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The Moments of Log-ACD Models

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Abstract

We provide existence conditions and analytical expressions of the moments of Log-ACD models. We focus on the dispersion index and the autocorrelation function and compare them with those of ACD and SCD models.

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Keywords: Duration model, overdispersion, autocorrelation function, high frequency financial data.

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1 Introduction

With the aim of modelling durations between events such as trades and quote updates that occur randomly during the market hours on stock exchanges, Engle and Russell (1998) introduced the autoregressive conditional duration (ACD) model. This model combines elements from transition analysis and Engle's (1982) autoregressive conditional heteroskedasticity (ARCH) model. One motivation behind the ACD and the ARCH model appears similar: market events, such as trades and quote arrivals, occur in clusters. The ACD model also makes it possible to test some implications of market microstructure models (see O'Hara, 1995, for a survey) through the introduction of conditioning information.

Following the contribution of Engle and Russell (1998), other duration models have been put forward. Bauwens and Giot (2000) introduced a logarithmic version of the ACD model, called the Log-ACD model, which is more convenient than the ACD model when conditioning variables are included in the model in order to test microstructure effects. The reason is that the ACD model practically requires to impose non-negativity restrictions on its parameters, whereas the Log-ACD model does not. As an alternative to the Weibull distribution used in the original ACD model, Grammig and Maurer (2000) introduced an ACD model based on the Burr distribution (which includes the Weibull as a particular case). Ghysels, Gouriéroux, and Jasiak (2004) proposed the stochastic volatility duration model, which accounts for stochastic volatility in the durations. Bauwens and Veredas (2004) put forward the stochastic conditional duration (SCD) model, which uses a stochastic volatility-type model instead of a GARCH-type model to model the durations.

Until the present contribution, one drawback of the Log-ACD model, with respect to the ACD and the SCD models, was that the unconditional moments implied by the model were not available analytically. Bauwens and Giot (2000) therefore relied on numerical simulations to compute the moments of several Log-ACD models, in particular their autocorrelation function (ACF) and dispersion index (i.e. the ratio of standard deviation to mean). This led them to conclude that Log-ACD models were able to fit the stylized facts of stock market durations 'as well' as ACD models. These facts are a rather slowly decreasing ACF that starts from a relatively low positive value, a consequence of the clustering of activity, and overdispersion. The latter implies that very small and very large durations occur in higher proportions than is compatible with an exponential distribution.

In this paper we thus provide analytical expressions for the unconditional moments and ACF for the models belonging to the Log-ACD class as defined in Bauwens and Giot (2000), focusing on its most general parametrization. The results of this paper are proved using the method that has been proposed by He, Teräsvirta, and Malmsten (2002) and He (2000) for the moments of exponential GARCH models. We also provide an empirical application in which we compute the unconditional moments and ACF for the ACD and Log-ACD models estimated on financial durations for several stocks traded on the New York Stock Exchange.

The paper is organized as follows. In Section 2, we define the class of Log-ACD models, we provide the conditions of existence and the general formulae of the moments. In Section 3, we look at the properties of the dispersion index and the ACF. In Section 4, a comparison between the conditions for the existence of moments and autocorrelations is carried out between Log-ACD, ACD and SCD models. Section 5 presents the comparison using real data. Section 6 concludes. Proofs are relegated to the Appendix.

2 Log-ACD Models: Definition and Moments

We denote by x_i the duration between two events that happened at times t_{i-1} and t_i, i.e. $x_i = t_i - t_{i-1}$. We assume that the stochastic process $\{x_i\}$ generating the durations is doubly infinite (i goes from $-\infty$ to $+\infty$).

A Log-ACD model specifies the observed duration as the mixing process

$$x_i = e^{\psi_i}\epsilon_i, \tag{1}$$

where the ϵ_i are independent and identically distributed, with

$$E\left(\epsilon_i\right) = \mu, \tag{2}$$

$$Var\left(\epsilon_i\right) = \sigma^2, \tag{3}$$

so that $E(x_i|\mathcal{H}_i) = \mu \exp(\psi_i)$, where $\mathcal{H}_i$ denotes the information set available at time t_{i-1} (the beginning of the duration x_i), which includes the past durations.

The important assumption, which is the same as for ACD models (see Engle and Russell 1998), is that the dependence in the duration process can be subsumed in the conditional expectation $E(x_i|\mathcal{H}_i)$, in such a way that $x_i/E(x_i|\mathcal{H}_i)$ is *iid*. For further reference, we define

$$\Psi_i = \exp(\psi_i). \tag{4}$$

To introduce dependence in the process, which can produce a clustering of durations, ψ_i is specified as an autoregressive equation,[1] which in its most general form (in this paper) is written as

$$\psi_i = \omega + \sum_{j=1}^{p} \alpha_j g(\epsilon_{i-j}) + \sum_{j=1}^{p} \beta_j \psi_{i-j}, \tag{5}$$

which is equivalent to

$$\Psi_i = e^{\omega} \prod_{j=1}^{p} e^{\alpha_j g(\epsilon_{i-j})} \prod_{j=1}^{p} \Psi_{i-j}^{\beta_j}. \tag{6}$$

Two choices of the function $g(\epsilon_{i-j})$ are $\ln \epsilon_{i-j}$ or ϵ_{i-j}. The first one corresponds to the Log-ACD$_1$ model, in which equation (5) becomes

$$\begin{aligned} \psi_i &= \omega + \textstyle\sum_{j=1}^{p} \alpha_j \ln \epsilon_{i-j} + \sum_{j=1}^{p} \beta_j \psi_{i-j} \\ &= \omega + \textstyle\sum_{j=1}^{p} \alpha_j \ln x_{i-j} + \sum_{j=1}^{p} \left(\beta_j - \alpha_j\right) \psi_{i-j}, \end{aligned} \tag{7}$$

and the second one to the Log-ACD$_2$ model, for which

$$\begin{aligned} \psi_i &= \omega + \textstyle\sum_{j=1}^{p} \alpha_j \epsilon_{i-j} + \sum_{j=1}^{p} \beta_j \psi_{i-j} \\ &= \omega + \textstyle\sum_{j=1}^{p} \alpha_j (x_{i-j} / \exp \psi_{i-j}) + \sum_{j=1}^{p} \beta_j \psi_{i-j}. \end{aligned} \tag{8}$$

Several choices are available for the distribution of ϵ_i: exponential, gamma, generalized gamma, Weibull, Burr, lognormal, Pareto..., in principle any distribution with positive support. The choice of a particular distribution should be guided by the desire to have a 'correct' specification, and perhaps by its convenience for estimation. Among the distributions cited above, the Burr and the Pareto do not necessarily have finite moments, so restrictions on their parameters must be imposed to ensure that the variance and the mean exist. The Burr family includes the Weibull (and the exponential) as a particular case, while the generalized gamma includes the gamma and the Weibull (hence the exponential). All these distributions depend on a scale parameter that we normalize at 1. For distributions that are indexed by a single shape parameter (gamma, Weibull), μ and σ^2 are linked through that parameter. For the exponential distribution, the parameter is fixed to 1 so that $\mu = \sigma^2 = 1$. The Burr and generalized gamma depend on two shape parameters, and are therefore more flexible; in particular they can have a non-monotonous hazard function. The moments of a Log-ACD model depend of course on the moments of ϵ_i.

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[1] The results derived for the Log-ACD(p, p) can be directly applied to any Log-ACD(r, q) model with $r \neq q$, as the latter specification can always be nested in the former by simply choosing $p = \max\{r, q\}$.

In order to proceed, let us introduce the matrix

$$\mathbf{\Omega} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{p-1} & \beta_p \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}, \tag{9}$$

and the coefficients

$$\phi_k = \boldsymbol{\beta}' \mathbf{\Omega}^{k-p-1} \boldsymbol{\phi} \quad \text{for } k > p, \tag{10}$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$, and $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)$ such that

$$\begin{aligned} &\phi_0 = 1, \\ &\phi_1 = \beta_1, \\ &\phi_s = \textstyle\sum_{j=1}^{s} \beta_j \phi_{s-j}, \qquad s = 2, \ldots, p, \\ &\phi_s = \textstyle\sum_{j=1}^{p} \beta_j \phi_{s-j}, \qquad s > p. \end{aligned} \tag{11}$$

Let $\lambda(\mathbf{\Omega})$ be the absolute value of the maximum eigenvalue of the matrix $\mathbf{\Omega}$. The unconditional moments of x_i exist and are independent of i as $k \to \infty$ if and only if $\lambda(\mathbf{\Omega}) < 1$. In this case, $\mathbf{\Omega}^k \to 0$ and $\sum_{j=0}^{k} \mathbf{\Omega}^j \to (I - \mathbf{\Omega})^{-1}$ as $k \to \infty$, which is necessary for the sequence $\{\phi_i\}$ to converge to a finite value (see for example Hamilton (1994) page 20).

Theorem 1 *Assume that $E\left[\exp\left(m\theta_j g(\epsilon_i)\right)\right]$ and $\mu_m = E\left|x_i^m\right|$ exist for an arbitrary $m \in \mathbb{R}_+$. For the Log-ACD process defined by equations (1)-(5), the condition $\lambda(\mathbf{\Omega}) < 1$ is necessary and sufficient for the existence of the m-th moment $E\left|x_i^m\right|$. Under this condition,*

$$E\left(x_i^m\right) = \mu_m \exp\left[m\omega(1 - \sum_{j=1}^{p} \beta_j)^{-1}\right] \prod_{j=1}^{\infty} E\left[\exp\left(m\theta_j g(\epsilon_i)\right)\right], \tag{12}$$

where

$$\begin{aligned} &\theta_1 = \alpha_1, \\ &\theta_s = \begin{cases} \sum_{j=1}^{s} \alpha_j \phi_{s-j}, & s = 2, \ldots, p, \\ \sum_{j=1}^{p} \alpha_j \phi_{s-j}, & s > p. \end{cases} \end{aligned} \tag{13}$$

In the following corollary, we adapt this result to the Log-ACD (1,1) case.

Corollary 1 *For the Log-ACD (1,1), the hypotheses of Theorem 1 reduce to the fol-*

lowing: $E\left[\exp\left(m\alpha\beta^{j-1}g(\epsilon_i)\right)\right] < \infty$, $\mu_m < \infty$ *for an arbitrary positive integer* m *and* $|\beta| < 1$. *Under these conditions,*

$$E\left(x_i^m\right) = \mu_m \exp(\frac{m\omega}{1-\beta}) \prod_{j=1}^{\infty} E\left[\exp\left(m\alpha\beta^{j-1}g(\epsilon_i)\right)\right]. \tag{14}$$

(Note that $\alpha = \alpha_1$, *and* $\beta = \beta_1$*).*

For the practical computation of (12), the infinite product that appears in the moment expression can be truncated after a sufficiently large number of terms since β^j tends to 0.[2] For example, if we use an exponential distribution, $E(\epsilon^{\alpha\beta^j}) = \Gamma(1+\alpha\beta^j)$ and $E\left(\exp\left(\epsilon\alpha\beta^j\right)\right) = 1/(1-\alpha\beta^j)$, so that both expectations tend to 1 when j tends to infinity.

