with respect of some dynamics then it contains a Mertens stable set.*

Proof Since the index is equal to degree, the component has nonzero degree too. One can then adapt the the proof of existence of stable sets (Theorem 1 in Mertens (1989)) as following, : If γ is the game identified with its payoff multimatrix and C is the component let D be a small disk around $\gamma \in \mathbb{R}^{\kappa n}$ and let U be a neighborhood of $C \times \gamma$ in $\Sigma \times \mathbb{R}^{\kappa n}$ so that $N_D = \eta \bigcap U$ is a neighborhood of C in η projecting onto$D$. By definition of degree this projection is essential (homologically non trivial). If W is the space of small enough strategy perturbations and N_W is the graph of all Nash equilibria of games in W , we use the notation of Mertens (1989), we can define define N'_W as $N_W \bigcap N_D$, this is the part of N_W that is close to C. The projection from $(N'_W, \partial N'_W)$ to $(W, \partial W)$ is not homologous to zero for the same reason as in Theorem 1 of Mertens (1989) , and the rest of the proof runs as there , after substituting N'_W to N_W. $\triangle$

An important consequence of corollary 2 is the following.

[4]Corollary 2 was conjectured by Govindan and Wilson (1997a) and proved by DeMichelis and Germano (1998), the special case of two players had been worked out in Govindan and Wilson (1997b). The advantage of the present approach is that the unknottedness theorem allows for a straightforward proof of the general n player case that avoids homology and cohomology theory which are essential to the proof of DeMichelis and Germano (1998).

Corollary 4 *Let $\sigma \in \Sigma$ be a Nash equilibrium of degree -1 of the game $\gamma \in I\!R^{\kappa n}$ then σ cannot be locally stable under any Nash dynamics for which it is a regular zero. This result can be extended to non regular zeroes and component using the theorem in Demichelis (2000).*

Proof. A regular zero of a dynamics F is stable if and only if all the eigenvalues in the Jacobian of F are less then zero, from the definition this implies that the index has to be one; since this is also the degree by Corollary 2 the result follows by contradiction.

$\triangle$

The result concerns stability properties of mixed strategy equilibria since regular equilibria of degree -1 are always mixed strategy equilibria, (Gül, Pearce, and Stacchetti (1993), Ritzberger (1994)). It says that if such an equilibrium has degree is -1, then it can never be locally stable under *any* Nash dynamics of which it is a regular zero (remark that generically dynamics have only regular zeroes on a generic game, for instance all dynamics in the litterature have regular zeroes on regular equilibria) . Since the total degree (i.e. the sum over all local degrees) of the projection map π is $+1$, (Kohlberg and Mertens (1986)), this implies that in a generic game roughly half of the equilibria (i.e., n out of $2n+1$) will be locally unstable. An example of a possible application is the fact that the mixed strategy equilibrium of a two player 2×2 coordination game cannot be locally stable under any Nash dynamics. We cannot say anything about equilibria of degree $+1$ like the (unique) mixed strategy equilibrium of a 2×2 matching pennies game.

Corollary 4 relates to recent literature on learning mixed strategy equilibria. Ritzberger and Weibull (1995) show that only strict equilibria are asymptotically stable under the (multi-population) replicator dynamics, therefore precluding the possibility that any mixed strategy equilibria may be asymptotically stable under that dynamics. Our result reinforces their result by showing that a subset of the mixed strategy equilibria is not only not asymptotically stable, but will actually be unstable under a large class of dynamics that includes the replicator dynamics (also see extensions below).

Fudenberg and Kreps (1993) exhibit learning processes under which all Nash equilibria are stable and also learning processes under which all strategy profiles that are not Nash equilibria are unstable. We can apply a stochastic approximation theorem of Pemantle (1990), Theorem 1, to show (roughly) that regular Nash equilibria of degree -1 continue to be unstable under discrete time processes of the form:

$$\sigma(t+1) = \sigma(t) + \delta_t \cdot F(\sigma(t), \gamma) + \epsilon_t,$$

where $\{\delta_t, \epsilon_t\}_t$ are random variables that satisfy certain minimal conditions and F is a sufficiently smooth Nash dynamics. This indicates that although all the mixed strategy Nash equilibria may be reached by some of the discrete time learning processes of Fudenberg and Kreps, some of them may be harder to reach than others when there is some further noise in the system. Our result is also in line with Kaniovski and Young (1995) and Oechssler (1997), Benaim and Hirsch (1999), and Hopkins (1999), who obtain convergence to mixed strategy Nash equilibria that in all the cases they consider are in fact equilibria of degree +1. In this sense it is also in line with Fudenberg and Kreps' global convergence result for 2×2 games with unique completely mixed Nash equilibrium.

Besides Shapley's famous example, Jordan (1993), Oechssler (1997), and Benaim and Hirsch (1999) on the other hand, also contain examples of mixed Nash equilibria that are the unique equilibria of the underlying games, (hence of degree +1), that are unstable under natural learning processes. Hopkins (1999) clarifies stability properties of such mixed Nash equilibria for a class of dynamics that is shown to unify continuous time best response and replicator type dynamics for two player games.

5 Appendices

In(A) we briefly sketch how to extend the definition of index and degree to cases where Nash equilibria are not necessarily regular, e.g. the case of nontrivial components of Nash equilibria, and in (B) we deal with dynamics that have more zeros than there are Nash equilibria, so as to include dynamics such as the replicator dynamics. In particular we show how to weaken condition (ii) in the definition of $\mathcal{F}$.

