Preliminary version

THE MOMENTS OF FIRST-ORDER LOG-ACD MODELS

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March 10, 2001

Abstract

We provide existence conditions and analytical expressions of the moments of logarithmic autoregressive conditional duration (Log-ACD) models. We focus on the dispersion index and the autocorrelation function and compare them with those of ACD models (Engle and Russell 1998). Using duration data for several stocks traded on the New York Stock Exchange (NYSE), we compare the models in terms of their ability at fitting some stylized facts.

Keywords: Duration model, overdispersion, autocorrelation function, high frequency financial data.

JEL classification: C41

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This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsability is assumed by the authors.

1 Introduction

With the objective to model durations between events like trades and quote updates that occur randomly during the market hours on stock exchanges, Engle and Russell (1998) introduced the autoregressive conditional duration (ACD) model. This model combines elements from transition analysis and Engle's (1982) autoregressive conditional heteroskedasticity (ARCH) model. One motivation behind the ACD and the ARCH model appears similar: market events, like trades and quote arrivals, occur in clusters. The ACD model also permits to test some implications of market microstructure models (see O'Hara 1995 for a survey) through the introduction of conditioning information.

Following the contribution of Engle and Russell (1998), other duration models have been put forward. Bauwens and Giot (2000) introduced a logarithmic version of the ACD model, called the Log-ACD model, which is more convenient than the ACD model when conditioning variables are included in the model in order to test microstructure effects. The reason is that the ACD model practically requires to impose non-negativity restrictions on its parameters, whereas the Log-ACD model does not. As an alternative to the Weibull distribution used in the original ACD model, Grammig and Maurer (1999) introduced an ACD model based on the Burr distribution (which includes the Weibull as a particular case). Ghysels, Gouriéroux, and Jasiak (1997) proposed the stochastic volatility duration model, which accounts for stochastic volatility in the durations. Bauwens and Veredas (1999) put forward the stochastic conditional duration (SCD) model, which uses a stochastic volatility-type model instead of a GARCH-type model to model the durations.

Until the present contribution, one drawback of the Log-ACD model, with respect to the ACD and the SCD models, was that the unconditional moments implied by the model were not available analytically. Bauwens and Giot (2000) relied therefore on numerical simulations to compute the moments of several Log-ACD models, in particular their autocorrelation function (ACF) and dispersion index (i.e. the ratio of standard deviation to mean). This led them to conclude that Log-ACD models were able to fit the stylized facts of stock market durations 'as well' as ACD models. These facts are a rather slowly decreasing ACF that starts from a relatively low positive value, a consequence of the clustering of activity, and overdispersion. The latter implies that very small and very large durations occur in higher proportions than is compatible with an exponential distribution.

In this paper we thus provide analytical expressions for the unconditional moments and ACF for the models belonging to the Log-ACD class as defined in Bauwens and Giot (2000). The results of this paper are proved using the method that has been proposed by He, Terasvirta, and Malmsten (1999) for the moments of a family of exponential GARCH models. We also provide an empirical application in which we compute the unconditional moments and ACF for the ACD and Log-ACD models estimated on financial durations for several stocks traded on the New York Stock Exchange.

The paper is organized as follows. In Section 2, we define the class of Log-ACD models, we provide the conditions of existence and the general formula of the moments. In Section 3, we look at the properties of the dispersion index and the

ACF. In Section 4, we pursue the comparison using real data. Section 5 concludes. Proofs are relegated in an appendix.

2 Log-ACD models: definition and moments

We denote by x_i the duration between two events that happened at times t_{i-1} and t_i, i.e. $x_i = t_i - t_{i-1}$. We assume that the stochastic process $\{x_i\}$ generating the durations is doubly infinite (i goes from $-\infty$ to $+\infty$).