If α and β are both positive (as is practically always the case), computing the moment given in the previous theorem requires knowledge of $E(\epsilon^p)$ for any positive p (not necessarily integer) in the Log-ACD$_1$ case, and $E\left(\exp(p\epsilon)\right)$ in the Log-ACD$_2$ case. The (non-integer) moments $E(\epsilon^p)$ are available for the generalized gamma and Burr distributions, and all their particular cases. The moment generating function which provides $E\left(\exp(p\epsilon)\right)$ is only available analytically for the gamma distribution (including the exponential).

To be able to obtain an approximation of the moment generating function for the other distributions considered, namely the Weibull, the Burr and the generalized gamma, one can notice that the following Taylor expansion can be used:

$$E\left(\exp(p\epsilon)\right) = \sum_{k=0}^{\infty} \frac{p^k}{k!} E\left(\epsilon^k\right). \tag{15}$$

For any of the p-th order moments $E\left(\exp(p\epsilon)\right)$ to exist, the infinite series of integer moments $E\left(\epsilon^k\right)$ must converge to a finite value. In the Burr case, this condition is never satisfied, as the maximum fractional finite moment is determined by the ratio of its two shape parameters. For the Weibull and the generalized gamma on the other hand, the infinite moment series converges only if the shape parameter common to the two distributions is larger than one. In this case, it is possible to truncate the infinite sum and obtain an approximation of the p-th moment $E\left(\exp(p\epsilon)\right)$.

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[2] In practice, we found that for first and second-order moments, truncation after 1000 terms was more than sufficient to get a high accuracy.

3 Dispersion and Autocorrelation Function

Durations between stock market events are often characterized by overdispersion, meaning that the standard deviation of the data is larger than their mean (see Section 5). Another important stylized fact is the shape of the ACF, which usually decreases slowly from a relatively low positive first-order autocorrelation. It is therefore essential that Log-ACD models be able to fit such stylized facts, for some parameter values.

Let us measure the degree of dispersion of the random variable x by the variation coefficient, or its square root (= standard deviation/mean) that we call the dispersion index and we denote by δ_x. This ratio is larger than 1 in the case of overdispersion. This measure is a direct by-product of Theorem 1, and we have the following result:

Corollary 2 *For the Log-ACD process defined by equations (1)-(5), assume that the hypotheses of Theorem 1 hold for* $m = 1, 2$. *Then*

$$1 + \delta_x^2 = \left(1 + \delta^2\right) \frac{\prod_{j=1}^{\infty} E\left(e^{2\theta_j g(\epsilon_i)}\right)}{\left[\prod_{j=1}^{\infty} E\left(e^{\theta_j g(\epsilon_i)}\right)\right]^2} \geq 1 + \delta^2, \tag{16}$$

where $\delta = \sigma/\mu$ *is the dispersion index of* ϵ_i.

The dispersion index of x_i cannot be smaller than that of ϵ_i. Thus, it suffices that ϵ_i be equidispersed ($\delta = 1$) for x_i to be overdispersed, as long as $\alpha \neq 0$. Figure 1 illustrates the variation of δ_x as a function of α (from 0 to 0.2) and β (from 0.8 to 0.98) when ϵ_i is exponential (so that $\delta = 1$) and the model is a Log-ACD$_1$(1,1). For the Log-ACD$_2$(1,1) model, the figure is almost identical, the difference being that the values of δ_x are slighlty smaller (except for the combinations $\alpha = 0.2$ and $0.8 < \beta < 0.94$).

The next theorem provides the autocorrelation function.

Theorem 2 *For the Log-ACD process defined by equations (1)-(5), assume that* $\mu < \infty$, $\lambda(\mathbf{\Omega}) < 1$, $E\left[e^{\delta g(\epsilon_i)}\right] < \infty$ *for any* $\delta \in R$, $E\left[\epsilon_{i-n} \exp\left(\theta_n g(\epsilon_{i-n})\right)\right] < \infty$ *for any* $n \in N_+$, *and* $E\left[\exp\left((\phi_{n-j}\alpha_{j+h} + \theta_{hn}^*)g(\epsilon_{i-n-h})\right)\right] < \infty$ *for* j *and* h *such that* $n \geq 1$. *Then, for* $n \geq 1$, *the* n*-th order autocorrelation of* $\{x_i\}$ *has the form*

$$\rho_n = \frac{\mu E\left[\epsilon_i e^{\theta_n g(\epsilon_i)}\right] \prod_{j=1}^{n-1} E\left[e^{\theta_j g(\epsilon_i)}\right] \prod_{j=p}^{\infty} E\left[e^{\theta_{jn}^* g(\epsilon_i)}\right] M_{n,p} - \mu^2 \left(\prod_{j=1}^{\infty} E\left[e^{\theta_j g(\epsilon_i)}\right]\right)^2}{\mu_2 \prod_{j=1}^{\infty} E\left[e^{2\theta_j g(\epsilon_i)}\right] - \mu^2 \left(\prod_{j=1}^{\infty} E\left[e^{\theta_j g(\epsilon_i)}\right]\right)^2}, \tag{17}$$

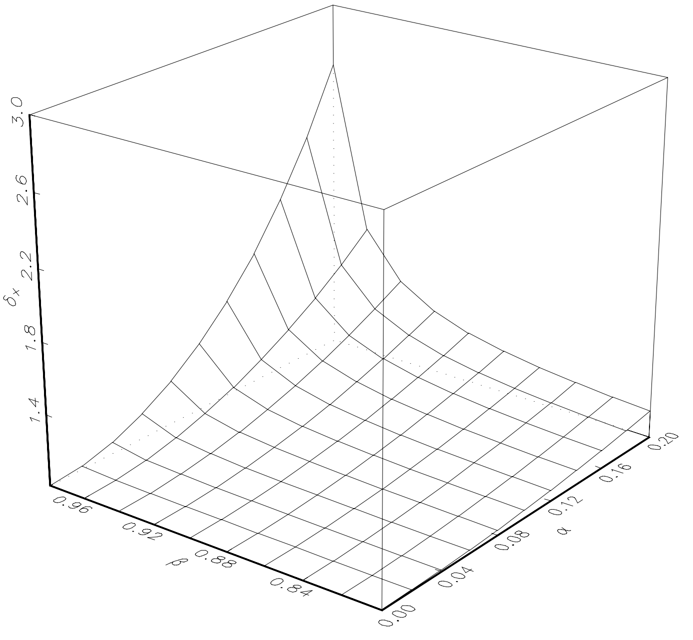
where

$$M_{n,p} = \begin{cases} = \prod_{h=1}^{p-n} E\left[e^{\left(\sum_{j=1}^{n}\phi_{n-j}\alpha_{h+j}+\theta^*_{hn}\right)g(\epsilon_{i-n-h})}\right] & \\ \quad \cdot \prod_{h=1}^{n-1} E\left[e^{\left(\sum_{j=1}^{h}\phi_{n-j}\alpha_{p-h+j}+\theta^*_{p-h,n}\right)g(\epsilon_{i-n-h})}\right] & \textit{for } 1 \le n \le p, \\ = \prod_{h=1}^{p-1} E\left[e^{\left(\sum_{j=1}^{p-h}\phi_{n-j}\alpha_{h+j}+\theta^*_{hn}\right)g(\epsilon_{i-n-h})}\right] & \textit{for } n > p, \end{cases} \tag{18}$$

θ^*_{jn} *is defined in equation (57) in the Appendix, and* $\mu_2 = \sigma^2 + \mu^2$.

Figure 1

Dispersion index of log-ACD$_1$ model (exponential distribution).



The following corollary is the special Log-ACD (1,1) case of Theorem 2.

Corollary 3 *For the Log-ACD (1,1) process, the hypotheses of Theorem 2 reduce to the following:* $|\beta| < 1$ *and* $E\left[\exp\left(2\alpha g(\epsilon_i)\right)\right] < \infty$. *Under these conditions*

$$\rho_n = \frac{\mu E\left[\epsilon_i e^{\alpha\beta^{n-1}g(\epsilon_i)}\right]\prod\limits_{j=1}^{n-1} E\left[e^{\alpha\beta^{j-1}g(\epsilon_i)}\right]\prod\limits_{j=1}^{\infty} E\left[e^{\alpha(1+\beta^n)\beta^{j-1}g(\epsilon_i)}\right]-\mu^2\left(\prod\limits_{j=1}^{\infty} E\left[e^{\alpha\beta^{j-1}g(\epsilon_i)}\right]\right)^2}{\mu_2\prod_{j=1}^{\infty} E\left[e^{2\alpha\beta^{j-1}g(\epsilon_i)}\right]-\mu^2\left(\prod_{j=1}^{\infty} E\left[e^{\alpha\beta^{j-1}g(\epsilon_i)}\right]\right)^2}. \tag{19}$$

Some remarks can be made on the features of the autocorrelation function provided by Theorem 2.

First, it is worthwhile to notice that $\lim_{n\to\infty}\rho_n = 0$. This can be easily seen, in the

Log-ACD(p,p) instance, by considering that, as $n \to \infty$,

$$\begin{aligned}
&E\left[\epsilon_{i-n} e^{\theta_n g(\epsilon_{i-n})}\right] \to \mu, \\
&\textstyle\prod_{j=1}^{n-1} E\left[e^{\theta_j g(\epsilon_i)}\right] \prod_{j=p}^{\infty} E\left[e^{\theta^*_{jn} g(\epsilon_i)}\right] \to \left(\prod_{j=1}^{n-1} E\left[e^{\theta_j g(\epsilon_i)}\right]\right)^2, \text{ and} \\
&M_{n,p} \to 1.
\end{aligned} \tag{20}$$

Hence, the numerator of equation (17) tends to zero.

Another remark is that the shape of ρ_n as a function of n in Theorem 2 is determined by the absolute value of the maximum eigenvalue of the $\mathbf{\Omega}$ matrix. The closer $\lambda(\mathbf{\Omega})$ to 1, the more persistent the autocorrelation. Notice that $\lambda(\mathbf{\Omega}) = \beta$ in the Log-ACD$_1$ case.

