
Modelling Financial High Frequency Data Using Point Processes

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1 Introduction

Since the seminal papers by Hasbrouck (1991) and Engle and Russell (1998) the modelling of financial data at the transaction level is an ongoing topic within financial econometrics. This has created a new literature, often referred to as "the econometrics of (ultra-)high-frequency finance" or "high-frequency econometrics". The peculiar properties of financial transaction data, such as the irregular spacing in time, the discreteness of price changes, the bid-ask bounce as well as the presence of strong intraday seasonalities and persistent dynamics, has spurred a surge in new econometric approaches. One important strand of the literature deals with the irregular spacing of data in time. Taking into account the latter is indispensable if one seeks to exploit the full amount of information in financial transaction data. For example, the timing and frequency of order and trade arrivals carry information on the state of the market and play an important role in market microstructure analysis, for the modelling of intraday volatility as well as the measurement and prediction of liquidity, transaction costs and implied liquidity risks.

Accounting for the irregular occurrence of transaction data requires us to model the series as so-called *financial point processes*. We label the inter-event waiting times as financial durations and classify them according to the events of interest, with the most common being *trade durations* and *quote durations*, defined by the time between two consecutive trade or quote arrivals, respectively. *Price durations* correspond to the time between cumulative absolute price changes of given size and can be used as a volatility measure. Similarly, a *volume duration* is defined as the time until a cumulative order volume of given size is traded and captures an important dimension of market liquidity. For more details and illustrations, see Bauwens and Giot (2001) or Hautsch (2004).

One important property of transaction data is that market events are clustered over time implying that financial durations follow positively auto-correlated processes with strong persistence. In fact, the dynamic properties of financial durations are quite similar to those of (daily) volatilities. These features may be captured in alternative ways through different dynamic models based on either duration, intensity or counting representations of a point process.

This chapter reviews duration- and intensity-based models of financial point processes. In Section 2, we introduce the fundamental concepts of point process theory and discuss major statistical tools. In Section 3, we review the class of *dynamic duration models*. Specifying a (dynamic) duration model is arguably the most intuitive way to characterize a point process in discrete time and has been suggested by Engle and Russell (1998), which inspired a large literature. Nevertheless, Russell (1999) realized that a continuous-time setting on the basis of the intensity function constitutes a more flexible framework which is particularly powerful for the modelling of multivariate processes. Different types of *dynamic intensity models* are presented in Section 4.

2 Fundamental Concepts of Point Process Theory

In this section, we discuss important concepts in point process theory which are needed throughout this chapter. In Section 2.1, we introduce the notation and basic definitions. The fundamental concepts of intensity functions, compensators and hazard rates are defined in Section 2.2, whereas in Section 2.3 different classes and representations of point processes are discussed. Finally, in Section 2.4, we present the random time change theorem which yields a powerful result for the construction of diagnostics for point process models. Most concepts discussed in this section are based upon Chapter 2 of Karr (1991).

2.1 Notation and Definitions

Let $\{t_i\}_{i \in \{1, \dots, n\}}$ denote a random sequence of increasing event times $0 < t_1 < \dots < t_n$ associated with an orderly (simple) point process. Then, $N(t) := \sum_{i \geq 1} \mathbb{1}_{\{t_i \leq t\}}$ defines the right-continuous (càdlàg) counting function. Throughout this chapter, we consider only point processes which are integrable, i.e. $E[N(t)] < \infty \forall t \geq 0$. Furthermore, $\{W_i\}_{i \in \{1, \dots, n\}}$ denotes a sequence of $\{1, \dots, K\}$ -valued random variables representing K different types of events. Then, we call the process $\{t_i, W_i\}_{i \in \{1, \dots, n\}}$ an K -variate *marked* point process on $(0, \infty)$ as represented by the K sequences of event-specific arrival times $\{t_i^k\}_{i \in \{1, \dots, n^k\}}$, $k = 1, \dots, K$, with counting functions $N^k(t) := \sum_{i \geq 1} \mathbb{1}_{\{t_i \leq t\}} \mathbb{1}_{\{W_i = k\}}$.

The internal history of an K -dimensional point process $N(t)$ is given by the filtration \mathcal{F}_t^N with $\mathcal{F}_t^N = \sigma(N^k(s) : 0 \leq s \leq t, k \in \Xi)$, $N^k(s) =$

$\sum_{i \geq 1} \mathbb{1}_{\{t_i \leq s\}} \mathbb{1}_{\{W_i \in \Xi\}}$, where Ξ denotes the σ -field of all subsets of $\{1, \dots, K\}$. More general filtrations, including e.g. also processes of explanatory variables (covariates) $\{z_i\}_{i \in \{1, \dots, n\}}$ are denoted by \mathcal{F}_t with $\mathcal{F}_t^N \subseteq \mathcal{F}_t$.

Define $x_i := t_i - t_{i-1}$ with $i = 1, \dots, n$ and $t_0 := 0$ as the inter-event duration from t_{i-1} until t_i . Furthermore, $x(t)$ with $x(t) := t - t_{\check{N}(t)}$, with $\check{N}(t) := \sum_{i \geq 1} \mathbb{1}_{\{t_i < t\}}$ denoting the left-continuous counting function, is called the backward recurrence time. It is a left-continuous function that grows linearly through time with discrete jumps back to zero *after* each arrival time t_i . Finally, let $\theta \in \Theta$ denote model parameters.

2.2 Compensators, Intensities, and Hazard Rates

In martingale-based point process theory, the concept of *compensators* plays an important role. Using the property that an \mathcal{F}_t -adapted point process $N(t)$ is a submartingale³, it can be decomposed into a zero mean martingale $M(t)$ and a (unique) \mathcal{F}_t -predictable increasing process, $\tilde{A}(t)$, which is called the compensator of $N(t)$ and can be interpreted as the local conditional mean of $N(t)$ given the past. In statistical theory, this decomposition is typically referred to as the Doob-Meyer decomposition.

Define $\lambda(t)$ as a scalar, positive \mathcal{F}_t -predictable process, i.e. $\lambda(t)$ is adapted to \mathcal{F}_t , and left-continuous with right hand limits. Then, $\lambda(t)$ is called the (\mathcal{F}_t -conditional) *intensity* of $N(t)$ if

$$\tilde{A}(t) = \int_0^t \lambda(u) du, \quad (1)$$

where $\tilde{A}(t)$ is the (unique) compensator of $N(t)$. This relationship emerges from the interpretation of the compensator as integrated (conditional) hazard function. Consequently, $\lambda(t)$ can be also defined by the relation

$$\mathbb{E}[N(s) - N(t) | \mathcal{F}_t] = \mathbb{E} \left[\int_t^s \lambda(u) du \middle| \mathcal{F}_t \right], \quad (2)$$

which has to hold (almost surely) for all t, s with $0 \leq t \leq s$. Letting $s \downarrow t$ leads to the heuristic representation which is more familiar in classical duration analysis. Then, $\lambda(t)$ is obtained by

$$\lambda(t+) := \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \mathbb{E}[N(t + \Delta) - N(t) | \mathcal{F}_t], \quad (3)$$

where $\lambda(t+) := \lim_{\Delta \downarrow 0} \lambda(t + \Delta)$. In case of a stationary point process, $\bar{\lambda} := \mathbb{E}[dN(t)]/dt = \mathbb{E}[\lambda(t)]$ is constant.