A Log-ACD model specifies the observed duration as the mixing process

$$x_i = e^{\psi_i}\epsilon_i, \tag{1}$$

where the ϵ_i are IID, with

$$\mathrm{E}\,\epsilon_i = \mu \tag{2}$$

$$\mathrm{Var}\,\epsilon_i = \sigma^2, \tag{3}$$

so that $\mathrm{E}(x_i|\mathcal{H}_i) = \mu\exp(\psi_i)$, where $\mathcal{H}_i$ denotes the information set available at time t_{i-1} (the beginning of the duration x_i), which includes the past durations.

The important assumption, which is the same as for ACD models (see Engle and Russell 1998), is that the dependence in the duration process can be subsumed in their conditional expectations $\mathrm{E}(x_i|\mathcal{H}_i)$, in such a way that $x_i/\mathrm{E}(x_i|\mathcal{H}_i)$ is independent and identically distributed. For further reference, we define

$$\Psi_i = \exp(\psi_i). \tag{4}$$

To introduce a dependence in the process, which can produce a clustering of durations, ψ_i is specified as an autoregressive equation,[1] which in its most general form (in this paper) is written as

$$\psi_i = \omega + \alpha g(\epsilon_{i-1}) + \beta\psi_{i-1}. \tag{5}$$

Two choices of the function $g(\epsilon_{i-1})$ are $\ln\epsilon_{i-1}$ or ϵ_{i-1}. The first one corresponds to the Log-ACD$_1$ model, in which (5) becomes

$$\begin{aligned}\psi_i &= \omega + \alpha\ln\epsilon_{i-1} + \beta\psi_{i-1} \\ &= \omega + \alpha\ln x_{i-1} + (\beta-\alpha)\psi_{i-1},\end{aligned} \tag{6}$$

and the second one to the Log-ACD$_2$ model, for which

$$\begin{aligned}\psi_i &= \omega + \alpha\epsilon_{i-1} + \beta\psi_{i-1} \\ &= \omega + \alpha(x_{i-1}/\exp\psi_{i-1}) + \beta\psi_{i-1}.\end{aligned} \tag{7}$$

Several choices are available for the distribution of ϵ_i: exponential, gamma, generalised gamma, Weibull, Burr, lognormal, Pareto... , in principle any distribution with positive support. The choice of a particular distribution should be

[1] In this paper we consider a first-order process, i.e. with one lag of ϵ_i and one lag of ψ_i, so that we are dealing with Log-ACD(1,1) models. Our motivation is to simplify the exposition, but the results of this paper can be generalized to Log-ACD(p,q) models.

guided by the desire of having a 'correct' specification, and perhaps by its convenience for estimation. Among the distributions cited above, the Burr and the Pareto do not necessarily have finite moments, so that restrictions on their parameters must be imposed to ensure that the variance and the mean exist. The Burr family includes the Weibull (and the exponential) as a particular case, while the generalized gamma includes the gamma and the Weibull (hence the exponential). All these distributions depend on a scale parameter that we normalize at 1. For distributions which are indexed by a single shape parameter (gamma, Weibull), μ and σ^2 are linked through that parameter. For the exponential distribution, the parameter is fixed to 1 so that $\mu = \sigma^2 = 1$. The Burr and generalized gamma depend on two shape parameters, and are therefore more flexible, in particular they can have a non-monotonous hazard function. The moments of a Log-ACD model depend of course on the moments of ϵ_i.

Theorem 1 *Assume that* $\mathrm{E}\exp[m\alpha g(\epsilon_i)]$ *and* $\mathrm{E}\epsilon_i^m$ *exist for an arbitrary* m. *For the Log-ACD process defined by (1)-(5), the condition* $|\beta| < 1$ *is necessary and sufficient for the existence of* $\mathrm{E}x_i^m$. *Under this condition,*

$$\begin{aligned} \mathrm{E}x_i^m &= \mathrm{E}\epsilon_i^m \, \mathrm{E}\Psi_i^m \\ &= \mathrm{E}\epsilon_i^m \exp\left(\frac{m\omega}{1-\beta}\right) \prod_{j=1}^{\infty} \mathrm{E}\exp[m\alpha\beta^{j-1} g(\epsilon_i)]. \end{aligned} \tag{8}$$