Figure 2

First autocorrelation of Log-ACD$_2$ model (exponential distribution)

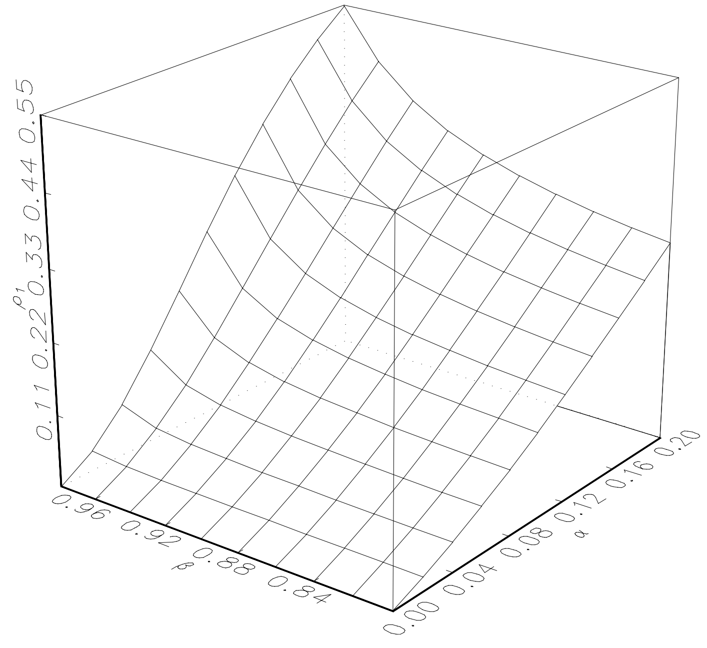


Figure 2 illustrates the variation of ρ_1 in the same setup as in Figure 1 (again with ϵ_i exponential, so that $\mu = \sigma = 1$). For the Log-ACD$_1$(1,1) model, the figure is almost the same, but the value of ρ_1 in the Log-ACD$_1$(1,1) case is larger than in the Log-ACD$_2$(1,1) whenever $\alpha < 0.08$ and smaller whenever $\alpha > 0.14$, while in the intermediate cases it is larger when $\beta > 0.9$ (approximately). However, the differences are never larger than 0.04. These features are not necessarily the same for other distributions of ϵ_i. From this Figure, we see that for $\alpha < 0.10$, ρ_1 does not exceed 0.20 (roughly) when β is smaller than 0.96.

Another feature of interest is the rate of decrease of the ACF. We assume that $0 < \beta < 1$ to avoid oscillation of the signs of the autocorrelations. For example, the Log-ACD$_1$(1,1) model can be written as the ARMA(1,1) process

$$\ln x_i = \omega + \beta \ln x_{i-1} + u_i - (\beta - \alpha) u_{i-1}, \tag{21}$$

where $u_i = \ln x_i - \psi_i$ is a martingale difference. The autocorrelations of the logarithm of the duration therefore decrease geometrically at the rate β. However, by computing equation (19) for many parameter configurations, we found that the autocorrelations of the duration decrease at the above rate only after a 'large' lag. For small lags, the rate of decrease is less than β, although not much. Table 1 provides, for several parameter values, the value of ρ_1, the ratio ρ_2/ρ_1, and the value of n from which the rate of decrease is equal to β (for a precision of 4 decimal digits). The results in the table show that i) for fixed β, the larger α, the larger the difference $\beta - \rho_2/\rho_1$ and the value of n, and ii) for fixed α, the larger β, the smaller the difference $\beta - \rho_2/\rho_1$ but the larger the value of n.

Table 1: Properties of the ACF of the Log-ACD$_2$ model
(exponential distribution)

	β					
	0.800	0.840	0.880	0.920	0.960	0.980
α						
0.04	*0.045*	*0.046*	*0.048*	*0.051*	*0.061*	*0.079*
	0.793	*0.834*	*0.875*	*0.917*	*0.958*	*0.979*
	24	*30*	*38*	*53*	*93*	*162*
0.08	*0.100*	*0.104*	*0.111*	*0.123*	*0.157*	0.213
	0.785	*0.827*	*0.869*	*0.912*	*0.955*	0.976
	28	*34*	*44*	*63*	*115*	212
0.12	*0.164*	*0.172*	*0.185*	*0.209*	0.270	0.353
	0.775	*0.818*	*0.861*	*0.905*	0.950	0.972
	30	37	48	69	129	244
0.16	0.234	0.247	0.267	0.302	0.380	0.467
	0.763	0.807	0.851	0.896	0.942	0.965
	31	39	51	74	140	268
0.20	0.306	0.324	0.350	0.392	0.474	0.548
	0.749	0.793	0.838	0.885	0.932	0.955
	33	41	53	78	149	288

Notes: In each cell from top to bottom, one finds the value of ρ_1, the ratio ρ_2/ρ_1, and the value of n which gives $\beta = \rho_{n+1}/\rho_n$ to four decimal places.

From Figure 2 and Table 1, we see that there is a region of parameter values for which the autocorrelation function starts at a low positive value (say less than about 0.2) and decreases "slowly" (see the italicized entries of Table 1).

4 Comparison with ACD and SCD Models

The ACD model, introduced by Engle and Russell (1998), is defined by the following equations:

$$\begin{aligned} &x_i = \Psi_i \epsilon_i, \\ &\Psi_i = \omega + \alpha x_{i-1} + \beta \Psi_{i-1}, \\ &\omega > 0, \quad \alpha \geq 0, \quad \beta \geq 0, \quad \beta = 0 \text{ if } \alpha = 0, \end{aligned} \tag{22}$$

where the baseline duration ϵ_i follows the same assumptions as in the Log-ACD case and $(\alpha + \beta)$ in the ACD conditional duration Ψ_i is analogous to the β term in the logarithmic specification.

For this class of models, computing moments and autocorrelation functions is easy and one can obtain the following simple expression in the ACD (1,1) instance:

$$\mu_x = E\left(x\right) = \frac{\mu\omega}{1-\mu\alpha-\beta} \qquad \text{if } 0 \leq \mu \text{ and } (\alpha+\beta) < 1, \tag{23}$$

$$\delta_x^2 = \frac{\sigma_x^2}{\mu_x^2} = \left(\frac{\sigma^2}{\mu^2}\right)\left(\frac{1-\beta^2-2\mu\alpha\beta}{1-\left(\mu\alpha+\beta\right)^2-\alpha^2\sigma^2}\right) \geq \delta^2, \tag{24}$$

$$\rho_1 = \frac{\alpha\left(1-\beta^2-\alpha\beta\right)}{1-\beta^2-2\alpha\beta}, \tag{25}$$

$$\rho_n = (\alpha+\beta)\rho_{n-1} \quad (n > 1). \tag{26}$$

It must, however, be noticed that the conditions for the existence of the moments of higher order involve the parameters α and β in the formula for the conditional duration, which is not the case for the Log-ACD model, where conditions on β do not change. Furthermore, the ACF of the durations decreases geometrically at the rate $\alpha + \beta$, since the ACD can be rewritten as an ARMA model with AR parameter $\alpha + \beta$.

Like the Log-ACD model, the SCD model, introduced by Bauwens and Veredas (2004), has a non linear expression for the conditional duration Ψ_i. The model has the following specification:

$$\begin{aligned} &x_i = \Psi_i\, \epsilon_i = e^{\psi_i}\, \epsilon_i, \\ &\psi_i = \omega + \beta\psi_{i-1} + \eta_i, \quad |\beta| < 1, \end{aligned} \tag{27}$$

where again the baseline duration term follows the same assumptions as in the Log-ACD case, but is independent of η_i, the other random term present in the model, characterized by an *iid* normal distribution with mean 0 and variance σ^2.

The SCD model allows for a simple structure for moments and ACF, which is

$$\begin{aligned}
\mu_x &= \mu e^{\frac{\omega}{1-\beta}+\frac{1}{2}\left(\frac{\sigma^2}{1-\beta^2}\right)}, \\
1+\delta_x^2 &= \left(1+\delta^2\right) e^{\frac{\sigma^2}{1-\beta^2}} \geq 1+\delta^2, \\
\rho_k &= \frac{e^{\frac{\sigma^2\beta^k}{1-\beta^2}}-1}{(1+\delta^2)\left(e^{\frac{\sigma^2}{1-\beta^2}}-1\right)} \approx \frac{\sigma^2\beta^k/\left(1-\beta^2\right)}{(1+\delta^2)\left(e^{\frac{\sigma^2}{1-\beta^2}}-1\right)} \approx \beta\rho_{k-1}.
\end{aligned} \tag{28}$$

A relevant remark is that, as in the Log-ACD case, the autocorrelation function ρ_k geometrically decreases at rate β only asymptotically, while for small k the rate of decrease is smaller.

5 Fitting the Stylized Facts

In this section, we consider an application to financial durations for stocks traded on the NYSE. The objective of this empirical application is to provide an illustrative example of the use of the formulae derived in the previous section. The possibility of calculating the moments that are implied by the estimated parameters allows us also to compare various specifications (ACD, Log-ACD$_1$ and Log-ACD$_2$) and distributions for the baseline durations in their ability to "fit" the sample moments of the data.

As reviewed by Giot (2000), while durations can simply be defined as the time elapsed between two market events, by judiciously defining the notion of market event one can highlight several important features of intraday market activity. For example, a duration between two quotes is a quote duration and the modelling of these using ACD or Log-ACD type models can quantify the notion of quoting activity, i.e. the rate at which the specialists post quotes.

Important extensions related to the quote process are the notions of price and volume durations. Price durations are defined as the minimum time for the stock price to escape from a given price interval. In our application, we focus on the mid-price of the specialist quote, i.e., the average of the bid and ask prices, and the price interval considered is \$0.125. It can be shown (see Giot, 2000) that there is a relationship between the volatility of the price process and the conditional hazard of the ACD or Log-ACD model. Thus this provides a strong motivation for the use of such high frequency duration models in the modelling of intraday volatility. A volume duration is defined as the time required for total traded volume to accumulate until a given amount (25000 shares in our application). This duration can be considered as a partial measure of market liquidity, as it indicates the time needed to trade a given amount of shares.

The data set considered in the empirical evaluation consists of series of price and volume durations of five stocks (Boeing, Coca Cola, Disney, Exxon and IBM) taken from the Trade and Quote (TAQ) database of the New York Stock Exchange. For each stock, we have considered two periods. The first period ranges from September to November 1996, while the second goes from January to April 1997.

To take into account the known seasonal effects, we followed Engle and Russell (1998) in computing adjusted durations as

$$x_i = X_i/\phi(t_i, j), \tag{29}$$

where X_i is the original duration (extracted from the data base) and $\phi(t_i, j)$ is the seasonal effect, considered as the function of the time (t_i) and the day of the week (j) of the transaction. The function $\phi(t_i, j)$ is estimated by averaging over thirty minute intervals for each day of the week and smoothing with a cubic spline. The resulting time-of-day and time-of-week adjusted duration is denoted by x_i.