Equation (3) manifests the close analogy between the intensity function and the *hazard function* which is given by

³ An \mathcal{F}_t -adapted càdlàg process $N(t)$ is a submartingale if $\mathbb{E}[|N(t)|] < \infty$ for each t and if $s < t$ implies that $\mathbb{E}[N(t) | \mathcal{F}_s] \geq N(s)$.

$$h(x) := f(x)/S(x) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \Pr[x \leq X < x + \Delta | X \geq x] \quad (4)$$

with x denoting the (inter-event) duration as represented by the realization of a random variable X with probability density function $f(x)$, survivor function $S(x) := 1 - F(x)$, and cumulative distribution function (cdf) $F(x) := \Pr[X \leq x]$. Whereas the intensity function is defined in (continuous) calendar time, the hazard rate is typically defined in terms of the length of a duration x and is a key concept in (cross-section) survival analysis.

2.3 Types and Representations of Point Processes

The simplest type of point process is the homogeneous Poisson process defined by

$$\Pr[(N(t + \Delta) - N(t)) = 1 | \mathcal{F}_t] = \lambda \Delta + o(\Delta), \quad (5)$$

$$\Pr[(N(t + \Delta) - N(t)) > 1 | \mathcal{F}_t] = o(\Delta), \quad (6)$$

with $\Delta \downarrow 0$. Then, $\lambda > 0$ is called the Poisson rate corresponding to the (constant) intensity. Accordingly, equations (5) and (6) define the *intensity representation* of a Poisson process. A well-known property of homogeneous Poisson processes is that the inter-event waiting times x_i are independently exponentially distributed, leading to the *duration representation*. In this context, λ is the hazard rate of the exponential distribution. Furthermore, it can be shown (see e.g. Lancaster (1997)) that the number of events in an interval $(a, b]$, $N(a, b) := N(b) - N(a)$ is Poisson distributed with $\Pr[N(a, b) = k] = \exp[-\lambda(b - a)] [\lambda(b - a)]^k / k!$, yielding the *counting representation*. All three representations of a Poisson process can be used as the starting point for the specification of a dynamic point process model.

Throughout this chapter we associate the term *duration models* to a model of the (discrete-time) duration process observable at the event-times $\{t_i\}_{i=1, \dots, n}$. Then, researchers parameterize the conditional distribution function $F(x_i | \mathcal{F}_{t_{i-1}})$ or, alternatively, the conditional hazard rate $h(x_i | \mathcal{F}_{t_{i-1}})$. Generally, such a model should aim, in particular, at fitting the dynamical and distributional properties of durations. The latter is often characterized by the excess dispersion, corresponding to the ratio between the standard deviation to the mean. In classical hazard rate models employed in traditional survival analysis, the hazard rate is typically parameterized in terms of covariates, see e.g. Kalbfleisch and Prentice (1980), Kiefer (1988) or Lancaster (1997). The most well-known hazard model is the *proportional hazard* model introduced by Cox (1972) and is given by

$$h(x|z; \theta) = h_0(x|\gamma_1)g(z, \gamma_2), \quad (7)$$

where $\theta = (\gamma_1, \gamma_2)$, $h_0(\cdot)$ denotes the so-called baseline hazard rate and $g(\cdot)$ is a function of the covariates z and parameters γ_2 . The baseline hazard rate

may be parameterized in accordance with a certain distribution, like e.g., a Weibull distribution with parameters $\lambda, p > 0$ implying

$$h_0(x|\gamma_1) = \lambda p(\lambda x)^{p-1}. \quad (8)$$

For $p = 1$ we obtain the exponential case $h_0(x|\gamma_1) = \lambda$, implying a constant hazard rate. Alternatively, if $p > 1$, $\partial h_0(x|\gamma_1)/\partial x > 0$, i.e. the hazard rate is increasing with the length of the spell which is referred to as "positive duration dependence". In contrast, $p < 1$ implies "negative duration dependence". Non-monotonic hazard rates can be obtained with more flexible distributions, like the generalized F and special cases thereof, including the generalized gamma, Burr, Weibull and log-logistic distributions. We refer to the Appendix to Chapter 3 of Bauwens and Giot (2001) and to the Appendix of Hautsch (2004) for definitions and properties. Alternatively, the baseline hazard may be left unspecified and can be estimated nonparametrically, see Cox (1975).

An alternative type of duration model is the class of *accelerated failure time* (AFT) models given by

$$h(x|z; \theta) = h_0[xg(z, \gamma_2)|\gamma_1]g(z, \gamma_2). \quad (9)$$

Here, the effect of the exogenous variables is to accelerate or to decelerate the time scale on which the baseline hazard h_0 is defined. As illustrated in Section 3.1, AFT-type approaches are particularly attractive to model autocorrelated duration processes.

Because of their discrete-time nature, duration models cannot be used whenever the information set has to be updated *within* a duration spell, e.g. caused by time-varying covariates or event arrivals in other point processes. For this reason, (discrete-time) duration models are typically used in a univariate framework.

Whenever a continuous-time modelling is preferential (as e.g. to account for the asynchronous event arrivals in a multivariate framework), it is more natural to specify the intensity function directly, leading to so-called *intensity models*. One important extension of a homogenous Poisson process is to allow the intensity to be directed by a real-valued, non-negative (stationary) random process $\lambda^*(t)$ with (internal) history \mathcal{F}_t^* leading to the class of *doubly stochastic Poisson processes* (Cox processes). In particular, $N(t)$ is called a Cox process directed by $\lambda^*(t)$ if conditional on $\lambda^*(t)$, $N(t)$ is a Poisson process with mean $\lambda^*(t)$, i.e. $\Pr[N(a, b) = k | \mathcal{F}_t^*] = \exp[-\lambda^*(t)] [\lambda^*(t)]^k / k!$. The doubly stochastic Poisson process yields a powerful class of probabilistic models with applications in seismology, biology and economics. For instance, specifying $\lambda^*(t)$ in terms of an autoregressive process yields a dynamic intensity model which is particularly useful to capture the clustering in financial point processes. For a special type of doubly stochastic Poisson process see Section 4.2.

A different generalization of the Poisson process is obtained by specifying $\lambda(t)$ as a (linear) self-exciting process given by

$$\lambda(t) = \omega + \int_0^t w(t-u) dN(u) = \omega + \sum_{t_i < t} w(t-t_i), \quad (10)$$

where ω is a constant, $w(s)$ denotes a non-negative weight function, and $\int_0^t w(s) dN(s)$ is the stochastic Stieltjes integral of the process w with respect to the counting process $N(t)$. The process (10) was proposed by Hawkes (1971) and is therefore named a *Hawkes process*. If $w(s)$ declines with s , then, the process is *self-exciting* in the sense that $\text{Cov}[N(a, b), N(b, c)] > 0$, where $0 < a \leq b < c$. Different types of Hawkes processes and their applications to financial point processes are presented in Section 4.1. In alternative specifications, the intensity is driven by an autoregressive process which is updated at each point of the process. This leads to a special type of point process models which does not originate from the classical point process literature but rather from the *autoregressive conditional duration* (ACD) literature reviewed in Section 3 and brings time series analysis into play. The resulting model is called an *autoregressive conditional intensity* model and is considered in Section 4.2.

Finally, starting from the counting representation of a Poisson process leads to the class of *count data models*. Dynamic extensions of Poisson processes in terms of counting representations are not surveyed in this chapter. Important references are e.g. Rydberg and Shephard (2003) and Heinen and Rengifo (2007).