If α and β are both positive (as is practically always the case), computing the moment given in the previous theorem requires to know $\mathrm{E}(\epsilon^p)$ for any positive p (not necessarily integer) in the Log-ACD$_1$ case, and $\mathrm{E}\exp(p\epsilon)$ in the Log-ACD$_2$ case. The (non-integer) moments $\mathrm{E}(\epsilon^p)$ are available for the generalized gamma and Burr distributions, and all their particular cases. The moment generating function which provides $\mathrm{E}\exp(p\epsilon)$ is only available for the gamma distribution (including the exponential), not for all the other distributions encompassed by the generalized gamma and the Burr classes (such as the Weibull). For the latter cases, one could compute $\mathrm{E}\exp(p\epsilon)$ for all the required values of p by a deterministic rule of numerical integration (such as Simpson's rule). As an alternative, one could proceed also by simulation of a very large sample of the process to estimate the moments.

For the practical computation of (8), the infinite product that appears in the moment expression can be truncated after a sufficiently large number of terms since β^j tends to 0.[2] For example, if we use an exponential distribution, $\mathrm{E}(\epsilon^{\alpha\beta^j}) = \Gamma(1+\alpha\beta^j)$ and $\mathrm{E}\exp(\epsilon\alpha\beta^j) = 1/(1-\alpha\beta^j)$, so that both expectations tend to 1 when j tends to infinity.

3 Dispersion and autocorrelation function

Durations between stock market events are often characterized by overdispersion, meaning that the standard deviation of the data is larger the their mean (see Section 5). Another important stylized fact is the shape of the ACF, which usually

[2]In practice, we found that for first and second-order moments, truncation after 1000 terms was more than sufficient to get a high accuracy.

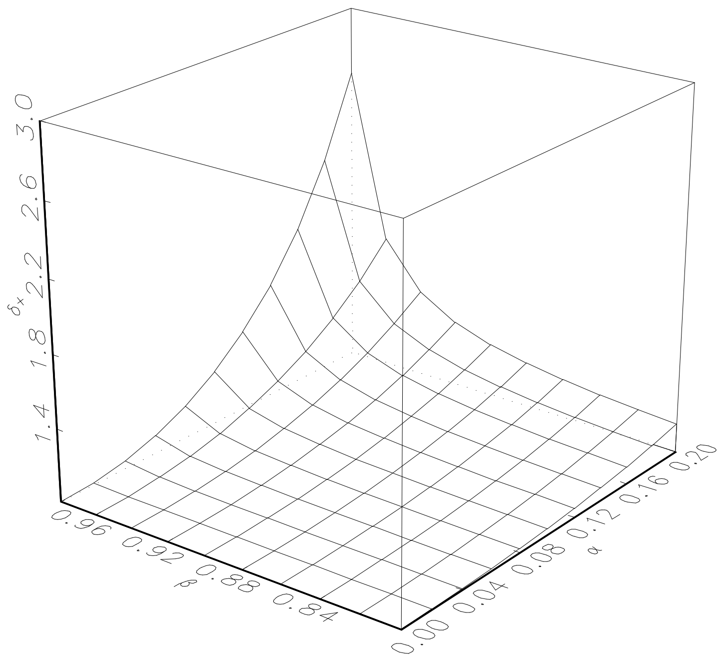

Figure 1: Dispersion Index of Log-ACD_2 Model (Exponential Distribution)

decreases slowly from a relatively low positive first-order autocorrelation. It is therefore essential that Log-ACD models be able to fit such stylized facts, for some parameter values.