Each deseasonalized sequence of data has been estimated by ACD(1,1), Log-ACD$_1$(1,1) and Log-ACD$_2$(1,1), and for each one of these models we have considered a series of distributions for the conditional durations, namely: exponential (0 shape parameters), Weibull and gamma (1 shape parameter), and Burr and generalized gamma (2 shape parameters). In all these distributions, a further parameter, the scale one, is present. We have chosen to constrain this parameter to the value such that the expectation of the baseline duration ϵ_i equals 1 in order to avoid an identification problem with the parameters of the autoregressive factor (another possible choice could have been to fix it to 1). The number of observations is different in each sequence of data, ranging from a minimum of 1609 (for the Coca Cola price durations of 1996) to a maximum of 19680 (for the IBM price durations of 1996). Table 2 reports the maximum likelihood (ML) estimates for the case of IBM price durations in the 1997 data set.[3] The ML estimates for each model, distribution and data sequence were then used to compute the analytical expressions for the unconditional moments and autocorrelation functions. The results based on the analytical expressions were then compared with the empirical (unconditional) moments and ACF.

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[3] We do not report standard errors since they are not needed in the following discussion.

Table 2: Point Estimates

	exponential	Weibull	gamma	Burr	generalised gamma
ACD					
ω	0.063	0.062	0.063	0.089	0.112
α	0.098	0.097	0.098	0.114	0.128
β	0.840	0.842	0.840	0.803	0.760
Log-ACD$_1$					
ω	0.042	0.042	0.042	0.057	0.050
α	0.090	0.089	0.090	0.109	0.108
β	0.928	0.929	0.928	0.900	0.900
Log-ACD$_2$					
ω	-0.084	-0.083	-0.084	-0.087	-0.089
α	0.082	0.082	0.082	0.089	0.087
β	0.938	0.939	0.938	0.920	0.919

Notes: ML estimates of the parameters of the ACD, Log-ACD$_1$ and Log-ACD$_2$ models, assuming various distributions for ϵ_i. Data: price durations (at \$1/8) for IBM, January-April 1997, 18878 observations.

Tables 3 and 4 report the first two empirical moments and the dispersion indices resulting from the analytical expressions for the three models. Broadly speaking, the unconditional moments computed from the analytical formulae. As one can see from the first moment, the second and the dispersion index, the models are quite capable of reproducing the empirical moments in the fitted distribution of the unconditional durations. The first moment and the dispersion ratio, in particular, seem to be the ones that can be better matched by the analytical values. Of course, some extreme cases arise, in which the estimation can not really catch many of the features of the data or the estimated parameters are very close to some conditions for the existence of moments in the conditional distribution (as could be the case with the Burr). The analytical (estimated) moments for the Log-ACD$_2$ model are not reported for the Burr and generalized gamma distribution. The reason is that the conditions on the convergence of the series in equation (15) to a finite value are never satisfied in the Burr case and were not satisfied by the parameters resulting from the estimations with the generalized gamma Log-ACD$_2$ model. Figure 3 reports as a graphical example the empirical ACF of a series of data (IBM price durations for the January-April 1997 period) and the ACF computed from the estimated parameters of various models.

Table 3: Volume durations - moments implied by point estimates

*BOEING v*96	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.006*	1.098	1.096	1.094	1.111	1.088	0.303	1.001	1.033	1.039	1.034	1.164	1.775	1.082
variance	*1.539*	10.09	2.242	2.469	2.504	2.306	0.273	1.582	1.723	1.738	1.702	4.627	5.257	3.992
dispersion	*0.731*	2.714	0.932	1.031	1.015	0.973	1.406	0.761	0.785	0.781	0.768	1.553	0.818	1.553
*COCA COLA v*96														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*0.999*	1.017	1.026	1.015	1.023	1.016	0.795	0.998	1.006	1.003	1.006	1.019	1.157	1.013
variance	*1.766*	2.320	1.894	1.894	1.898	1.889	1.383	1.758	1.816	1.801	1.826	2.287	2.392	2.258
dispersion	*0.876*	1.115	0.894	0.916	0.902	0.911	1.089	0.875	0.891	0.889	0.897	1.097	0.887	1.097
*DISNEY v*96														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*0.999*	1.003	1.003	1.002	1.001	0.999	0.538	0.994	1.002	0.999	1.002	1.046	1.279	1.042
variance	*1.414*	2.353	1.425	1.491	1.432	1.426	0.653	1.388	1.455	1.414	1.420	2.555	2.341	2.540
dispersion	*0.644*	1.157	0.646	0.697	0.656	0.653	1.120	0.637	0.671	0.645	0.643	1.156	0.655	1.156
*EXXON v*96														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.000*	1.010	1.016	1.009	1.012	1.013	0.694	1.000	1.005	1.006	1.006	1.030	1.193	1.029
variance	*1.433*	2.246	1.484	1.524	1.479	1.480	1.049	1.430	1.484	1.451	1.451	2.319	2.060	2.318
dispersion	*0.658*	1.096	0.662	0.705	0.666	0.665	1.084	0.656	0.685	0.659	0.656	1.089	0.669	1.089
*IBM v*96														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.006*	19.27	1.638	1.256	1.379	1.305	0.318	4.906	2.855	3.423	1.006	1.251	1.056	0.259
variance	*1.656*	NA	94.74	5.983	11.04	7.115	0.441	46.30	15.07	22.42	1.826	6.572	2.067	0.274
dispersion	*0.798*	10.00	5.857	1.670	2.193	1.783	1.834	0.962	0.921	0.955	0.896	1.788	0.924	1.763
*BOEING v*97														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.001*	1.081	1.079	1.077	1.089	1.072	0.308	0.999	1.030	1.033	1.031	1.156	1.736	1.084
variance	*1.532*	7.579	2.096	2.290	2.279	2.144	0.278	1.564	1.702	1.703	1.677	4.474	4.979	3.931
dispersion	*0.727*	2.342	0.895	0.986	0.958	0.929	1.392	0.753	0.777	0.771	0.759	1.530	0.807	1.530
*DISNEY v*97														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*0.999*	1.001	0.999	0.999	0.999	0.997	0.531	0.995	1.003	1.001	1.003	1.042	1.268	1.038
variance	*1.399*	2.318	1.405	1.473	1.410	1.405	0.631	1.382	1.449	1.405	1.410	2.518	2.279	2.499
dispersion	*0.635*	1.147	0.636	0.689	0.644	0.642	1.114	0.628	0.664	0.635	0.634	1.148	0.646	1.148
*EXXON v*97														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.000*	1.002	1.008	1.000	1.004	0.999	0.401	1.000	1.062	1.038	1.053	1.056	1.515	0.849
variance	*1.486*	2.594	1.520	1.559	1.543	1.521	0.386	1.469	1.691	1.606	1.646	2.769	3.449	1.789
dispersion	*0.697*	1.258	0.704	0.748	0.729	0.724	1.183	0.685	0.707	0.700	0.696	1.218	0.709	1.217
*IBM v*97														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.003*	1.025	1.030	1.024	1.039	1.024	0.288	0.959	1.015	1.013	1.014	1.118	1.643	1.074
variance	*1.495*	3.759	1.631	1.666	1.762	1.658	0.243	1.381	1.538	1.562	1.540	3.689	4.084	3.399
dispersion	*0.698*	1.606	0.732	0.768	0.794	0.763	1.392	0.706	0.703	0.723	0.705	1.397	0.717	1.396

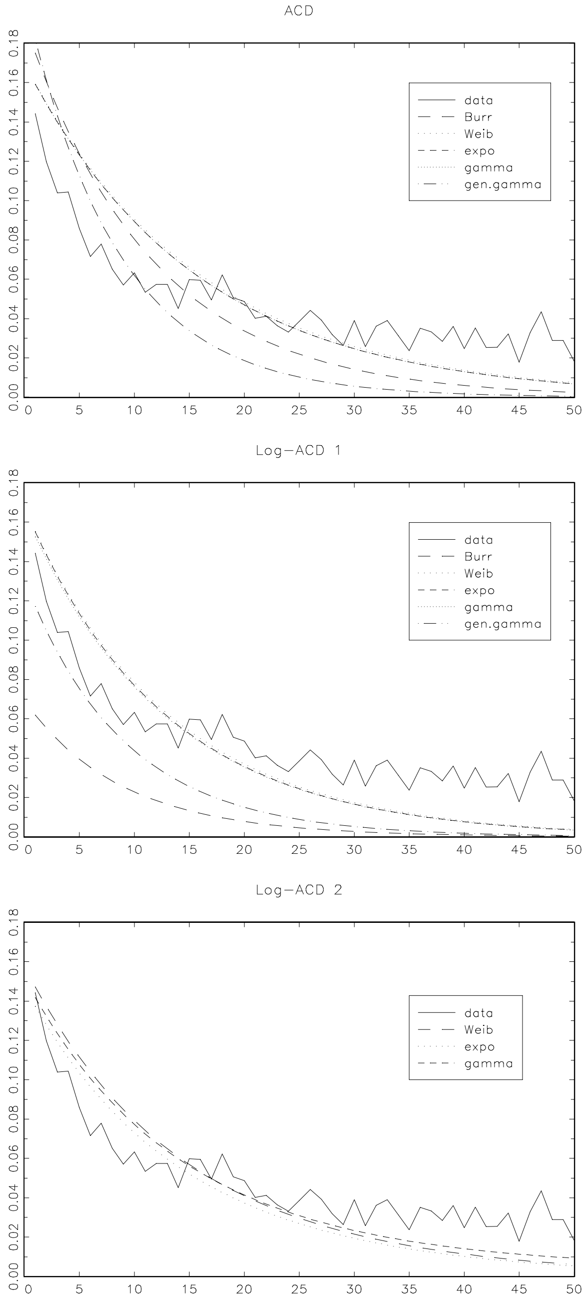
Notes: Unconditional moments for the ACD, Log-ACD$_1$ (LACD$_1$) and Log-ACD$_2$ (LACD$_1$) models computed by applying the analytical expressions with the estimated parameters. The first column (in italics) gives the empirical moments computed from the data. The capital letters denote the model (ACD, Log-ACD$_1$ or Log-ACD$_2$) while the small ones denote the conditional distribution (e for exponential, w for Weibull, g for gamma, b for Burr and gg for generalized gamma).