2.4 The Random Time Change Theorem

One fundamental result of martingale-based point process theory is the (multivariate) random time change theorem by Meyer (1971) which allows to transform a wide class of point processes to a homogeneous Poisson process:

Theorem: *Assume a multivariate point process $(N^1(t), \dots, N^K(t))$ is formed from the event times $\{t_i^k\}_{i \in \{1, \dots, n^k\}}$, $k = 1, \dots, K$, and has continuous compensators $(\tilde{\Lambda}^1(t), \dots, \tilde{\Lambda}^K(t))$ with $\tilde{\Lambda}^k(\infty) = \infty$ for each $k = 1, \dots, K$. Then, the point processes formed from $\{\tilde{\Lambda}^k(t_i^k)\}_{i=1, \dots, n^k}$, $k = 1, \dots, K$, are independent Poisson processes with unit intensity.*

Proof: See Meyer (1971) or Brown and Nair (1988) for a more accessible and elegant proof.

Define $\tau^k(t)$ as the (\mathcal{F}_t) -stopping time obtained by the solution of $\int_0^{\tau^k(t)} \lambda^k(s) ds = t$. Applying the random time change theorem to (1) implies that the point processes $\tilde{N}^k(t)$ with $\tilde{N}^k(t) := N^k(\tau^k(t))$ are independent Poisson processes with unit intensity and event times $\{\tilde{\Lambda}^k(t_i^k)\}_{i=1, \dots, n^k}$ for $k = 1, \dots, K$. Then, the so-called integrated intensities

$$\Lambda^k(t_{i-1}^k, t_i^k) := \int_{t_{i-1}^k}^{t_i^k} \lambda^k(s) ds = \tilde{\Lambda}^k(t_i^k) - \tilde{\Lambda}^k(t_{i-1}^k) \quad (11)$$

correspond to the increments of independent Poisson processes for $k = 1, \dots, K$. Consequently, they are independently standard exponentially distributed across i and k . For more details, see Bowsher (2007). The random time change theorem plays an important role in order to construct diagnostic tests for point process models (see Section 4.3) or to simulate point processes.

3 Dynamic Duration Models

In this section, we discuss univariate dynamic models for the durations between consecutive (financial) events. In Section 3.1, we review the class of ACD models, which is by far the most used class in the literature on financial point processes. In Section 3.2, we briefly discuss statistical inference for ACD models. In Section 3.3, we present other dynamic duration models, and in the last section we review some applications.

3.1 ACD Models

The class of ACD models has been introduced by Engle and Russell (1997, 1998) and Engle (2000). In order to keep the notation simple, define x_i in the following as the inter-event duration which is standardized by a seasonality function $s(t_i)$, i.e. $x_i := (t_i - t_{i-1})/s(t_i)$. The function $s(t_i)$ is typically parameterized according to a spline function capturing time-of-day or day-of-week effects. Time-of-day effects arise because of systematic changes of the market activity throughout the day and due to openings of other related markets. In most approaches $s(t_i)$ is specified according to a linear or cubic spline function and is estimated separately in a first step yielding seasonality adjusted durations x_i . Alternatively, a non-parametric approach has been proposed by Veredas et al. (2002). For more details and examples regarding seasonality effects in financial duration processes, we refer the reader to Chapter 2 of Bauwens and Giot (2001) or to Chapter 3 of Hautsch (2004).

The key idea of the ACD model is to model the (seasonally adjusted) durations $\{x_i\}_{i=1, \dots, n}$ in terms of a multiplicative error model in the spirit of Engle (2002), i.e.

$$x_i = \Psi_i \varepsilon_i, \quad (12)$$

where Ψ_i denotes a function of the past durations (and possible covariates), and ε_i defines an i.i.d. random variable. It is assumed that

$$\mathbb{E}[\varepsilon_i] = 1, \quad (13)$$

so that Ψ_i corresponds to the conditional duration mean (the so-called "conditional duration") with $\Psi_i := \mathbb{E}[x_i | \mathcal{F}_{t_{i-1}}]$. The ACD model can be rewritten in terms of the intensity function as

$$\lambda(t | \mathcal{F}_t) = \lambda_\varepsilon \left(\frac{x(t)}{\Psi_{\check{N}(t)+1}} \right) \frac{1}{\Psi_{\check{N}(t)+1}}, \quad (14)$$

where $\lambda_\epsilon(s)$ denotes the hazard function of the ACD error term. This formulation shows that the ACD model belongs to the class of AFT models. Assuming ϵ_i to be standard exponentially distributed yields the so-called *Exponential ACD* model. More flexible specifications arise by assuming ϵ_i to follow a more general distribution, see the discussion after equation (8). It is evident that the ACD model is the counter-part to the GARCH model (Bollerslev (1986)) for duration processes. Not surprisingly, many results and specifications from the GARCH literature have been adapted to the ACD literature.

The conditional duration, Ψ_i , is defined as a function Ψ of the information set $\mathcal{F}_{t_{i-1}}$ and provides therefore the vehicle for incorporating the dynamics of the duration process. In this respect it is convenient to use an ARMA-type structure of order (p, q) , whereby

$$\Psi_i = \Psi(\Psi_{i-1}, \dots, \Psi_{i-q}, x_{i-1}, \dots, x_{i-p}). \quad (15)$$

For simplicity, we limit the exposition in the sequel to the case $p = q = 1$.

The first model put forward in the literature is the *linear ACD model*, which specializes (15) as

$$\Psi_i = \omega + \beta\Psi_{i-1} + \alpha x_{i-1}. \quad (16)$$

Since Ψ_i must be positive, the restrictions $\omega > 0$, $\alpha \geq 0$ and $\beta \geq 0$ are usually imposed. It is also assumed that $\beta = 0$ if $\alpha = 0$, otherwise β is not identified. The process defined by (12), (13) and (16) is covariance-stationary if

$$(\alpha + \beta)^2 - \alpha^2\sigma^2 < 1, \quad (17)$$

where $\sigma^2 := \text{Var}[\epsilon_i] < \infty$, and has the following moments and autocorrelations:

- (1) $E[x_i] := \mu_x = \omega / (1 - \alpha - \beta)$,
- (2) $\text{Var}[x_i] := \sigma_x^2 = \mu_x^2 \sigma^2 \frac{1 - \beta^2 - 2\alpha\beta}{1 - (\alpha + \beta)^2 - \alpha^2\sigma^2}$,
- (3) $\rho_1 = \frac{\alpha(1 - \beta^2 - \alpha\beta)}{1 - \beta^2 - 2\alpha\beta}$ and $\rho_n = (\alpha + \beta)\rho_{n-1}$ for $n \geq 2$.

The condition (17) ensures the existence of the variance. These results are akin to those for the GARCH(1,1) zero-mean process. They can be generalized to ACD(p, q) processes with $p, q > 1$. In applications, estimates of $\alpha + \beta$ are typically found to be in the interval (0.85, 1) with α lying in the interval (0.01, 0.15). Since the ACD(1,1) model can be written as

$$x_i = \omega + (\alpha + \beta)x_{i-1} + u_i - \beta u_{i-1}, \quad (18)$$

where $u_i := x_i - \Psi_i$ is a martingale difference innovation, the resulting autocorrelation function (ACF) is that of an ARMA(1,1) process that has AR and MA roots close to each other. This type of parameter configuration generates the typical ACF shape of clustered data. Nevertheless, the ACF decreases at a geometric rate, though it is not uncommon to find duration series with

an ACF that decreases at a hyperbolic rate. This tends to happen for long series and may be due to instabilities of parameters which give the illusion of long memory in the process. In order to allow for long range dependence in financial duration processes, Jasiak (1998) extends the ACD model to a *fractionally integrated ACD* model. For alternative ways to specify long memory ACD models, see Koulikov (2002).