Let us measure the degree of dispersion of the random variable x by the variation coefficient, or its square root (= standard deviation/mean) that we call the dispersion index and we denote by δ_x. This ratio is larger than 1 in the case of overdispersion. This measure is a direct by-product of Theorem 1, and we have the following result:

Corollary 1 *For the Log-ACD process defined by (1)-(5), assume that* $|\beta| < 1$ *and* $\mathrm{E}\exp[2\alpha g(\epsilon_i)] < \infty$. *Then*

$$1 + \delta_x^2 = (1 + \delta^2) \prod_{j=1}^{\infty} \frac{\mathrm{E}\exp[2\alpha\beta^{j-1} g(\epsilon_i)]}{\{\mathrm{E}\exp[\alpha\beta^{j-1} g(\epsilon_i)]\}^2} \tag{9}$$

(where $\delta = \sigma/\mu$ *is the dispersion index of* ϵ_i*), and* $\delta_x \geq \delta$ *(with equality if* $\alpha = 0$*).*

The dispersion index of x_i cannot be smaller than that of ϵ_i. Thus, it suffices that ϵ_i be equidispersed ($\delta = 1$) for x_i to be overdispersed, as long as $\alpha \neq 0$. Figure 1 illustrates the variation of δ_x as a function of α (from 0 to 0.2) and β (from 0.8 to 0.98) when ϵ_i is exponential (so that $\delta = 1$) and the model is a Log-ACD_2. For the Log-ACD_1 model, the figure is almost identical, the difference

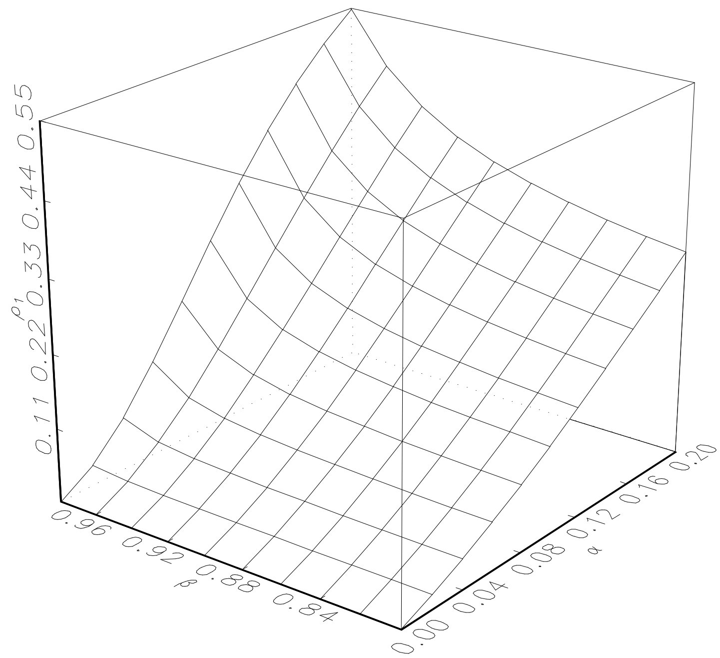

Figure 2: First Autocorrelation of Log-ACD$_2$ Model (Exponential Distribution)

being that the values of δ_x are slighlty larger (except for the combinations $\alpha = 0.2$ and $0.8 < \beta < 0.94$).

The next theorem provides the autocorrelation function.

Theorem 2 *For the Log-ACD process defined by (1)-(5), assume that* $|\beta| < 1$, $\mathrm{E}\exp[2\alpha g(\epsilon_i)] < \infty$, *and* $\mathrm{E}\{\epsilon_i \exp[\alpha g(\epsilon_i)]\} < \infty$. *Then*

$$\rho_n = \frac{\mu \mathrm{E}[\epsilon_i e^{\alpha\beta^{n-1} g(\epsilon_i)}] \prod_{j=1}^{n-1} \mathrm{E}[e^{\alpha\beta^{j-1} g(\epsilon_i)}] \prod_{j=1}^{\infty} \mathrm{E}[e^{\alpha(1+\beta^n)\beta^{j-1} g(\epsilon_i)}] - \mu^2 \prod_{j=1}^{\infty} [\mathrm{E}e^{\alpha\beta^{j-1} g(\epsilon_i)}]^2}{(\sigma^2 + \mu^2) \prod_{j=1}^{\infty} \mathrm{E}\exp[2\alpha\beta^{j-1} g(\epsilon_i)] - \mu^2 \prod_{j=1}^{\infty} [\mathrm{E}e^{\alpha\beta^{j-1} g(\epsilon_i)}]^2} \tag{10}$$

for $n \geq 1$.