(A) *Nash equilibria that are not necessarily regular.* To extend the definitions of index and degree to components of equilibria that are not necessarily regular Nash equilibria, let $C \subset E$ be an isolated, compact and connected component of Nash equilibria of the game $\gamma \in I\!R^{\kappa n}$, let $\gamma_\nu \in I\!R^{\kappa n}$ be a regular game sufficiently close γ, and let $N_C \subset \eta$ be a sufficiently small neighborhood of C in η. The **index** of a Nash dynamics F at the component C is defined as the integer:

$$i(C,F) \quad = \sum_{e\in\pi^{-1}(\gamma_\nu)\cap N_C} i(e,F^\delta),$$

where, we take a sufficiently small perturbation F^δ of the $F_\epsilon \in \mathcal{F}$ extending F so that F^δ is smooth and has regular zeroes (In general F^δ will not be in $\mathcal{F}$) . Similarly, the **(local) degree** of the component is defined as the integer:

$$d(C,\pi) \quad = \sum_{e\in\pi^{-1}(\gamma_\nu)\cap N_C} d(e,\pi),$$

It can be verified that perturbations of games, γ_ν, and dynamics, F^δ, that are nicely behaved always exist and that the numbers $i(C,F)$ and $d(C,\pi)$ do not depend on the perturbations or on the (sufficiently small) neighborhood N_C chosen.

(B) *Dynamics that have more zeros than Nash equilibria.* Corollary 2 can be extended to other interesting dynamics, such as the replicator and aggregate monotonic dynamics, by observing that a little perturbation of them make all their zeroes that are not Nash equilibria disappear. Formally we first define the following larger class of dynamics:

Definition 3 *A* **quasi-Nash dynamics** *is a map* $F : E \to I\!R^{K-n}$ *satisfying: (i)* F *is continuous, (ii')* $F^{-1}(0) \supset \eta$*, (iii)* F *points weakly to the interior of* Σ *along the boundary* ∂E*, and (iv) there exists a one parameter family* F_t *such that* F_t *still satisfies (i), (ii) and (iii),* $F_0 = F$*, and moreover* $F_t^{-1}(0) = \eta$ *for* $t > 0$*.*

The replicator dynamics and more generally the class of aggregate monotone selection dynamics are examples of such dynamics. It is easy to see that the definitions imply that the index of a connected component of zeroes of a quasi-Nash dynamics is the sum of of the indices of the components of Nash equilibria it contains, in particular Corollary 2 holds for isolated zeroes of a quasi-Nash dynamics. So if an equilibrium has degree -1 it will be unstable for any dynamics of which it is a regular zero, this is the case for example for the replicator dynamics (see Ritzberger (1994)).

References

[1] Benaim, M., and M.W. Hirsch (1999) "Mixed Equilibria and Dynamical Systems Arising from Fictitious Play in Perturbed Games," *Games and Economic Behavior*, **29**: 36-72.

[2] Demichelis, S. (2000) "On the Index and Stability", preprint.

[3] DeMichelis, S. and F. Germano (1998) "On the Indices of Zeros of Nash Fields," mimeo CORE and University of Pavia.

[4] Fudenberg, D., and D.M. Kreps (1993) "Learning Mixed Equilibria," *Games and Economic Behavior*, **5**: 320-367.

[5] Fudenberg, D., and D.K. Levine (1998) *The Theory of Learning in Games*, MIT Press, Cambridge, MA.

[6] Govindan, S. and R. Wilson (1995) "A Sufficient Condition for Invariance of Essential Components," *Duke Mathematical Journal*, **81**: 39-46.

[7] Govindan, S. and R. Wilson (1997a) "Equivalence and Invariance of the Index and the Degree of Nash Equilibria," *Games and Economic Behavior*, **21**: 56-61.

[8] Govindan, S. and R. Wilson (1997b) "Uniqueness of the Index for Nash Equilibria of Two-Player Games," *Economic Theory*, **10**: 541-49.

[9] Gül, F., Pearce, D., and E. Stacchetti (1993) "A Bound on the Proportion of Pure Strategy Equilibria in Generic Games," *Mathematics of Operations Research*, **18**: 548-52.

[10] Hopkins, E. (1999) "A Note on Best Response Dynamics," *Games and Economic Behavior*, **29**: 138-150.

[11] Jordan, J.S. (1993) "Three Problems with Learning Mixed-Strategy Nash Equilibria," *Games and Economic Behavior*, **5**: 368-386.

[12] Kaniovski, Y.M., and H.P. Young (1995) "Learning Dynamics with Stochastic Perturbations," *Games and Economic Behavior*, **11**: 330-363.

[13] Kohlberg, E. and J.F. Mertens (1986) “On the Strategic Stability of Equilibria,” *Econometrica*, **54**: 1003-37.

[14] Mertens, J. F. (1989) “Stable Equilibria- A reformulation Part 1,” *Mathematics of Operations Research*, **14**: 575-625.

[15] Mertens, J. F. (1991) “Stable Equilibria- A reformulation Part 2,” *Mathematics of Operations Research*, **14**: 694-752.

[16] Oechssler, J. (1997) “An Evolutionary Interpretation of Mixed-Strategy Equilibria,” *Games and Economic Behavior*, **21**: 203-237.

[17] Pemantle, R. (1990) “Nonconvergence to Unstable Points in Urn Models and Stochastic Approximations,” *The Annals of Probability*, **18**: 698-712.

[18] Ritzberger, K. (1994) “The Theory of Normal Form Games from the Differentiable Viewpoint,” *International Journal of Game Theory*, **23**: 207-36.

[19] Ritzberger, K. and J.W. Weibull (1995) “Evolutionary Selection in Normal-Form Games,” *Econometrica*, **63**: 1371-99.

[20] Samuelson, L., and J. Zhang (1992) “Evolutionary Stability in Asymmetric Games,” *Journal of Economic Theory*, **57**: 363-91.