A drawback of the linear ACD model is that it is difficult to allow Ψ_i to depend on functions of covariates without violating the non-negativity restriction. For this reason, Bauwens and Giot (2000) propose a class of *logarithmic ACD models*, where no parametric restrictions are needed to ensure positiveness of the process:

$$\ln \Psi_i = \omega + \beta \ln \Psi_{i-1} + \alpha g(\epsilon_{i-1}), \quad (19)$$

where $g(\epsilon_{i-1})$ is either $\ln \epsilon_{i-1}$ (log-ACD of type I) or ϵ_{i-1} (type II). Using this setting, it is convenient to augment Ψ_i by functions of covariates, see e.g. Bauwens and Giot (2000). The stochastic process defined by (12), (13) and (19) is covariance-stationary if

$$\beta < 1, \quad \mathbb{E}[\epsilon_i \exp\{\alpha g(\epsilon_i)\}] < \infty, \quad \mathbb{E}[\exp\{2\alpha g(\epsilon_i)\}] < \infty. \quad (20)$$

Its mean, variance and autocorrelations are given in Section 3.2 in Bauwens and Giot (2001), see also Fernandes and Grammig (2006) and Bauwens et al. (2008). Drost and Werker (2004) propose to combine one of the previous ACD equations for the conditional duration mean with an unspecified distribution for ϵ_i , yielding a class of *semi-parametric ACD models*.

The *augmented ACD* (AACD) model introduced by Fernandes and Grammig (2006) provides a more flexible specification of the conditional duration equation than the previous models. Here, Ψ_i is specified in terms of a power transformation yielding

$$\Psi_i^{\delta_1} = \omega + \beta \Psi_{i-1}^{\delta_1} + \alpha \Psi_{i-1}^{\delta_1} [|\epsilon_{i-1} - \xi| - \rho(\epsilon_{i-1} - \xi)]^{\delta_2}, \quad (21)$$

where $\delta_1 > 0$, $\delta_2 > 0$, ξ , and ρ are parameters. The so-called news impact function $[|\epsilon_{i-1} - \xi| - \rho(\epsilon_{i-1} - \xi)]^{\delta_2}$ allows a wide variety of shapes of the curve tracing the impact of ϵ_{i-1} on Ψ_i for a given value of Ψ_{i-1} and the remaining parameters. The parameters ξ and ρ are shift and rotation parameters, respectively. If $\xi = \rho = 0$, the linear ACD model is obtained by setting $\delta_1 = \delta_2 = 1$, the type I logarithmic ACD model by letting δ_1 and δ_2 tend to 0, and the type II version by letting δ_1 tend to 0 and setting $\delta_2 = 1$. Fernandes and Grammig (2006) compare different versions of the AACD model using IBM price durations arising from trading at the New York Stock Exchange (NYSE). Their main finding is that "letting δ_1 free to vary and accounting for asymmetric effects (by letting ξ and ρ free) seem to operate as substitute sources of flexibility". Hautsch (2006) proposes an even more general augmented ACD model that nests in particular the so-called *EXponential ACD model* proposed by

Dufour and Engle (2000) implying a kinked news impact function. As a counterpart to the so-called semiparametric GARCH model proposed by Engle and Ng (1993), Hautsch (2006) suggests specifying the news impact function in terms of a linear spline function based on the support of ε_i . He illustrates that the high flexibility of this model is needed in order to appropriately capture the dynamic properties of financial durations.

Another way to achieve flexibility in ACD models is to use the idea of mixtures. The mixture may apply to the error distribution alone, as in De Luca and Zuccolotto (2003), De Luca and Gallo (2004) and Hujer and Vuletic (2007), or may involve the dynamic component as well. Zhang et al. (2001) propose a *threshold ACD model* (TACD), wherein the ACD equation and the error distribution change according to a threshold variable such as the previous duration. For J regimes indexed by $j = 1, \dots, J$, the model is defined as

$$x_i = \Psi_i \varepsilon_i^{(j)}, \quad (22)$$

$$\Psi_i = \omega^{(j)} + \beta^{(j)}\Psi_{i-1} + \alpha^{(j)}x_{i-1} \quad (23)$$

when $x_{i-1} \in [r_{j-1}, r_j)$, and $0 = r_0 < r_1 < \dots < r_J = \infty$ are the threshold parameters. The superscript (j) indicates that the distribution or the model parameters can vary with the regime operating at observation i . This model can be viewed as a mixture of J ACD models, where the probability to be in regime j at i is equal to 1 and the probabilities to be in each of the other regimes are equal to 0. Hujer et al. (2002) extend this model to let the regime changes be governed by a hidden Markov chain.

While the TACD model implies discrete transitions between the individual regimes, Meitz and Teräsvirta (2006) propose a class of *smooth transition ACD (STACD) models* generalizing linear and logarithmic ACD models. Conditions for strict stationarity, ergodicity, and existence of moments for this model and other ACD models are provided in Meitz and Saikkonen (2004) using the theory of Markov chains. A motivation for the STACD model is, like for the AACD model, to allow for a nonlinear impact of the past duration on the next expected duration.

3.2 Statistical Inference

The estimation of most ACD models can be easily performed by maximum likelihood (ML). Engle (2000) demonstrates that the results by Bollerslev and Wooldridge (1992) on the quasi-maximum likelihood (QML) property of the Gaussian GARCH(1,1) model extend to the Exponential-ACD(1,1) model. QML estimates are obtained by maximizing the quasi-loglikelihood function given by

$$\ln \mathcal{L}(\theta; \{x_i\}_{i=1, \dots, n}) = - \sum_{i=1}^n \left[\ln \Psi_i + \frac{x_i}{\Psi_i} \right]. \quad (24)$$

For more details we refer to Chapter 3 of Bauwens and Giot (2001), Chapter 5 of Hautsch (2004), and to the survey of Engle and Russell (2005).

Residual diagnostics and goodness-of-fit tests can be performed by evaluating the stochastic properties of the ACD residuals $\hat{\epsilon}_i = x_i/\hat{\Psi}_i$. The dynamic properties are easily analyzed based on Portmanteau statistics or tests against independence such as proposed by Brock et al. (1996). The distributional properties can be evaluated by Engle and Russell's (1998) test for no excess dispersion using the asymptotically standard normal test statistic $\sqrt{n/8} \hat{\sigma}^2$, where $\hat{\sigma}^2$ denotes the empirical variance of the residual series. Dufour and Engle (2000) and Bauwens et al. (2004) evaluate the models' goodness-of-fit based on density forecasts using the probability integral transform as proposed by Diebold et al. (1998). A nonparametric test against distributional misspecification is proposed by Fernandes and Grammig (2005) based on the work by Aït-Sahalia (1996). Statistics that exclusively test for misspecifications of the conditional mean function Ψ_i have been worked out by Meitz and Teräsvirta (2006) using the Lagrange Multiplier principle and by Hautsch (2006) using (integrated) conditional moment tests. A common result is that too simple ACD specifications, such as the ACD or Log-ACD model are not flexible enough to adequately capture, even in-sample, the properties of observed financial durations. However, in order to avoid the problem of potential over-fitting a serious comparison of ACD specifications should rely on out-of-sample evaluations.