Figure 2 illustrates the variation of ρ_1 in the same setup as in Figure 1 (again with ϵ_i exponential, so that $\mu = \sigma = 1$). For the Log-ACD$_1$ model, the figure is almost the same, but the value od ρ_1 in the Log-ACD$_1$ case is larger than in the Log-ACD$_2$ whenever $\alpha < 0.08$ and smaller whenever $\alpha > 0.14$, while in the intermediate cases it is larger when $\beta > 0.9$ (approximately). However, the differences are never larger than 0.04. These features are not necessarily the same for other distributions of ϵ_i.

Table 1: Properties of the ACF of Log-ACD$_2$ Model
(Exponential Distribution)

	β					
	0.800	0.840	0.880	0.920	0.960	0.980
α						
0.04	0.045	0.046	0.048	0.051	0.061	0.079
	0.793	0.834	0.875	0.917	0.958	0.979
	24	30	38	53	93	162
0.08	0.100	0.104	0.111	0.123	0.157	0.213
	0.785	0.827	0.869	0.912	0.955	0.976
	28	34	44	63	115	212
0.12	0.164	0.172	0.185	0.209	0.270	0.353
	0.775	0.818	0.861	0.905	0.950	0.972
	30	37	48	69	129	244
0.16	0.234	0.247	0.267	0.302	0.380	0.467
	0.763	0.807	0.851	0.896	0.942	0.965
	31	39	51	74	140	268
0.20	0.306	0.324	0.350	0.392	0.474	0.548
	0.749	0.793	0.838	0.885	0.932	0.955
	33	41	53	78	149	288

In each cell, from top to bottom, one finds the value of ρ_1, the ratio ρ_2/ρ_1, and the value of n from which $\rho_{n+1}/\rho_n = \beta$ to four decimal places.

Another feature of interest is the rate of decrease of the ACF. We assume that $0 < \beta < 1$ to avoid oscillation of the signs of the autocorrelations. If we consider for example the Log-ACD$_1$ model, it can be written as the ARMA(1,1) process

$$\ln x_i = \omega + \beta \ln x_{i-1} + u_i - (\beta - \alpha) u_{i-1} \tag{11}$$

where $u_i = \ln x_i - \psi_i$ is a martingale difference. The autocorrelations of the logarithm of the duration therefore decrease geometrically at the rate β. However, by computing (10) for many parameter configurations, we found that the autocorrelations of the duration decrease at the above rate only after a 'large' lag. For small lags, the rate of decrease is less than β, although not much. The following table provides, for several parameter values, the value of ρ_1, the ratio ρ_2/ρ_1, and the value of n from which the rate of decrease is equal to β (for a precision to 4 decimal digits). The results in the table show that i) for fixed β, the larger α, the larger the difference $\beta - \rho_2/\rho_1$ and the value of n, and ii) for fixed α, the larger β, the smaller the difference $\beta - \rho_2/\rho_1$ but the larger the value of n.

4 Fitting the stylized facts

In this section, we consider an application to financial durations for stocks traded on the NYSE. As reviewed by Giot (2000), while durations can simply be defined

Table 2: Point Estimates

	ACD	Log-ACD$_1$	Log-ACD$_2$
		Boeing	
ω	0.045	0.057	-0.103
α	0.126	0.101	0.100
β	0.832	0.857	0.950
		IBM	
ω	0.029	0.035	-0.073
α	0.084	0.084	0.071
β	0.888	0.888	0.968

Maximum likelihood estimates of the parameters of the ACD, Log-ACD$_1$ and Log-ACD$_2$ models, assuming an exponential distribution for ϵ_i. Data: price durations (at \$1/8) of Boeing and IBM.

as the time elapsed between two market events, by judiciously defining the notion of market event one can highlight several important features of intraday market activity. For example, a duration between two quotes is a quote duration and the modelling of these using ACD or Log-ACD type models can quantify the notion of quoting activity, i.e. the rate at which the specialist post quotes.