Table 4: Price durations - moments implied by point estimates

BOEING v96	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.006*	1.070	1.055	1.070	1.200	1.021	1.096	0.938	1.060	1.038	1.017	0.988	0.928	0.972
variance	*2.938*	2.894	3.072	2.947	NA	3.488	2.915	2.325	2.776	5.836	3.345	2.203	2.094	2.138
dispersion	*1.380*	1.236	1.327	1.254	4.008	1.532	1.195	1.283	1.213	2.102	1.494	1.124	1.197	1.124
COCA COLA v96														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.001*	1.008	1.005	1.008	1.034	0.995	1.012	0.986	1.006	1.019	1.011	0.997	0.983	0.997
variance	*2.377*	2.116	2.216	2.139	3.212	2.435	2.136	2.137	2.134	3.163	2.714	2.048	2.098	2.047
dispersion	*1.171*	1.040	1.092	1.051	1.415	1.207	1.041	1.094	1.052	1.430	1.286	1.029	1.080	1.029
DISNEY v96														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.001*	1.017	1.016	1.017	1.055	0.990	0.962	0.952	1.013	1.017	1.003	0.995	0.991	0.991
variance	*2.517*	2.260	2.269	2.176	4.084	2.391	2.036	2.012	2.169	3.707	2.619	2.112	2.109	2.096
dispersion	*1.229*	1.088	1.093	1.049	1.631	1.199	1.094	1.103	1.054	1.606	1.265	1.064	1.070	1.064
EXXON v96														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.000*	1.004	1.002	1.004	1.036	0.988	0.999	0.974	1.014	1.023	1.007	0.998	0.989	0.998
variance	*2.432*	2.056	2.106	2.033	3.708	2.277	2.077	2.030	2.105	3.694	2.614	2.030	2.056	2.029
dispersion	*1.196*	1.019	1.047	1.008	1.566	1.153	1.038	1.066	1.023	1.591	1.256	1.018	1.049	1.018
IBM v96														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.002*	1.029	1.035	1.029	1.126	1.004	0.888	0.945	1.037	1.031	1.016	0.997	1.032	0.914
variance	*2.429*	2.421	2.351	2.224	5.876	2.512	1.780	1.943	2.225	4.312	2.632	2.173	2.244	1.826
dispersion	*1.192*	1.134	1.091	1.049	1.908	1.222	1.121	1.083	1.034	1.748	1.244	1.089	1.052	1.089
BOEING v97														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.007*	1.062	1.049	1.062	1.186	1.018	1.087	0.940	1.058	1.038	1.018	0.988	0.934	0.977
variance	*2.910*	2.795	2.960	2.837	NA	3.374	2.850	2.312	2.736	5.647	3.314	2.199	2.105	2.149
dispersion	*1.367*	1.216	1.301	1.231	3.441	1.502	1.188	1.270	1.203	2.059	1.482	1.118	1.188	1.118
DISNEY v97														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.000*	1.011	1.012	1.011	1.045	0.991	0.955	0.959	1.008	1.014	1.002	0.995	0.998	0.991
variance	*2.415*	2.201	2.191	2.106	3.574	2.327	1.976	1.983	2.101	3.348	2.533	2.095	2.097	2.076
dispersion	*1.190*	1.073	1.067	1.029	1.507	1.170	1.079	1.076	1.033	1.501	1.233	1.056	1.052	1.056
EXXON v97														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.000*	1.014	0.923	1.016	NA	0.746	1.468	1.204	1.230	1.014	NA	0.584	0.148	0.032
variance	*3.257*	2.326	2.421	2.595	NA	2.406	4.522	3.706	3.507	3.348	NA	0.736	0.058	0.002
dispersion	*1.502*	1.124	1.357	1.229	NA	1.822	1.054	1.247	1.147	1.501	1.705	1.078	1.287	1.049
IBM v97														
	data	ACD					LACD$_1$					LACD$_2$		
		e	w	g	b	gg	e	w	g	b	gg	e	w	g
mean	*1.002*	1.012	1.017	1.012	1.077	0.994	0.910	0.952	1.008	1.029	1.004	0.998	1.023	0.995
variance	*2.376*	2.222	2.143	2.041	4.904	2.326	1.811	1.899	2.032	4.427	2.496	2.119	2.136	2.110
dispersion	*1.169*	1.082	1.035	0.997	1.795	1.163	1.088	1.046	1.000	1.781	1.214	1.063	1.019	1.06

Notes: Unconditional moments for the ACD, Log-ACD$_1$ and Log-ACD$_2$ models computed by applying the analytical expressions with the estimated parameters. The first column (in italics) gives the empirical moments computed from the data. The capital letters denote the model (ACD, Log-ACD$_1$ or Log-ACD$_2$) while the small ones denote the conditional distribution (e for exponential, w for Weibull, g for gamma, b for Burr and gg for generalized gamma).

Figure 3



Notes: ACF for the ACD, Log-ACD$_1$, Log-ACD$_2$ models with various conditional distributions (using the analytical expressions computed for the estimated parameters) and empirical data (price duration at \$1/8 for IBM, 1997 data).

In order to summarize the large number of empirical results obtained, we make a ranking of models. The results of this ranking may serve as a guide for the interpretation of the results. The steps followed have been kept as simple as possible. First, for each stock, period and distribution we computed the percentage difference between the empirical and the theoretical (i.e. resulting from the estimated parameters) first moment and dispersion index (which is also a function of the second moment). We also computed a weighted sum of the absolute difference between the values taken by the empirical and theoretical autocorrelations. Only the first 50 values were considered and we assigned decreasing weights (0.975^n to the n-th autocorrelation, which assignes a weight of 0.28 to the 50-th lag). Second, the stocks, for each period, were then ranked for each one of the three considered criteria (deviation of the first moment, dispersion and autocorrelations) and the numbers denoting their positions in the rankings were added to provide a global ranking. Third, the resulting ranks were finally added for all the stocks and periods, keeping the distinction between price and volume durations. In the resulting ranks, the models and distributions with the lowest values are the ones that better perform globally on the three criteria together.

Table 5: Ranking results

Price		**Volume**	
Model	*sum of ranks*	*Model*	*sum of ranks*
eLACD$_2$	24	ggLACD$_1$	33
ggLACD$_1$	27	wLACD$_1$	36
ggACD	38	bLACD$_1$	38
gLACD$_2$	46	gLACD$_1$	46
gLACD$_1$	57	ggACD	49
wLACD$_2$	57	gACD	58
wACD	58	wACD	63
eACD	66	bACD	79
wLACD$_2$	67	eLACD$_2$	79
eLACD$_1$	76	gLACD$_2$	80
gACD	86	wLACD$_2$	82
bLACD$_1$	105	eLACD$_1$	83
bACD	112	eACD	93

Notes: Sum of rank points for first moment, dispersion and autocorrelation for all the stocks and periods. A lower value of the sum indicates a better performance. The capital letters denote the model (ACD, Log-ACD$_1$ or Log-ACD$_2$) while the small ones denote the conditional distribution (e for exponential, w for Weibull, g for gamma, b for Burr and gg for generalized gamma).

Table 5 displays the results of the rank computation. It is quite evident that the performance of the models and distributions considered varies with the kind of duration, price or volume that we fitted. For price durations, the generalized gamma seems to be the best distribution, followed by the Weibull. The Burr is strongly penalized by its constraint on the number of existing moments, often failing to correctly model the second moment, which is reflected in a poorly fitted dispersion index and ACF. The ranking does not seem to give many hints about what specification (ACD, Log-ACD$_1$ or Log-ACD$_2$) may be preferable, though the exponential Log-ACD$_2$ is the model that performs the best. The results on volume durations lead instead to a marked preference the Log-ACD$_1$ specification, followed by the ACD one. Here again, one can see that the generalized gamma seems to grant a significant gain over other distributions. This should not come as a surprise, as its parametrization is richer than the one of Weibull and gamma and it does not suffer from the constraints for the existence of moments that characterize the Burr.

6 Conclusion

We have provided analytical formulae for the moments of Log-ACD(p,p) models. The formulae are more complex than for the ACD model, since the ACD model is actually a linear process (ARMA) whereas the Log-ACD is non-linear. We have shown that the shape of the autocorrelation function of Log-ACD models is different from the shape of the ACF of the ACD model. The formulae can be used to check implied moments from parameter estimates, as in the illustration of this paper. They could also be used to select parameter values in order to match desired moments (e.g. for designing a Monte Carlo experiment). Finally, we have illustrated the different aptitudes of various models and distributions by estimating the empirical moments.

Appendix

Proof of Theorem 1

For simplicity in the notation, let us define the vectors

$$\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_p)', \mathbf{g}_i = (g(\epsilon_{i-1}), ..., g(\epsilon_{i-p}))'.$$

Suppose that $1 \leq k \leq p$. If we apply the definition of Ψ_i in equation (6) to Ψ_{i-1} in (6), after rearranging and substituting with ϕ_1 we can write

$$\Psi_i = \exp\left[\omega(1+\phi_1)\right] \cdot \exp\left(\boldsymbol{\alpha}'\mathbf{g}_i + \phi_1 \boldsymbol{\alpha}'\mathbf{g}_{i-1}\right) \cdot \prod_{j=1}^{p-1} \Psi_{i-j-1}^{\phi_1\beta_j+\beta_{j+1}} \cdot \Psi_{i-p-1}^{\phi_1\beta_p} . \tag{30}$$

If we apply it again to Ψ_{i-2} in equation (30) and substitute with ϕ_2, we get

$$\begin{aligned}\Psi_i = \;& \exp\left[\omega(1+\phi_1+\phi_2)\right]\cdot\exp\left(\boldsymbol{\alpha}'\mathbf{g}_i+\phi_1\boldsymbol{\alpha}'\mathbf{g}_{i-1}+\phi_2\boldsymbol{\alpha}'\mathbf{g}_{i-2}\right)\\ &\cdot\textstyle\prod_{j=1}^{p-2}\Psi_{i-j-2}^{\beta_j\phi_2+(\beta_{j+1}\phi_1+\beta_{j+2})}\cdot\;\Psi_{i-p-1}^{\beta_{p-1}\phi_2+\beta_p\phi_1}\cdot\;\Psi_{i-p-2}^{\beta_p\phi_2}.\end{aligned}\tag{31}$$

Continuing applying the definition given in equation (6) and substituting with ϕ_k until $k=p$, yields

$$\Psi_i=\prod_{j=0}^{p}e^{\omega\phi_j}\prod_{j=0}^{p}e^{\phi_j\boldsymbol{\alpha}'\mathbf{g}_{i-j}}\cdot\Psi_{i-p-1}^{\sum_{j=1}^{p}\beta_j\phi_{p-j+1}}\cdot\Psi_{i-p-2}^{\sum_{j=2}^{p}\beta_j\phi_{p-j+2}}\cdot\ldots\cdot\Psi_{i-p-p-1}^{\beta_{p-1}\phi_p+\beta_p\phi_{p-1}}\cdot\Psi_{i-p-p}^{\beta_p\phi_p}.\tag{32}$$