3.3 Other Models

ACD models strongly resemble ARCH models. Therefore it is not surprising that Taylor's (1986) stochastic volatility model for financial returns has been a source of inspiration of corresponding duration models. Bauwens and Veredas (2004) propose the *stochastic conditional duration (SCD) model* as an alternative to ACD-type models. The SCD model relates to the logarithmic ACD model in the same way as the stochastic volatility model relates to (a restricted version of) the exponential GARCH model by Nelson (1991). Thus the model is defined by equations (12), (13), and

$$\ln \Psi_i = \omega + \beta \ln \Psi_{i-1} + \gamma \epsilon_{i-1} + u_i, \quad (25)$$

where u_i is iid $N(0, \sigma_u^2)$. The process $\{u_i\}$ is assumed to be independent of the process $\{\epsilon_i\}$. The set of possible distributions for the duration innovations ϵ_i is the same as that for ACD models. This model generates a rich class of hazard functions for x_i through the interplay of two distributions. The latent variable Ψ_i may be interpreted as being inversely related to the information arrival process which triggers bursts of activity on financial markets. The "leverage" term $\gamma \epsilon_{i-1}$ in (25) is added by Feng et al. (2004) to allow for an intertemporal correlation between the observable duration and the conditional duration. Bauwens and Veredas (2004) use a logarithmic transformation of (12) and

employ QML estimation based on the Kalman filter. Strickland et al. (2006) use Bayesian estimation with a Markov chain Monte Carlo algorithm. For ML estimation, Feng et al. (2004) use the Monte Carlo method of Durbin and Koopman (2004), and Bauwens and Galli (2008) use efficient importance sampling.

The ACD and SCD models reviewed above share the property that the dynamics of higher moments of the duration process are governed by the dynamics of the conditional mean. Ghysels et al. (2004) argue that this feature is restrictive and introduce a nonlinear two factor model that disentangles the movements of the mean and that of the variance of durations. Since the second factor is responsible for the variance heterogeneity, the model is named the *stochastic volatility duration (SVD) model*. The departure point for this model is a standard static duration model in which the durations are independently and exponentially distributed with a gamma heterogeneity, i.e.

$$x_i = \frac{U_i}{aV_i} = \frac{H(1, F_{1i})}{aH(b, F_{2i})}, \quad (26)$$

where U_i and V_i are two independent variables which are gamma(1,1) (i.e. exponential) and gamma(b, b) distributed, respectively. The last ratio in (26) uses two independent Gaussian factors F_{1i} and F_{2i} , and $H(b, F) = G(b, \varphi(F))$, where $G(b, \cdot)$ is the quantile function of the gamma(b, b) distribution and $\varphi(\cdot)$ the cdf of the standard normal distribution. Ghysels et al. (2004) extend this model to a dynamic setup through a VAR model for the two underlying Gaussian factors. The estimation of the model requires simulation methods.

3.4 Applications

ACD models can be used to estimate and predict the intra-day volatility of returns from the intensity of price durations. As shown by Engle and Russell (1998), a price intensity is closely linked to the instantaneous price change volatility. The latter is given by

$$\tilde{\sigma}^2(t) := \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \mathbb{E} \left[\left(\frac{p(t+\Delta) - p(t)}{p(t)} \right)^2 \middle| \mathcal{F}_t \right], \quad (27)$$

where $p(t)$ denotes the price (or midquote) at t . By denoting the counting process associated with the event times of cumulated absolute price changes of size dp by $N^{dp}(t)$, we can formulate (27) in terms of the intensity function of the process of dp -price changes. Then, the dp -price change instantaneous volatility can be computed as

$$\begin{aligned} \tilde{\sigma}_{(dp)}^2(t) &= \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \Pr [|p(t+\Delta) - p(t)| \geq dp | \mathcal{F}_t] \cdot \left[\frac{dp}{p(t)} \right]^2 \\ &= \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \Pr [\{ N^{dp}(t+\Delta) - N^{dp}(t) \} > 0 | \mathcal{F}_t] \cdot \left[\frac{dp}{p(t)} \right]^2 \end{aligned}$$

$$:= \lambda^{dp}(t) \cdot \left[\frac{dp}{p(t)} \right]^2, \quad (28)$$

where $\lambda^{dp}(t)$ denotes the corresponding dp -price change intensity. Hence, using (14), one can estimate or predict the instantaneous volatility of the price process $p(t)$ at any time point. Giot (2005) compares these estimates with usual GARCH based estimates obtained by interpolating the prices on a grid of regularly spaced time points. He finds that GARCH based predictions are better measures of risk than ACD based ones in a Value-at-Risk (VaR) evaluation study.

ACD and related models have been typically used to test implications of asymmetric information models of price formation. For example, the model of Easley and O'Hara (1992) implies that the number of transactions influences the price process through information based clustering of transactions. Then, including lags as well as expectations of the trading intensity as explanatory variables in a model for the price process allows to test such theoretical predictions. For a variety of different applications in market microstructure research, see Engle and Russell (1998), Engle (2000), Bauwens and Giot (2000), Engle and Lunde (2003), and Hafner (2005) among others. Several authors have combined an ACD model with a model for the marks of a financial point process. The basic idea is to model the duration process by an ACD model, and conditionally on the durations, to model the process of marks. Bauwens and Giot (2003) model the direction of the price change between two consecutive trades by formulating a competing risks model, where the direction of the price movement is triggered by a Bernoulli process. Then, the parameters of the ACD process depend on the direction of the previous price change, leading to an asymmetric ACD model. A related type of competing risks model is specified by Bisière and Kamionka (2000). Prigent et al. (2001) use a similar model for option pricing. Russell and Engle (2005) develop an autoregressive conditional multinomial model to simultaneously model the time between trades and the dynamic evolution of (discrete) price changes. An alternative approach based on a dynamic integer count model is proposed by Liesenfeld et al. (2006).

A related strand of the literature studies the interaction between the trading intensity and the trade-to-trade return volatility. Engle (2000) augments a GARCH equation for returns per time unit by the impact of the inverse of the observed and expected durations (x_i and Ψ_i), and of the surprise x_i/Ψ_i . A decrease in x_i or Ψ_i has a positive impact on volatility while the surprise has a reverse influence. Ghysels and Jasiak (1998) and Grammig and Wellner (2002) study a GARCH process for trade-to-trade returns with time-varying parameters which are triggered by the trading intensity. Meddahi et al. (2006) derive a discrete time GARCH model for irregularly spaced data from a continuous time volatility process and compare it to the ACD-GARCH models by Engle (2000) and Ghysels and Jasiak (1998).

4 Dynamic Intensity Models

In this section, we review the most important types of dynamic intensity models which are applied to model financial point processes. The class of Hawkes processes and extensions thereof are discussed in Section 4.1. In Section 4.2, we survey different autoregressive intensity models. Statistical inference for intensity models is presented in Section 4.3, whereas the most important applications in the recent literature are briefly discussed in Section 4.4.

4.1 Hawkes Processes

Hawkes processes originate from the statistical literature in seismology and are used to model the occurrence of earthquakes, see e.g. Vere-Jones (1970), Vere-Jones and Ozaki (1982), and Ogata (1988) among others. Bowsher (2007) was the first applying Hawkes models to financial point processes. As explained in Section 3.2, Hawkes processes belong to the class of self-exciting processes, where the intensity is driven by a weighted function of the time distance to previous points of the process. A general class of univariate Hawkes processes is given by

$$\lambda(t) = \varphi \left(\mu(t) + \sum_{t_i < t} w(t - t_i) \right), \quad (29)$$

where φ denotes a possibly nonlinear function, $\mu(t)$ is a deterministic function of time, and $w(s)$ denotes a weight function. If φ is a positive function, we obtain the class of nonlinear Hawkes processes considered by Brémaud and Massoulié (1996). In this case, $\mu(t)$ and $w(t)$ can take negative values since the transformation $\varphi(\cdot)$ preserves the non-negativity of the process. Such a specification is useful whenever the intensity may be negatively affected by the process history or covariates. For instance, in the context of financial duration processes, $\mu(t)$ can be parameterized as a function of covariates. Stability conditions for nonlinear Hawkes processes are derived by Brémaud and Massoulié (1996). For the special case where φ is a linear function, we obtain the class of linear Hawkes processes originally considered by Hawkes (1971). They are analytically and computationally more tractable than their nonlinear counterparts, however, they require $\mu(t) > 0$ and $w(t) > 0$ in order to ensure non-negativity.