An important extension related to the quote process is the notion of price duration. Because price durations are defined as the minimum time for the stock price to escape from a given price interval, it can be shown (see Giot, 2000) that there is a relationship between the volatility of the price process and the conditional hazard of the ACD or Log-ACD model. Thus this provides a strong motivation for the use of such high frequency duration models in the modelling of intraday volatility.

We focus on two stocks traded on the NYSE, Boeing and IBM. Using data from the Trades and Quotes database (TAQ database), we computed price durations for the two stocks with the price threshold level c equal to \$1/8. These durations are the times needed for the price process (the price is computed on the mid point of the bid and ask prices) to change by at least \$1/8. As detailed extensively in the recent literature (see Engle and Russell, 1998, or Bauwens and Giot, 2000), the price durations feature a strong intraday seasonality (due to characteristics of the stock exchange) which is removed from the data prior to estimating the duration models. If the raw observed price duration is noted X_i, the deseasonalized price duration x_i is computed as $x_i = X_i/\phi(t_i)$, where $\phi(t_i)$ is the time-of-day effect. The latter is computed as the expected duration conditioned on time-of-day and on the day of the week, where the expectation is computed by averaging the durations over thirty minutes intervals for each day of the week.

For each set of deseasonalized price durations, we estimated the ACD, Log-ACD$_1$ and Log-ACD$_2$ models with an exponential distribution for the error term. The ML estimates, reported in Table 2, take values similar to what is usually

Table 3: Moments Implied by Point Estimates

	ACD	Log-ACD$_1$	Log-ACD$_2$	Data
		Boeing		
Mean	1.059	1.021	0.990	1.007
Variance	1.658	1.166	1.226	1.895
dispersion index	1.215	1.057	1.118	1.367
		IBM		
Mean	1.028	0.952	0.998	1.002
Variance	1.358	0.982	1.182	1.426
dispersion index	1.133	1.041	1.089	1.192

Unconditional moments for the ACD, Log-ACD$_1$ and Log-ACD$_2$ models computed by applying the analytical expressions with the estimated parameters reported in Table 2. The last column gives the empirical moments computed from the data.

found for such data (see Bauwens and Giot, 2000).[3] We then used the estimated parameters to compute the analytical expressions for the unconditional moments and ACF. The results based on the analytical expressions can then be compared to the empirical (unconditional) moments and ACF.

Table 3 reports the first two empirical moments and the moments resulting from the analytical expressions for the three models. Broadly speaking, the unconditional moments computed from the analytical formulas are in line with the empirical ones. Nevertheless, for the two stocks considered in this illustration, the variance and the dispersion index implied match better the empirical values for the ACD model than for the Log-ACD models. It remains to be checked if these differences would remain if the distribution of the error term (ϵ_i) is more flexible than the exponential one (results to be added in a next version of the paper).

Figures 3 and 4 give the ACF of the data and those implied by the theoretical formulas for the three models. A graphical comparison of the four curves in both figures indicates that the Log-ACD$_2$ model provides the best fit to the empirical ACF. In particular, the Log-ACD$_2$ model provides the best match for the first autocorrelation coefficient.

5 Conclusion

We provide analytical formulas for the moments of Log-ACD(1,1) models. Work is in progress to generalize the results to Log-ACD(p,q) models. The formulas are more complex than for the ACD model, since the ACD model is actually a linear process (ARMA) whereas the Log-ACD is non-linear. We have shown that the shape of the autocorrelation function of Log-ACD models is different from the shape of the ACF of the ACD model. The formulas can be used to check implied moments from parameter estimates, as in the illustration of this paper. They could also be used to select parameter values in order to match desired moments

[3] We do not report standard errors since they are not needed in the following discussion.