In order to be able to iterate further, we need to derive an expression of ϕ_k when $k>p$, given equations (9)-(11). This can be done by noticing that the following equalities

$$\begin{aligned}\boldsymbol{\Omega}^p\left(\phi_0,0,0,...,0\right)' &= \boldsymbol{\Omega}^{p-1}\left(\phi_1,\phi_0,0,\ldots,0\right)'\\ &= \ldots\\ &= \boldsymbol{\Omega}\left(\phi_{p-1},\phi_{p-2},\ldots,\phi_1,\phi_0\right)'\\ &= \left(\phi_p,\phi_{p-1},\ldots,\phi_2,\phi_1\right)'\end{aligned}\tag{33}$$

hold and by applying them to equation (10), to show that

$$\begin{aligned}\phi_k &= \boldsymbol{\beta}'\boldsymbol{\Omega}^{k-p-2}\boldsymbol{\Omega}\left(\phi_p,\phi_{p-1},\ldots,\phi_2,\phi_1\right)'\\ &= \beta_1\boldsymbol{\beta}'\boldsymbol{\Omega}^{k-p-2}\left(\phi_p,\phi_{p-1},\ldots,\phi_2,\phi_1\right)+\\ &\quad\;\beta_2\boldsymbol{\beta}'\boldsymbol{\Omega}^{k-p-2}\left(\phi_{p-1},\phi_{p-2},\ldots,\phi_1,\phi_0\right)+\\ &\quad\;\ldots+\beta_p\boldsymbol{\beta}'\boldsymbol{\Omega}^{k-p-2}\left(\phi_1,\phi_0,0,\ldots,0\right)\\ &= \beta_1\phi_{k-1}+\beta_2\phi_{k-2}+...+\beta_p\phi_{k-p}\\ &= \textstyle\sum_{j=1}^{p}\beta_j\phi_{k-j}.\end{aligned}\tag{34}$$

Let us consider the case $k=p+1$. Applying the definition of Ψ_i in equation (6) to Ψ_{i-p-1} in equation (32), we get

$$\begin{aligned}\Psi_i = \;& \textstyle\prod_{j=0}^{p+1}e^{\omega\phi_j}\prod_{j=0}^{p+1}e^{\phi_j\boldsymbol{\alpha}'\mathbf{g}_{i-j}}\cdot\\ &\Psi_{i-p-2}^{\sum_{j=1}^{p}\beta_j\phi_{p-j+2}}\cdot\Psi_{i-p-3}^{\sum_{j=2}^{p}\beta_j\phi_{p-j+3}}\cdot\ldots\cdot\Psi_{i-p-p}^{\beta_{p-1}\phi_{p+1}+\beta_p\phi_p}\cdot\Psi_{i-p-p-1}^{\beta_p\phi_{p+1}}.\end{aligned}\tag{35}$$

For notational simplicity again, let us define the parameters

$$\begin{aligned}
&\xi_{k+1} = \phi_{k+1} \\
&\xi_{k+2} = \beta_2\phi_k + ... + \beta_p\phi_{k-p+2} \\
&\xi_{k+3} = \beta_3\phi_k + ... + \beta_p\phi_{k-p+3} \\
&\cdots \\
&\xi_{k+p} = \beta_p\phi_k,
\end{aligned} \tag{36}$$

which enable us to write equation (35) as

$$\Psi_i = \prod_{j=0}^{p+1} \exp\left(\omega\phi_j\right) \cdot \prod_{j=0}^{p+1} \exp\left(\phi_j \boldsymbol{\alpha}'\mathbf{g}_{i-j}\right) \cdot \prod_{j=1}^{p} \Psi_{i-j-p-1}^{\xi_{p+1+j}}. \tag{37}$$

Let us consider now the case $k = m > p + 1$. By recursively applying the definition of Ψ_i in equation (6) to $\Psi_{i-p-1}, \cdots, \Psi_{i-m+1}$, Ψ_{i-m}, and substituting with the ξ_j's we can write

$$\Psi_i = \prod_{j=0}^{m} \exp\left(\omega\phi_j\right) \cdot \prod_{j=0}^{m} \exp\left(\phi_j \boldsymbol{\alpha}'\mathbf{g}_{i-j}\right) \cdot \prod_{j=1}^{p} \Psi_{i-j-m}^{\xi_{m+j}}. \tag{38}$$

So, if $k > p$, we can use the following general form to express Ψ_i:

$$\Psi_i = \prod_{j=0}^{k} \exp\left(\omega\phi_j\right) \cdot \prod_{j=0}^{k} \exp\left(\phi_j \boldsymbol{\alpha}'\mathbf{g}_{i-j}\right) \cdot \prod_{j=1}^{p} \Psi_{i-j-k}^{\xi_{k+j}}. \tag{39}$$

In order to compute the first unconditional moment of x_i, we can multiply equation (39) by ϵ_i and take expectations on both sides, which yields:

$$E(x_i) = \mu_1 \exp\left(\omega \sum_{j=0}^{k} \phi_j\right) \cdot E\left[\prod_{j=0}^{k} \exp\left(\phi_j \boldsymbol{\alpha}'\mathbf{g}_{i-j}\right) \cdot \prod_{j=1}^{p} \Psi_{i-j-k}^{\xi_{k+j}}\right]. \tag{40}$$

As $k \to \infty$, noting that $\lim_{k\to\infty} \xi_{k+i} = 0$, if and only if $\lambda(\boldsymbol{\Omega}) < 1$, and that the ϵ_i's are *iid*, we obtain

$$\begin{aligned}
E(x_i) &= \mu_1 \exp\left(\omega \textstyle\sum_{j=0}^{\infty} \phi_j\right) \cdot E\left[\textstyle\prod_{j=0}^{\infty} \exp\left(\phi_j \boldsymbol{\alpha}'\mathbf{g}_{i-j}\right)\right] = \\
&= \mu_1 \exp\left(\omega \textstyle\sum_{j=0}^{\infty} \phi_j\right) \cdot [E(\exp\left(\alpha_1\phi_0 g(\epsilon_{i-1})\right) \cdot \\
&\quad E\left(\exp\left((\alpha_1\phi_1 + \alpha_2\phi_0)g(\epsilon_{i-2})\right)\right) \cdot ... \cdot E\left(\exp\left(\textstyle\sum_{j=1}^{p} \alpha_j\phi_{s-j} g(\epsilon_{i-j})\right)\right) \cdot ...].
\end{aligned} \tag{41}$$

If we define θ_j, $j \geq 1$ as the coefficients of $g(\epsilon_{i-j})$ in equation (41), we can see that equation (13) holds and that the first moment of x_i can be written as

$$E(x_i) = \mu_1 \exp\left(\omega \sum_{j=0}^{\infty} \phi_j\right) \cdot \prod_{j=0}^{\infty} E\left(\exp\left(\theta_j g(\epsilon_{i-j})\right)\right). \tag{42}$$

In order to complete the proof, we must show that, from equations (9) and (10) $\sum_{j=0}^{\infty} \phi_j = (1 - \sum_{j=1}^{p} \beta_j)^{-1}$ if and only if $\lambda(\mathbf{\Omega}) < 1$. In fact, from equation (10) it follows that

$$\begin{aligned} \textstyle\sum_{j=0}^{\infty} \phi_j &= \textstyle\sum_{j=0}^{p} \phi_j + \sum_{j=p+1}^{\infty} \phi_j = \sum_{j=0}^{p} \phi_j + \boldsymbol{\beta}' \sum_{j=p+1}^{\infty} \mathbf{\Omega}^{j-p-1} \boldsymbol{\phi} = \\ &= \textstyle\sum_{j=0}^{p} \phi_j + \boldsymbol{\beta}'(\mathbf{I} - \mathbf{\Omega})^{-1} \boldsymbol{\phi} = (1 - \sum_{j=1}^{p} \beta_j)^{-1}, \end{aligned} \tag{43}$$

if and only if $\lambda(\mathbf{\Omega}) < 1$, since

$$(\mathbf{I} - \mathbf{\Omega})^{-1} = (1 - \sum_{j=1}^{p} \beta_j)^{-1} \begin{bmatrix} 1 & \sum_{j=2}^{p} \beta_j & \sum_{j=3}^{p} \beta_j & \dots & \beta_{p-1} & \beta_p \\ 1 & 1 - \beta_1 & \sum_{j=3}^{p} \beta_j & \dots & \dots & \dots \\ 1 & 1 - \beta_1 & 1 - \beta_1 - \beta_2 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \beta_{p-1} + \beta_p & \beta_p \\ 1 & 1 - \beta_1 & 1 - \beta_1 - \beta_2 & \dots & 1 - \sum_{j=1}^{p-2} \beta_j & \beta_p \\ 1 & 1 - \beta_1 & 1 - \beta_1 - \beta_2 & \dots & 1 - \sum_{j=1}^{p-2} \beta_j & 1 - \sum_{j=1}^{p-1} \beta_j \end{bmatrix}. \tag{44}$$

As the proof was given for $m = 1$, it must be noted that the same results for $m > 1$ can be derived by raising both sides of equation (40) to the power m. ■

Proof of Corollary 1

In the Log-ACD (1,1), $\beta_s = \begin{cases} \beta & s = 1 \\ 0 & 1 < s \leq p \end{cases}$ and $\alpha_s = \begin{cases} \alpha & s = 1 \\ 0 & 1 < s \leq p \end{cases}$.
Then $\lambda(\mathbf{\Omega}) = |\beta|$.
Furthermore, equation (14) implies that in equation (34)

$$\phi_k = \beta \phi_{k-1} = \beta^2 \phi_{k-2} = \cdots = \beta^{k-1} \phi_1 = \beta^k. \tag{45}$$

Therefore, in equation (13), θ_s reduces to

$$\theta_s = \alpha \phi_{s-1} = \alpha \beta^{s-1}. \tag{46}$$

■

Proof of Corollary 2

Equation (16) follows directly from equation (12). Since $E\left(y^2\right) \geq \left(E\left(y\right)\right)^2$, defining y as $\exp[\alpha\beta^{j-1}g(\epsilon_i)]$, we see that each term of the infinite product in equation (16) is not smaller than 1, and equal to 1 if $\alpha = 0$. This implies that $\delta_x \geq \delta$. ■

Proof of Theorem 2

For notational simplicity, let us define the following parameters

$$\begin{array}{l}
\beta^*_{1n} = \phi_n + 1 \quad \text{for } n \geq 1, \\
\beta^*_{jn} = \sum_{h=1}^{n} \beta_{h+j-1}\phi_{n-h} \quad \text{for } 1 \leq n \leq p-j+1 \text{ and } 2 \leq j \leq p-1, \\
\beta^*_{jn} = \sum_{h=j}^{p} \beta_{h}\phi_{n+j-1-h} \quad \text{for } p-j+2 \leq n \leq p \text{ and } 2 \leq j \leq p-1, \\
\beta^*_{pn} = \beta_p\phi_{n+1} \quad \text{for } 1 \leq n \leq p, \\
\beta^*_{jn} = \sum_{h=1}^{p+1-j} \beta_{j+h-1}\phi_{n-h} \quad \text{for } n \geq p \text{ and } 2 \leq j \leq p,
\end{array} \tag{47}$$

and show how they are determined.