As pointed out by Hawkes and Oakes (1974), linear self-exciting processes can be viewed as clusters of Poisson processes. Then, each event is one of two types: an immigrant process or an offspring process. The immigrants follow a Poisson process and define the centers of so-called Poisson clusters. If we condition on the arrival time, say t_i , of an immigrant, then independently of the previous history, t_i is the center of a Poisson process, $\mathcal{Y}(t_i)$, of offspring on (t_i, ∞) with intensity function $\lambda_i(t) = \lambda(t - t_i)$, where λ is a non-negative function. The process $\mathcal{Y}(t_i)$ defines the first generation offspring process with respect to t_i . Furthermore, if we condition on the process $\mathcal{Y}(t_i)$, then each of

the events in $\mathcal{T}(t_i)$, say t_j , generates a Poisson process with intensity $\lambda_j(t) = \lambda(t - t_j)$. These independent Poisson processes build the second generation of offspring with respect to t_i . Similarly, further generations arise. The set of all offspring points arising from one immigrant are called a Poisson cluster. Exploiting the branching and conditional independence structure of a (linear) Hawkes process, Møller and Rasmussen (2004) develop a simulation algorithm as an alternative to the Shedler-Lewis thinning algorithm or the modified thinning algorithm by Ogata (1981) (see e.g. Daley and Vere-Jones (2003)). The immigrants and offsprings can be referred to as "main shocks" and "after shocks" respectively. This admits an interesting interpretation which is useful not only in seismology but also in high-frequency finance. Bousher (2007), Hautsch (2004) and Large (2007) illustrate that Hawkes processes capture the dynamics of financial point processes remarkably well. This indicates that the cluster structure implied by the self-exciting nature of Hawkes processes seem to be a reasonable description of the timing structure of events on financial markets.

The most common parameterization of $w(t)$ has been suggested by Hawkes (1971) and is given by

$$w(t) = \sum_{j=1}^P \alpha_j e^{-\beta_j t}, \quad (30)$$

where $\alpha_j \geq 0$, $\beta_j > 0$ for $j = 1, \dots, P$ are model parameters, and P denotes the order of the process and is selected exogenously (or by means of information criteria). The parameters α_j are scale parameters, whereas β_j drive the strength of the time decay. For $P > 1$, the intensity is driven by the superposition of different exponentially decaying weighted sums of the backward times to all previous points. In order to ensure identification we impose the constraint $\beta_1 > \dots > \beta_P$. It can be shown that the stationarity of the process requires $0 < \int_0^\infty w(s) ds < 1$, which is ensured only for $\sum_{j=1}^P \alpha_j / \beta_j < 1$, see Hawkes (1971).

While (30) implies an exponential decay, the alternative parameterization

$$w(t) = \frac{H}{(t + \kappa)^p}, \quad (31)$$

with parameters H , κ , and $p > 1$ features a hyperbolic decay. Such weight functions are typically applied in seismology and allow to capture long range dependence. Since financial duration processes also tend to reveal long memory behavior (see Jasiak (1998)), this specification may be interesting in financial applications.

Multivariate Hawkes models are obtained by a generalization of (29). Then, $\lambda(t)$ is given by the $(K \times 1)$ -vector $\lambda(t) = (\lambda^1(t), \dots, \lambda^K(t))'$ with

$$\lambda^k(t) = \varphi \left(\mu^k(t) + \sum_{r=1}^K \sum_{t_i^r < t} w_r^k(t - t_i^r) \right), \quad (32)$$

where $w_r^k(s)$ is a k -type weight function of the backward time to all r -type events. Using an exponential decay function, Hawkes (1971) suggests to parameterize $w_r^k(s)$ as

$$w_r^k(t) = \sum_{j=1}^P \alpha_{r,j}^k e^{-\beta_{r,j}^k t}, \quad (33)$$

where $\alpha_{r,j}^k \geq 0$ and $\beta_{r,1}^k > \dots > \beta_{r,P}^k > 0$ determine the influence of the time distance to past r -type events on the k -type intensity. Thus, in the multivariate case, $\lambda^k(t)$ depends not only on the distance to all k -type points, but also on the distance to all other points of the pooled process. Hawkes (1971) provides a set of linear parameter restrictions ensuring the stationarity of the process.

Bowsher (2007) proposes a generalization of the Hawkes model which allows to model point processes that are interrupted by time periods where no activity takes place. In high-frequency financial time series these effects occur because of trading breaks due to trading halts, nights, weekends or holidays. In order to account for such effects, Bowsher proposes to remove all non-activity periods and to concatenate consecutive activity periods by a spill-over function.

4.2 Autoregressive Intensity Processes

Hamilton and Jordà (2002) establish a natural link between ACD models and intensity models by allowing the ACD model to include covariates that may change during a duration spell (time-varying covariates). Their so-called *autoregressive conditional hazard (ACH) model* relies on the idea that in the Exponential ACD model, the intensity corresponds to the inverse of the conditional duration, i.e. $\lambda(t) = \Psi_{\tilde{N}(t)+1}^{-1}$. They extend this expression by a function of time-varying regressors $z_{\tilde{t}_j}$, where \tilde{t}_j denotes the arrival times in the covariate process and j is such that $t_{\tilde{N}(t)} < \tilde{t}_j < t$. Then,

$$\lambda(t) = \frac{1}{\Psi_{\tilde{N}(t)+1} + z_{\tilde{t}_j}' \gamma}, \quad (34)$$

where γ is a vector of unknown parameters.

An alternative model which can be seen as a combination of a duration model and an intensity model is introduced by Gerhard and Hautsch (2007). They propose a dynamic extension of a proportional intensity model due to Cox (1972), where the baseline intensity $\lambda_0(t)$ is not specified. Their key idea is to exploit the stochastic properties of the integrated intensity and to reformulate the model in terms of a regression model with unknown left-hand variable and Gumbel distributed error terms. See Kiefer (1988) for a nice illustration of this relation. To identify the unknown baseline intensity at discrete points, Gerhard and Hautsch follow the idea of Han and Hausman (1990) and

formulate the model in terms of an ordered response model based on categorized durations. In order to allow for serial dependence in the duration process, the model is extended by an observation-driven ARMA structure based on generalized errors. As a result, the resulting *semiparametric autoregressive conditional proportional intensity model* allows to capture serial dependence in durations and to estimate conditional failure probabilities without requiring explicit distributional assumptions.

In the *autoregressive conditional intensity (ACI) models* introduced by Russell (1999), the intensity function is directly modeled as an autoregressive process which is updated by past realizations of the integrated intensity. Let $\lambda(t) = (\lambda^1(t), \dots, \lambda^K(t))'$. Russell (1999) proposes to specify $\lambda^k(t)$ in terms of a proportional intensity structure given by

$$\lambda^k(t) = \Phi_{\check{N}(t)+1}^k \lambda_0^k(t) s^k(t), \quad k = 1, \dots, K, \quad (35)$$

where $\Phi_{\check{N}(t)+1}$ captures the dynamic structure, $\lambda_0^k(t)$ is a baseline intensity component capturing the (deterministic) evolution of the intensity between two consecutive points and $s^k(t)$ denotes a deterministic function of t capturing, for instance, possible seasonality effects. The function $\Phi_{\check{N}(t)}$ is indexed by the left-continuous counting function and is updated instantaneously *after* the arrival of a new point. Hence, Φ_i is constant for $t_{i-1} < t \leq t_i$. Then, the evolution of the intensity function between two consecutive arrival times is governed by $\lambda_0^k(t)$ and $s^k(t)$.