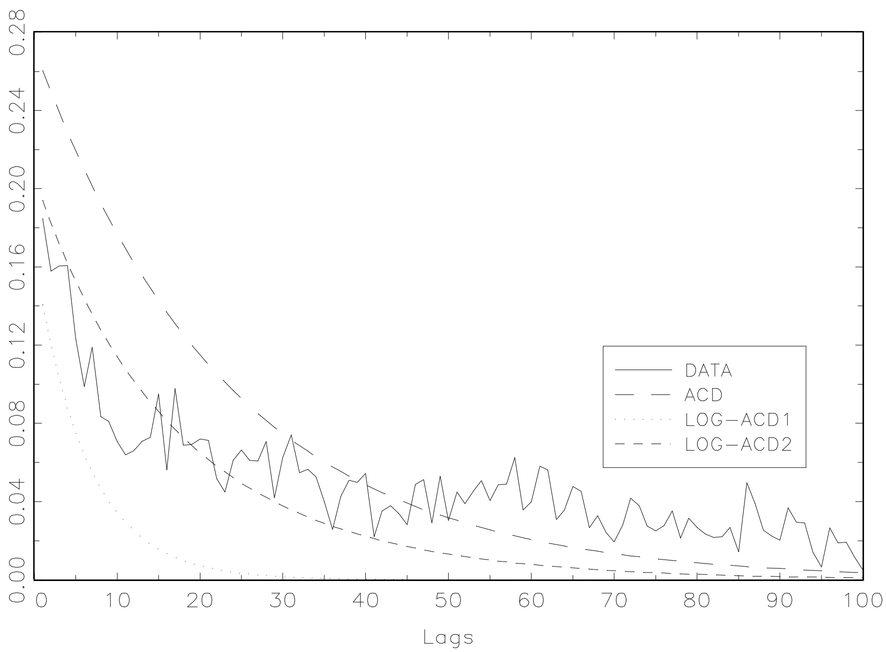

Figure 3: ACF for the ACD, Log-ACD$_1$, Log-ACD$_2$ models (using the analytical expressions computed for the estimated parameters) and empirical data (price duration at \$1/8 for Boeing).

(e.g. for designing a Monte Carlo experiment).

Appendix

Proof: [Theorem 1] From (4) and (5), we have

$$\Psi_i = \exp(\omega) \exp[\alpha g(\epsilon_{i-1})] \, \Psi_{i-1}^{\beta}. \tag{12}$$

By recursive substitutions on lagged Ψ_i, one obtains

$$\Psi_i = \exp\left(\omega \sum_{j=0}^{n-1} \beta^j\right) \exp\left[\alpha \sum_{j=1}^{n} \beta^{j-1} g(\epsilon_{i-j})\right] \Psi_{i-n}^{\beta^n}. \tag{13}$$

Raising both sides to the power m, multiplying by ϵ_i^m, and taking the unconditional expectation gives

$$\mathrm{E}(x_i^m) = \mathrm{E}(\epsilon_i^m) \exp\left(m\omega \sum_{j=0}^{n-1} \beta^j\right) \prod_{j=1}^{n} \mathrm{E}\{\exp[m\alpha\beta^{j-1} g(\epsilon_i)]\} \, \mathrm{E}[\Psi_{i-n}^{m\beta^n}], \tag{14}$$

using the hypothesis that $\{\epsilon_i\}$ is IID, and the existence of $\mathrm{E}\exp[m\alpha g(\epsilon_i)]$ which implies that of $\mathrm{E}\exp[m\alpha\beta^{j-1} g(\epsilon_i)]$ (since $|\beta| < 1$). Letting $n \to \infty$, one gets the