We can start by considering the product $(\Psi_i\Psi_{i-n})$ for $1 \leq n \leq p$. If we apply equation (6) to Ψ_i in the product and make use of the results of the first part of the proof of Theorem 1, we obtain

$$\Psi_i\Psi_{i-n} = \prod_{j=0}^{n-1} \exp\left(\omega\phi_j\boldsymbol{\alpha}'\mathbf{g}_{i-j}\right) \cdot \prod_{h=1}^{p-n+1} \Psi_{i-j-n+1}^{\sum_{j=1}^{n}\beta_{h+j-1}\phi_{n-j}} \cdot \prod_{h=1}^{n-1} \Psi_{i-p-h}^{\sum_{j=1}^{h}\beta_{p-h+j}\phi_{n-j}} \Psi_{i-n}. \tag{48}$$

If we suppose that $h = 1$ in the second product term of equation (48) and multiply by Ψ_{i-n}, it takes the form

$$\Psi_{i-n}^{\sum_{j=1}^{n}\beta_j\phi_{n-j}+1}, \tag{49}$$

which implies

$$\beta^*_{1n} = \sum_{j=1}^{n} \beta_j\phi_{n-j} + 1 = \phi_n + 1 \qquad \text{for } 1 \leq n \leq p, \tag{50}$$

which shows how the first expression of equation (47) is determined.

If we suppose that $h = 1$ in the third product term of equation (48), it yields $\Psi_{i-p-1}^{\beta_p\phi_{n-1}}$. So,

$$\beta^*_{pn} = \beta_p\phi_{n-1} \quad \text{for } 1 \leq n \leq p. \tag{51}$$

This shows how the fourth expression of equation (47) is determined.

Next, we consider the remaining cases defined by $h = 2, ..., p-n+1$ in the second product term and $h = 2, ..., n-1$ in the third. Thus

$$\begin{array}{ll} \Psi_{i-h-n+1}^{\sum_{j=1}^{n}\beta_{h+j}\phi_{n-j}} & \text{for } 2 \leq h \leq p-n+1 \\ \Psi_{i-p-h}^{\sum_{j=1}^{h}\beta_{p-h}\phi_{n-j}} & \text{for } 2 \leq h \leq n-1. \end{array} \tag{52}$$

Equation (52) indicates that β^*_{jn}, $j = 2, ..., p-n+1$, can be defined by setting $h = 2, ..., p-n+1$ in the first expression of equation (52) and β^*_{jn}, $p-n-2 \leq j \leq p-1$, can be defined by setting $h = n-1, ..., 2$ in the second. Analogously, for $2 \leq j \leq p-1$,

$$\beta^*_{jn} = \left\{ \begin{array}{ll} \sum_{h=1}^{n}\beta_{h+j-1}\phi_{n-h} & 1 \leq n \leq p-j+1 \\ \sum_{h=1}^{p}\beta_{h}\phi_{n+j-h-1} & p-j+2 \leq n \leq p. \end{array} \right. \tag{53}$$

Thus the second and the third expressions of equation (47) are derived.
If we finally consider the case of $n > p$. If we set $k = n-1$ in equations (30) and (31), we obtain a corresponding representation of $(\Psi_i\Psi_{i-n})$ which reads:

$$\Psi_i\Psi_{i-n} = \prod_{j=0}^{n-1}\exp\left(\omega\phi_j\boldsymbol{\alpha}'\mathbf{g}_{i-j}\right)\cdot\prod_{j=1}^{p}\Psi_{i-j-n+1}^{\xi_{n+j-1}}\Psi_{i-n}, \tag{54}$$

where

$$\begin{array}{l} \xi_n = \phi_n = \beta^*_{1n}, \\ \xi_{n+1} = \beta_2\phi_{n-1} + ... + \beta_p\phi_{n-p+1} = \beta^*_{2n}, \\ \xi_{n+2} = \beta_3\phi_{n-1} + ... + \beta_p\phi_{n-p+2} = \beta^*_{2n}, \\ ... \\ \xi_{n+p-1} = \beta_p\phi_{n-1} = \beta^*_{pn}. \end{array} \tag{55}$$

This shows how the first expression for $n \geq p+1$ and the fifth for $n > p$ of equation (47) are determined. If now we substitute β_j with β^*_{jn} in equation (34), and suppose that

$$\begin{array}{l} \phi^*_{0n} = 1, \\ \phi^*_{1n} = \beta^*_1, \\ \phi^*_{kn} = \sum_{j=1}^{j-1}\beta_j\phi^*_{k-1,n} + \beta^*_{kn} \quad j = 2, \cdots, p, \text{ and} \\ \phi^*_k = \boldsymbol{\beta}'\boldsymbol{\Gamma}^{\mathbf{k-p-1}}\boldsymbol{\phi}^* \quad j > p, \end{array} \tag{56}$$

we obtain an analogous expression for the parameter ϕ^*_{jn}.
Let us then define the following parameters, which will be useful in the remainder of the proof:

$$\theta^*_{hn} = \left\{ \begin{array}{ll} \sum_{j=1}^{h}\alpha_j\phi^*_{h+1-j,n} & h = 1, ..., p \\ \sum_{j=1}^{p}\alpha_j\phi^*_{h+1-j,n} & h > p \end{array} \right. . \tag{57}$$

We now take the expected value of $x_i x_{i-n}$, and we obtain the following expression:

$$\begin{aligned} E(x_i x_{i-n}) &= E(\epsilon_i \epsilon_{i-n} \Psi_i \Psi_{i-n}) = \\ &= E\left(\epsilon_i \epsilon_{i-n} \prod_{j=0}^{n-1} \exp\left(\omega \phi_j\right) \cdot \prod_{j=0}^{n-1} \exp\left(\phi_j \boldsymbol{\alpha}' \mathbf{g}_{i-j}\right) \cdot \prod_{j=1}^{p} \Psi_{i-j-n+1}^{\beta^*_{jn}}\right). \end{aligned} \tag{58}$$

If $n \geq p+1$ we can write equation (58) as

$$\begin{aligned} E(x_i x_{i-n}) &= E(\epsilon_i \epsilon_{i-n} \Psi_i \Psi_{i-n}) \\ &= \mu \prod_{j=0}^{n-1} \exp\left(\omega \phi_j\right) E[\epsilon_{i-n} \prod_{j=1}^{n} \exp\left(\phi_j g(\epsilon_{i-j})\right) \cdot \\ &\quad \cdot \prod_{j=1}^{p-1} \exp\left(\left(\sum_{j=1}^{p-h} \phi_{n-j} \alpha_{h+j}\right) g(\epsilon_{i-n-h})\right) \cdot \prod_{j=1}^{p} \Psi_{i-j-n+1}^{\beta^*_{jn}}]. \end{aligned} \tag{59}$$

If we apply the result in equation (39) to the last two products of the right hand side of equation (59) and let $k \to \infty$, we obtain

$$\begin{aligned} & E\left[\prod_{h=1}^{p-1} \exp\left(\left(\sum_{j=1}^{p-h} \phi_{n-j} \alpha_{h+j}\right) g(\epsilon_{i-n-j})\right)\right] \cdot \left(\prod_{j=1}^{p} \Psi_{i-j-n+1}^{\beta^*_{jn}}\right) = \\ &\quad = \left(\prod_{j=1}^{\infty} \exp\left(\omega \phi^*_{jn}\right)\right) \cdot \\ &\qquad \cdot E\left[\prod_{h=1}^{P-1} \exp\left(\left(\sum_{j=1}^{p-h} \phi_{n-j} \alpha_{h+j}\right) g(\epsilon_{i-n-j})\right) \cdot \prod_{j=1}^{p} \exp\left(\phi^*_{jn} g(\epsilon_{i-n-j})\right)\right] \cdot \\ &\qquad \cdot E\left[\prod_{j=p}^{\infty} \exp\left(\phi^*_{jn} g(\epsilon_{i-n-j})\right)\right] \\ &\quad = \prod_{j=1}^{\infty} \exp\left(\omega \phi^*_{jn}\right) \cdot E\left[\prod_{h=1}^{P-1} \exp\left(\left(\sum_{j=1}^{p-h} \phi_{n-j} \alpha_{h+j} + \theta^*_{hn}\right) g(\epsilon_{i-n-j})\right)\right] \cdot \\ &\qquad \cdot E\left[\prod_{j=p}^{\infty} \exp\left(\phi^*_{jn} g(\epsilon_{i-n-j})\right)\right] \\ &\quad = \prod_{j=1}^{\infty} \exp\left(\omega \phi^*_{jn}\right) \cdot E\left[\prod_{j=p}^{\infty} \exp\left(\theta^*_{jn} g(\epsilon_i)\right)\right] \cdot \\ &\qquad \cdot E\left[\prod_{h=1}^{p-1} \exp\left(\left(\sum_{j=1}^{p-h} \phi_{n-j} \alpha_{h+j} + \theta^*_{hn}\right) g(\epsilon_{i-n-h})\right)\right]. \end{aligned} \tag{60}$$

Hence, we can rewrite equation (59) in the following form:

$$\begin{aligned} E(x_i x_{i-n}) = \quad & \mu E\left[\epsilon_{i-n} \exp\left(\theta_n g(\epsilon_{i-n})\right)\right] \cdot \left(\prod_{j=1}^{\infty} \exp\left(\omega \phi^*_{jn}\right)\right) \cdot \left(\prod_{j=1}^{n-1} \exp\left(\omega \phi_j\right)\right) \cdot \\ & \cdot E\left[\prod_{j=1}^{n-1} \exp\left(\theta_j g(\epsilon_i)\right)\right] \cdot E\left[\prod_{j=p}^{\infty} \exp\left(\theta^*_{jn} g(\epsilon_i)\right)\right] \cdot \\ & \cdot E\left[\prod_{h=1}^{p-1} \exp\left(\left(\sum_{j=1}^{p-h} \phi_{n-j} \alpha_{h+j} + \theta^*_{hn}\right) g(\epsilon_{i-n-h})\right)\right]. \end{aligned} \tag{61}$$