In order to ensure the non-negativity of the process, the dynamic component Φ_i^k is specified in log-linear form, i.e.

$$\Phi_i^k = \exp\left(\tilde{\Phi}_i^k + z'_{i-1} \gamma^k\right), \quad (36)$$

where z_i denotes a vector of explanatory variables observed at arrival time t_i and γ^k is the corresponding parameter vector. Define ε_i as a scalar innovation term which is computed from the integrated intensity function associated with the most recently observed process, i.e.

$$\varepsilon_i := \sum_{k=1}^K \left(1 - \int_{t_{N^k(t_i)-1}^k}^{t_{N^k(t_i)}^k} \lambda^k(s) ds\right) y_i^k, \quad (37)$$

where y_i^k defines an indicator variable that takes on the value one if the i -th point of the pooled process is of type k and zero otherwise. According to the random time change argument presented in Section 2.4, ε_i corresponds to a random mixture of i.i.d. centered standard exponential variates and thus is itself an i.i.d. zero mean random variable. Then, the $(K \times 1)$ vector $\tilde{\Phi}_i = (\tilde{\Phi}_i^1, \dots, \tilde{\Phi}_i^K)'$ is parameterized as

$$\tilde{\Phi}_i = \sum_{k=1}^K \left(A^k \varepsilon_{i-1} + B^k \tilde{\Phi}_{i-1}\right) y_{i-1}^k, \quad (38)$$

where $A^k = \{a_j^k\}$ denotes a $(K \times 1)$ parameter vector and $B^k = \{b_{ij}^k\}$ is a $(K \times K)$ matrix of persistence parameters. Hence, the fundamental principle of the ACI model is that at each event t_i all K processes are updated by the realization of the integrated intensity with respect to the most recent process, where the impact of the innovation on the K processes can be different and also varies with the type of the most recent point. As suggested by Bowsher (2007), an alternative specification of the ACI innovation term could be $\tilde{\varepsilon}_i = 1 - \Lambda(t_{i-1}, t_i)$, where $\Lambda(t_{i-1}, t_i) := \sum_{k=1}^K \Lambda^k(t_{i-1}, t_i)$ denotes the integrated intensity of the pooled process computed between the two most recent points. Then, following the arguments above, $\tilde{\varepsilon}_i$ is a zero mean i.i.d. innovation term. Because of the regime-switching nature of the persistence matrix, the derivation of stationarity conditions is difficult. However, a sufficient (but not necessary) condition is that the eigenvalues of the matrices B^k for all $k = 1, \dots, K$ lie inside the unit circle.

As proposed by Hautsch (2004), the baseline intensity function $\lambda_0^k(t)$ can be specified as the product of K different Burr hazard rates, i.e.

$$\lambda_0^k(t) = \exp(\omega^k) \prod_{r=1}^K \frac{x^r(t)^{p_r^s - 1}}{1 + \eta_r^s x^r(t)^{p_r^s}}, \quad p_r^s > 0, \eta_r^s \geq 0. \quad (39)$$

According to this specification $\lambda^k(t)$ is driven not only by the k -type backward recurrence time but also by the time distance to the most recent point in all other processes $r = 1, \dots, K$ with $r \neq k$. A special case occurs when $p_r^s = 1$ and $\eta_r^s = 0, \forall r \neq s$. Then, the k -th process is affected only by its own history.

Finally, $s^k(t)$ is typically specified as a spline function in order to capture intraday seasonalities. A simple parameterization which is used in most studies is given by a linear spline function of the form $s^k(t) = 1 + \sum_{j=1}^S \nu_j^k (t - \tau_j) \cdot \mathbb{1}_{\{t > \tau_j\}}$, where $\tau_j, j = 1 \dots, S$, denote S nodes within a trading period and ν_j the corresponding parameters. A more flexible parameterization is e.g. given by a flexible Fourier form (Gallant (1981)) as used by Andersen and Bollerslev (1998) or Gerhard and Hautsch (2002) among others.

If $K = 1$ and $\eta_1^1 = 0$, the ACI model corresponds to a re-parameterized form of the Log-ACD model. If the ACI model is extended to include time-varying covariates (see Hall and Hautsch (2007)), it generalizes the approach by Hamilton and Jordà (2002). In this case, all event times associated with (discrete time) changes of time-varying covariates are treated as another point process that is not explicitly modelled. Then, at each event time of the covariate process, the multivariate intensity is updated, which requires a piecewise computation of the corresponding integrated intensities.

A generalization of the ACI model has been proposed by Bauwens and Hautsch (2006). The key idea is that the multivariate intensity function $\lambda(t) = (\lambda^1(t), \dots, \lambda^K(t))'$ is driven not only by the observable history of the process but also by a common component. The latter may be considered as a way to capture the unobservable general information flow in a financial market. By assuming the existence of a common unobservable factor $\lambda^*(t)$ following

a pre-assigned structure in the spirit of a doubly stochastic Poisson process (see Section 2.3), we define the internal (unobservable) history of $\lambda^*(t)$ as \mathcal{F}_t^* . We assume that $\lambda(t)$ is adapted to the filtration $\mathcal{F}_t := \sigma(\mathcal{F}_t^o \cup \mathcal{F}_t^*)$, where \mathcal{F}_t^o denotes some *observable* filtration. Then, the so-called *stochastic conditional intensity (SCI) model* is given by

$$\lambda^k(t) = \lambda^{o,k}(t) \left(\lambda_{\tilde{N}(t)+1}^* \right)^{\sigma_k^*}, \quad (40)$$

where $\lambda_{\tilde{N}(t)+1}^* := \lambda^*(t_{\tilde{N}(t)+1})$ denotes the common latent component which is updated at each point of the (pooled) process $\{t_i\}_{i \in \{1, \dots, n\}}$. The direction and magnitude of the process-specific impact of λ^* is driven by the parameters σ_k^* . The process-specific function $\lambda^{o,k}(t)$ denotes a conditionally deterministic idiosyncratic k -type intensity component given the *observable* history, \mathcal{F}_t^o .

Bauwens and Hautsch (2006) assume that λ_i^* has left-continuous sample paths with right-hand limits and follows a log-linear zero mean AR(1) process given by

$$\ln \lambda_i^* = a^* \ln \lambda_{i-1}^* + u_i^*, \quad u_i^* \sim iid N(0, 1). \quad (41)$$

Because of the symmetry of the distribution of $\ln \lambda_i^*$, Bauwens and Hautsch impose an identification assumption which restricts the sign of one of the scaling parameters σ_k^* . The observation-driven component $\lambda^{o,k}(t)$ is specified as the ACI model described above. However, in contrast to the basic ACI model, in the SCI model, the innovation term is computed based on the *observable* history of the process, i.e.

$$\varepsilon_i = \sum_{k=1}^K \left\{ -\varpi - \ln \Lambda^{o,k} \left(t_{N^k(t_i)-1}^k, t_{N^k(t_i)}^k \right) \right\} y_i^k, \quad (42)$$

where ϖ denotes Euler's constant, $\varpi = 0.5772$. Here, $\Lambda^{o,k} (t_{i-1}^k, t_i^k)$ is given by

$$\begin{aligned} \Lambda^{o,k} (t_{i-1}^k, t_i^k) &:= \sum_{j=N(t_{i-1}^k)}^{N(t_i^k)-1} \int_{t_j}^{t_{j+1}} \lambda^{o,k}(u) du \\ &= \sum_{j=N(t_{i-1}^k)}^{N(t_i^k)-1} (\lambda_j^*)^{-\sigma_k^*} \Lambda^k(t_j, t_{j+1}) \end{aligned} \quad (43)$$

corresponding to the sum of (piecewise) integrated k -type intensities which are observed through the duration spell and are standardized by the corresponding (scaled) realizations of the latent component. This specification ensures that ε_i can be computed exclusively based on past observables implying a separation between the observation-driven and the parameter-driven components

of the model. Bauwens and Hautsch (2006) analyze the probabilistic properties of the model and illustrate that the SCI model generates a wide range of (cross-)autocorrelation structures in multivariate point processes. In an application to a multivariate process of price intensities, they find that the latent component captures a substantial part of the cross-dependencies between the individual processes resulting in a quite parsimonious model. An extension of the SCI model to the case of multiple states is proposed by Koopman et al. (2008) and is applied to the modelling of credit rating transitions.