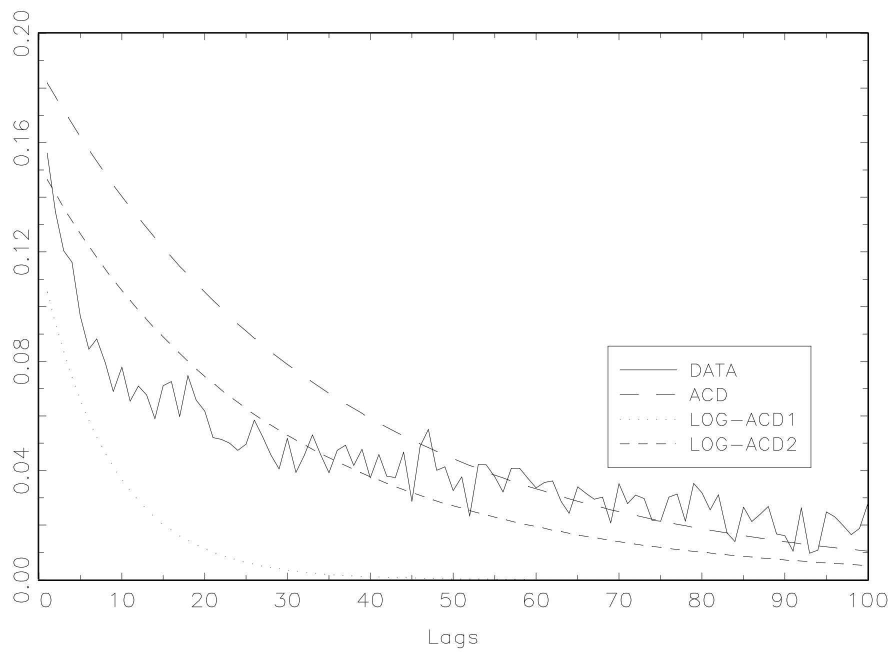

Figure 4: ACF for the ACD, Log-ACD$_1$, Log-ACD$_2$ models (using the analytical expressions computed for the estimated parameters) and empirical data (price duration at \$1/8 for IBM).

moment as given in the theorem, if and only if $|\beta| < 1$ (the last expectation in (14) tends to 1). $\diamond$

Proof: [Corollary 1] (9) follows directly from (8). Since $Ey^2 \geq (\mathrm{E}y)^2$, defining y as $\exp[\alpha\beta^{j-1}g(\epsilon_i)]$, we see that each term of the infinite product in (9) is not smaller than 1, and equal to 1 if $\alpha = 0$. This implies that $\delta_x \geq \delta$. $\diamond$

Proof: [Theorem 2] To obtain the ACF, we need $\mathrm{E}(x_i x_{i-n})$ since $\rho_n = [\mathrm{E}(x_i x_{i-n}) - \mu_x^2]/\sigma_x^2$, where μ_x and σ_x^2 are the mean and the variance of x_i implied by (8). Using (13), we get

$$\Psi_i \Psi_{i-n} = \exp\left(\omega \sum_{j=0}^{n-1} \beta^j\right) \exp\left[\alpha \sum_{j=1}^{n} \beta^{j-1} g(\epsilon_{i-j})\right] \Psi_{i-n}^{1+\beta^n}. \tag{15}$$

If we multiply both sides of this equality by $\epsilon_i \epsilon_{i-n}$ and we take the expectation, we obtain

$$\begin{aligned} \mathrm{E}(x_i x_{i-n}) = \quad & \mathrm{E}(\epsilon_i) \exp\left(\omega \tfrac{1-\beta^n}{1-\beta}\right) \textstyle\prod_{j=1}^{n-1} \mathrm{E}\{\exp[\alpha\beta^{j-1}g(\epsilon_i)]\} \\ & \mathrm{E}\{\epsilon_{i-n} \exp[\alpha\beta^{n-1}g(\epsilon_{i-n})]\}\, \mathrm{E}[\Psi_{i-n}^{1+\beta^n}], \end{aligned} \tag{16}$$

which exists using the assumption that $\mathrm{E}\{\epsilon_i \exp[\alpha g(\epsilon_i)]\}$ exists. From (8) when $m = 1$, we know that $\mathrm{E}(\Psi_i) = \exp[\omega/(1-\beta)] \prod_{j=1}^{\infty} \mathrm{E}[e^{\alpha\beta^{j-1}g(\epsilon_i)}]$. Applying this to $\mathrm{E}[\Psi_{i-n}^{1+\beta^n}]$ and substituting in the right-hand side of (16), we get the first term of the numerator of (10) multiplied by $\exp[2\omega/(1-\beta)]$, which can be simplified in ρ_n since it appears also in μ_x^2 and σ_x^2. $\diamond$

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