If $1 \leq n \leq p$ equation (58) reads

$$\begin{aligned} E(x_i x_{i-n}) &= E(\epsilon_i \epsilon_{i-n} \Psi_i \Psi_{i-n}) = \\ &= \mu \textstyle\prod_{j=0}^{n-1} \exp\left(\omega \phi_j\right) E[\epsilon_{i-n} \prod_{j=1}^{n} \exp\left(\phi_j g(\epsilon_{i-j})\right) \cdot \\ &\quad \cdot \textstyle\prod_{h=1}^{n} \exp\left(\left(\phi_{n-h} \sum_{j=1}^{p-h} \alpha_{h+j}\right) g(\epsilon_{i-n-j})\right) \cdot \prod_{j=1}^{p} \Psi_{i-j-n+1}^{\beta^*_{jn}}]. \end{aligned} \tag{62}$$

If again we apply the result in equation (39) to the last two products of the left hand side of equation (62) and let $k \to \infty$, we obtain

$$\begin{aligned} & E\left[\textstyle\prod_{h=1}^{n} \exp\left((\phi_{n-h} \sum_{j=1}^{p-h} \alpha_{h+j}) g(\epsilon_{i-n-j})\right)\right] \cdot (\prod_{j=1}^{p} \Psi_{i-j-n+1}^{\beta^*_{jn}}) = \\ & \quad = \textstyle\prod_{j=1}^{\infty} \exp\left(\omega \phi^*_{jn}\right) \cdot E\left[\prod_{h=1}^{p-n} \exp\left(\left(\sum_{j=1}^{n} \phi_{n-j} \alpha_{h+j} + \theta^*_{hn}\right) g(\epsilon_{i-n-j})\right)\right] \cdot \\ & \quad \cdot E\left[\textstyle\prod_{h=1}^{n-1} \exp\left((e^g \sum_{j=1}^{h} \phi_{n-j} \alpha_{p-h+j} + \theta^*_{p-h,n}) g(\epsilon_{i-n-j})\right)\right] \cdot \\ & \quad \cdot E\left[\textstyle\prod_{j=p}^{\infty} \exp\left(\theta^*_{jn} g(\epsilon_i)\right)\right]. \end{aligned} \tag{63}$$

Hence, we get the following expression for equation (62)

$$\begin{aligned} E(x_i x_{i-n}) = \; & \mu E\left[\epsilon_{i-n} \exp\left(\theta_n g(\epsilon_{i-n})\right)\right] \cdot \textstyle\prod_{j=1}^{\infty} \exp\left(\omega \phi^*_{jn}\right) \cdot \prod_{j=1}^{n-1} \exp\left(\omega \phi_j\right) \cdot \\ & \cdot E\left[\textstyle\prod_{j=1}^{n-1} \exp\left(\theta_j g(\epsilon_i)\right)\right] \cdot E\left[\prod_{j=p}^{\infty} \exp\left(\theta^*_{jn} g(\epsilon_i)\right)\right] \cdot \\ & \cdot E\left[\textstyle\prod_{h=1}^{p-n} \exp\left(\left(\sum_{j=1}^{n} \phi_{n-j} \alpha_{h+j} + \theta^*_{hn}\right) g(\epsilon_{i-n-j})\right)\right] \cdot \\ & \cdot E\left[\textstyle\prod_{h=1}^{n-1} \exp\left(\left(\sum_{j=1}^{h} \phi_{n-j} \alpha_{p-h+j} + \theta^*_{p-h,n}\right) g(\epsilon_{i-n-j})\right)\right]. \end{aligned} \tag{64}$$

Finally, to be able to simplify and derive equations (17)-(18), we need to show that, for any $n \geq 1$,

$$\prod_{j=0}^{n-1} \exp\left(\omega \phi_j\right) \prod_{j=1}^{\infty} \exp\left(\omega \phi^*_{jn}\right) = \exp\left(2\omega \left(1 - \sum_{j=1}^{p} \beta_j\right)^{-1}\right) \tag{65}$$

holds. To do so, we can first show that

$$\sum_{j=1}^{\infty} \phi^*_{jn} = \left(\sum_{j=1}^{p} \beta^*_{jn}\right) \left(1 - \sum_{j=1}^{p} \beta_j\right)^{-1}. \tag{66}$$

Let us consider

$$\begin{aligned} \textstyle\sum_{j=1}^{\infty} \phi^*_{jn} &= \textstyle\sum_{j=1}^{p} \phi^*_{jn} + \sum_{j=p+1}^{\infty} \phi^*_{jn} \\ &= \textstyle\sum_{j=1}^{p} \phi^*_{jn} + \boldsymbol{\beta}' \sum_{j=p+1}^{\infty} \boldsymbol{\Omega}^{j-p-1} \boldsymbol{\phi}^*_n \\ &= \textstyle\sum_{j=1}^{p} \phi^*_{jn} + \boldsymbol{\beta}' (\mathbf{I} - \boldsymbol{\Omega})^{-1} \boldsymbol{\phi}^*_n. \end{aligned} \tag{67}$$

Since $(\mathbf{I} - \boldsymbol{\Omega})^{-1}$ is known from equation (44), it is sufficient to consider the case $p = 2$.

Then equation (67) becomes,

$$\begin{aligned}\textstyle\sum_{j=1}^{\infty}\phi_{jn}^{*} &= \phi_{1n}^{*}+\phi_{2n}^{*}+\tfrac{1}{1-\beta_1-\beta_2}\begin{pmatrix}\beta_1 & \beta_2\end{pmatrix}\begin{pmatrix}1 & \beta_2\\ 1 & 1-\beta_1\end{pmatrix}\begin{pmatrix}\phi_{1n}^{*}\\ \phi_{2n}^{*}\end{pmatrix}=\\ &= \tfrac{\phi_{1n}^{*}+\phi_{2n}^{*}-\beta_1\phi_{1n}^{*}}{1-\beta_1-\beta_2}=\\ &= \tfrac{\beta_{1n}^{*}+\beta_{2n}^{*}}{1-\beta_1-\beta_2}.\end{aligned} \tag{68}$$

Next, we can show that equation (65) holds for any $n \geq 1$.
Let $n = 1$, then equation (65) has the form

$$\exp(\omega)\prod_{j=1}^{\infty}\exp\left(\omega\phi_{j1}^{*}\right)=\exp(\omega)\exp\left(\omega\frac{\beta_{11}^{*}+\beta_{21}^{*}}{1-\beta_1-\beta_2}\right)=\exp\left(\frac{2\omega}{1-\beta_1-\beta_2}\right). \tag{69}$$

Similarly, we can check for $n = 2$.
Assume now that equation (65) holds for $n = m > 2$, that is,

$$\prod_{j=0}^{m-1}\exp\left(\omega\phi_j\right)\prod_{j=1}^{\infty}\exp\left(\omega\phi_{jm}^{*}\right)=\exp\left(2\omega\left(1-\sum_{j=1}^{2}\beta_j\right)^{-1}\right), \tag{70}$$

we can show that it holds for $n = m + 1$. From equation (70) we have

$$\begin{aligned}\textstyle\sum_{j=0}^{m}\phi_j &\textstyle +\sum_{j=1}^{\infty}\phi_{j,m+1}^{*}\\ &\textstyle =\sum_{j=0}^{m-1}\phi_j+\phi_m+\frac{\beta_{1,m+1}^{*}+\beta_{2,m+1}^{*}}{1-\beta_1-\beta_2}\\ &\textstyle =\sum_{j=0}^{m-1}\phi_j+\frac{\beta_{1m}^{*}+\beta_{2m}^{*}}{1-\beta_1-\beta_2}+\left(\phi_m+\frac{\beta_{1,m+1}^{*}+\beta_{2,m+1}^{*}}{1-\beta_1-\beta_2}-\frac{\beta_{1m}^{*}+\beta_{2m}^{*}}{1-\beta_1-\beta_2}\right)\\ &\textstyle =2\omega(1-\sum_{j=1}^{p}\beta_j)^{-1}+\left(\phi_m+\frac{\beta_{1,m+1}^{*}+\beta_{2,m+1}^{*}}{1-\beta_1-\beta_2}-\frac{\beta_{1m}^{*}+\beta_{2m}^{*}}{1-\beta_1-\beta_2}\right).\end{aligned} \tag{71}$$

Now, the second term on the right-hand of equation (71) equals zero, because $\beta_{1,m+1}^{*} = \phi_{m+1} + 1$, $\beta_{2,m+1}^{*} = \beta_2\phi_m$ and $\phi_{m+1} = \beta_1\phi_m + \beta_2\phi_{m+1}$. Thus, equation (65) holds for any $n \geq 1$. ■

Proof of Corollary 3

As in Corollary 2, if $p = 1$, then $\lambda(\mathbf{\Omega}) = |\beta|$ and $\theta_s = \alpha\beta^{s-1}$. Hence $E(\epsilon_i e^{\theta_n g(\epsilon_i)})$ in equation (17) reduces to $E(\epsilon_i e^{\alpha\beta^{n-1}g(\epsilon_i)})$, which is finite if $E(\epsilon_i e^{\alpha g(\epsilon_i)}) < \infty$. For the same reason $E(e^{\theta_j g(\epsilon_i)})$ reduces to $E(e^{\alpha\beta_{j-1}g(\epsilon_i)})$, which is finite if $E(e^{2\alpha g(\epsilon_i)}) < \infty$. This last condition also ensures the existence of the second moment of x_i. Then, as

$$\theta_{jn}^{*}=\begin{array}{ll}\alpha\phi_{1n}^{*}=\alpha(\phi_n+1)=\alpha\beta^{j-1}(\beta^n+1) & j=1\\ \alpha\phi_{j}^{*}=\alpha\beta^{j-1}\phi_{1n}^{*}=\alpha\beta^{j-1}(\beta^n+1) & j>1,\end{array} \tag{72}$$

the factor $\prod_{j=p}^{\infty} E\left[e^{\theta^*_{jn} g(\epsilon_i)}\right]$ reduces to $\prod_{j=1}^{\infty} E\left[e^{\alpha\beta^{j-1}(\beta^n+1)g(\epsilon_i)}\right]$, which is finite if $E\left[e^{2\alpha g(\epsilon_i)}\right] < \infty$, as $\lim_{j\to\infty}\beta^{j-1} = 0$. Noticing that the products of $M_{n,p}$ reduce to 1 if $p = 1$ completes the proof. ■

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