4.3 Statistical Inference

Karr shows that valid statistical inference can be performed based on the intensity function solely, see Theorem 5.2. in Karr (1991) or Bowsher (2007). Assume a K -variate point process $N(t) = \{N^k(t)\}_{k=1}^K$ on $(0, T]$ with $0 < T < \infty$, and the existence of a K -variate \mathcal{F}_t -predictable process $\lambda(t)$ that depends on the parameters θ . Then, it can be shown that a genuine log likelihood function is given by

$$\ln \mathcal{L}(\theta; \{N(t)\}_{t \in (0, T]}) = \sum_{k=1}^K \left[\int_0^T (1 - \lambda^k(s)) ds + \int_{(0, T]} \ln \lambda^k(s) dN^k(s) \right],$$

which can be alternatively computed by

$$\ln \mathcal{L}(\theta; \{N(t)\}_{t \in (0, T]}) = \sum_{i=1}^n \sum_{k=1}^K (-\Lambda^k(t_{i-1}, t_i)) + y_i^k \ln [\lambda^k(t_i)] + TK. \quad (44)$$

Note that (44) differs from the standard log likelihood function of duration models by the additive (integrating) constant TK which can be ignored for ML estimation. By applying the so-called exponential formula (Yashin and Arjas (1988)), the relation between the integrated intensity function and the conditional survivor function is given by

$$S(x|\mathcal{F}_t) = \exp[-\Lambda(t, t+x)], \quad (45)$$

where $S(x|\mathcal{F}_t) := \Pr[(t_{\tilde{N}(t)+1} - t) \geq x|\mathcal{F}_t]$. This is the continuous counterpart to the well-known relation between the survivor function and the hazard rate, $S(x_i) = \exp(-\int_0^{x_i} h(u) du)$. Hence, by ignoring the term TK , (44) corresponds to the sum of the conditional survivor function and the conditional intensity function. However, according to Yashin and Arjas (1988), the exponential formula (45) is only valid if $S(x|\mathcal{F}_t)$ is absolutely continuous in \mathcal{F}_t , which excludes jumps of the conditional survivor function induced by changes of the information set during a spell. Therefore, in a continuous, dynamic setting, the interpretation of $\exp(-\Lambda(t_{i-1}, t_i))$ as a survivor function should be done with caution.

The computation of (44) for a Hawkes model is straightforward. In the case of an exponential decay function, the resulting log likelihood function can be

even computed in a recursive way (see e.g. Bowsher (2007)). An important advantage of Hawkes processes is that the individual intensities $\lambda^k(t)$ do not have parameters in common and the parameter vector can be expressed as $\theta = (\theta^1, \dots, \theta^K)$, where θ^k denotes the parameters associated with the k -type intensity component. Given that the parameters are variation free, the log likelihood function can be computed as $\ln \mathcal{L}(\theta; \{N(t)\}_{t \in (0, T]}) = \sum_{k=1}^K l^k(\theta^k)$ and can be maximized by maximizing the individual k -type components $l^k(\theta^k)$ separately. This facilitates the estimation particularly when K is large. In contrast, ACI models require to maximize the log likelihood function with respect to all the parameters jointly. This is due to the fact that the ACI innovations are based on the integrated intensities which depend on all individual parameters. The estimation of SCI models is computationally even more demanding since the latent factor has to be integrated out resulting in an n -dimensional integral. Bauwens and Hautsch (2006) suggest to evaluate the likelihood function numerically using the efficient importance sampling procedure introduced by Richard and Zhang (2007). Regularity conditions for the maximum likelihood estimation of stationary simple point processes are established by Ogata (1981). For more details, see also Bowsher (2007).

Diagnostics for intensity based point process models can be performed by exploiting the stochastic properties of compensators and integrated intensities given in Section 2.4. The model goodness-of-fit can be straightforwardly evaluated through the estimated integrated intensities of the K individual processes, $e_{i,1}^k := \hat{\Lambda}^k(t_{i-1}^k, t_i^k)$, the integrated intensity of the pooled process $e_{i,2} := \hat{\Lambda}(t_{i-1}, t_i) = \sum_{k=1}^K \hat{\Lambda}^k(t_{i-1}, t_i)$, or of the (non-centered) ACI residuals $e_{i,3} := \sum_{k=1}^K \left(\hat{\Lambda}^k(t_{i-1}^k, t_i^k) \right) y_i^k$. Under correct model specification, all three types of residuals must be i.i.d. standard exponentially distributed. Then, model evaluation is done by testing the dynamic and distributional properties. The dynamic properties are easily evaluated with Portmanteau statistics or tests against independence such as proposed by Brock et al. (1996). The distributional properties can be evaluated using Engle and Russell's (1998) test against excess dispersion (see Section 3.2). Other alternatives are goodness-of-fit tests based on the probability integral transform (PIT) as employed for diagnostics on ACD models by Bauwens et al. (2004).

4.4 Applications

Dynamic intensity models are primarily applied in multivariate financial point processes or whenever a continuous-time setting is particularly required, e.g. to account for time-varying covariates. One strand of applications focusses on the modelling of trading intensities of different types of orders in limit order books. Hall and Hautsch (2007) apply a bivariate ACI model to study the intensities of buy and sell transactions in the electronic limit order book market of the Australian Stock Exchange (ASX). The buy and sell intensities are specified to depend on time-varying covariates capturing the state of the market. On

the basis of the buy and sell intensities, denoted by $\lambda^B(t)$ and $\lambda^S(t)$, Hall and Hautsch (2007) propose a measure of the continuous net buy pressure defined by $\Delta^B(t) := \ln \lambda^B(t) - \ln \lambda^S(t)$. Because of the log-linear structure of the ACI model, the marginal change of $\Delta^B(t)$ induced by a change of the covariates is computed as $\gamma^B - \gamma^S$, where γ^B and γ^S denote the coefficients associated with covariates affecting the buy and sell intensity, respectively (see eq. (36)). Hall and Hautsch (2006) study the determinants of order aggressiveness and traders' order submission strategy at the ASX by applying a six-dimensional ACI model to study the arrival rates of aggressive market orders, limit orders as well as cancellations on both sides of the market. In a related paper, Large (2007) studies the resiliency of an electronic limit order book by modelling the processes of orders and cancellations on the London Stock Exchange using a ten-dimensional Hawkes process. Russell (1999) analyzes the dynamic interdependence between the supply and demand for liquidity by modelling transaction and limit order arrival times at the NYSE using a bivariate ACI model.

Another branch of the literature focusses on the modelling of the instantaneous price change volatility which is estimated on the basis of price durations, see (28) in Section 3.4. This relation is used by Bauwens and Hautsch (2006) to study the interdependence between instantaneous price change volatilities of several blue chip stocks traded at the NYSE based on a SCI model. In this setting, they find a strong evidence for the existence of a common latent component as a major driving force of the instantaneous volatilities on the market. In a different framework, Bowsher (2007) analyzes the two-way interaction of trades and quote changes using a two-dimensional generalized Hawkes process.

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